A NOTE ON EPI-CONVERGENCE OF SUMS UNDER THE INF-ADDITION RULE

JAMES V. BURKE AND TIM HOHEISEL

Abstract. We consider sums of sequences of extended real-valued functions focusing on the epigraphical liminf and limsup inequalities. Specifically, we note a deficiency in the statement of Theorem 7.46 in *Variational Analysis*, Springer, Berlin, Heidelberg, 1998. An elementary counterexample is provided, and a remedy that is well suited to the class of DC functions is established.

Version: June 23, 2015

1 Introduction

We consider sums of sequences of extended real-valued functions, i.e., functions that can take the values $+\infty$ and $-\infty$. For such functions the addition operation can lead to ambiguities and so rules for this operation need to be given in the infinite valued cases. In some cases, there is no ambiguity in the appropriate definition; for example,

$$\infty + \infty := \infty, \quad -\infty + (-\infty) := -\infty, \quad a + \infty := \infty, \quad a + (-\infty) := -\infty$$

for all $a \in \mathbb{R}$. The troublesome cases are $\infty + (-\infty)$ and $-\infty + \infty$. In the optimization community, the so-called *inf-addition rule*,

$$\infty + (-\infty) := \infty \tag{1}$$

is broadly used, e.g., see [4, Page 15]. This rule is usually applied in the context of minimization where it has a number of useful consequences. The rule is particularly useful in dealing with DC (difference of convex) functions , where the case $\infty+(-\infty)$ naturally occurs, e.g., see [2] or [5].

The primary issue addressed by this note concerns the analysis of the liminf and limsup inequalities for sums of extended real-valued functions. Specifically, we observe a flaw in the statement of [4, Theorem 7.46] in the case of the epi-liminf inequality for sums (see Example 3.3). The focus of this note is to suggest a remedy that is well suited to the class of DC functions (see Theorem 3.2).
The notation we employ follows that given in [4].

2 Extended real-valued sequences and inf-addition

We begin by first considering such limits for sequences of extended real-valued numbers.

2010 *Mathematics Subject Classification.* 49J52, 49J53, 90C25, 90C26, 90C46.
Key words and phrases. epi-convergence, epi-liminf, DC functions.
This research was partially supported by the DFG (Deutsche Forschungsgemeinschaft) under grant HO 4739/2-1, and the *Prodi-Chair* program at the University of Würzburg.

Lemma 2.1 (Extended lim inf and lim sup inequalities under inf-addition)**.** *Let* $\{a_k \in \overline{\mathbb{R}}\}, \{b_k \in \overline{\mathbb{R}}\}$. *Then the following hold:*

a) *We have* $\limsup_{k\to\infty} a_k + b_k \leq \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k$.

b) *Defining*

$$\underline{a} := \liminf_{k\to\infty} a_k, \quad \underline{b} := \liminf_{k\to\infty} b_k, \quad \text{and} \quad \underline{c} := \liminf_{k\to\infty} a_k + b_k$$

then either

$$\liminf_{k\to\infty} a_k + b_k \geq \liminf_{k\to\infty} a_k + \liminf b_k. \tag{2}$$

or

$$\underline{c} < \infty = \max\{\underline{a}, \underline{b}\} \tag{3}$$

holds true, but not both. In particular, the inequality (2) *holds under any of the conditions*

$$\underline{a} \in \mathbb{R}, \quad \underline{b} \in \mathbb{R} \quad \text{and} \quad \underline{a} \neq -\underline{b}. \tag{4}$$

Proof. a) For the cases that no inf-addition rule is needed for the right-hand side of the inequality in a), this is covered by [1, Propositions 5.6 and 4.5]. For the case that inf-addition occurs, the follows trivially, since in this case the right-hand side is ∞.

b) If (3) holds then, obviously, (2) is violated.

On the other hand, suppose that (3) is violated: If $\underline{c} = +\infty$, then (5) trivially holds; otherwise, $\max\{\underline{a}, \underline{b}\} < \infty$, and the inf-addition rule does not apply in which case (2) follows from [1, Propositions 5.6 and 4.5].

The remainder is also covered by the latter reference, but can also be obtained by showing that (4) implies the negation of (3).

□

The fact that there is an asymmetry in Lemma 2.1 with respect to liminf and limsup (no assumptions needed in a)) is due to the fact that inf-addition is an asymmetric notion.

The necessity of the assumptions in Lemma 2.1 b) is illustrated by the following example.

Example 2.2. *Consider the sequences* $\{a_k \in \overline{\mathbb{R}}\}, \{b_k \in \overline{\mathbb{R}}\}$ *with* $a_k := k, b_k := -k$ $(k \in \mathbb{N})$. *Then we have* $\lim_{k\to\infty} a_k = +\infty$, $\lim_{k\to\infty} b_k = -\infty$, *hence, by inf-addtion,*

$$\lim_{k\to\infty} a_k + \lim_{k\to\infty} b_k = \infty > 0 = \lim_{k\to\infty} a_k + b_k.$$

3 Epi-convergence of sequences of sums

Let $f_k, f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\bar{x} \in \mathbb{R}^n$ $(k \in \mathbb{N})$. We define the *epigraphical limit inferior* and *limit superior*, repsectively, of $\{f_k\}$ at $\bar{x}$ by

$$(\text{e}-\liminf_{k\to\infty} f_k)(\bar{x}) \quad := \quad \min\left\{\alpha \;\middle|\; \exists\{x^k\} \to \bar{x} : \liminf f_k(x^k) = \alpha\right\}, \tag{5}$$

$$(\text{e}-\limsup_{k\to\infty} f_k)(\bar{x}) \quad := \quad \min\left\{\alpha \;\middle|\; \exists\{x^k\} \to \bar{x} : \limsup f_k(x^k) = \alpha\right\}. \tag{6}$$

Clearly, we always have

$$(\text{e}-\liminf_{k\to\infty} f_k)(\bar{x}) \leq (\text{e}-\limsup_{k\to\infty} f_k)(\bar{x}).$$

The sequence $\{f_k\}$ is said to converge *epigraphically* at $\bar{x}$ to the function f if

$$(\text{e}-\liminf_{k\to\infty} f_k)(\bar{x}) \geq f(\bar{x}) \geq (\text{e}-\limsup_{k\to\infty} f_k)(\bar{x})$$

and we write $f_k \xrightarrow{e} f$ at $\bar{x}$. In view of (5) and (6) this can be characterized by

$$\begin{aligned} \forall\{x^k\} \to \bar{x}: &\qquad \liminf_{k\to\infty} f_k(x^k) \geq f(\bar{x}), \\ \exists\{x^k\} \to \bar{x}: &\qquad \limsup_{k\to\infty} f_k(x^k) \leq f(\bar{x}). \end{aligned}$$

Moreover, we say that $\{f_k\}$ converges *continuously* to f (and write $f_k \xrightarrow{c} f$) at $\bar{x}$ if

$$\forall\{x^k\} \to \bar{x}: \quad \lim_{k\to\infty} f_k(x^k) = f(\bar{x}).$$

In addition, we say that $\{f_k\}$ converges *pointwise* to f (and write $f_k \xrightarrow{p} f$) at $\bar{x}$ if

$$\lim_{k\to\infty} f_k(\bar{x}) = \bar{x}.$$

Proposition 3.1. *Let* $\phi_k, \phi, \rho_k, \rho : \mathbb{R}^n \to \overline{\mathbb{R}}$. *If*

$$\liminf_{k\to\infty} \phi_k(x^k) + \rho_k(x^k) \geq \liminf_{k\to\infty} \phi_k(x^k) + \liminf_{k\to\infty} \rho_k(x^k) \quad \forall\ \{x^k\} \to \bar{x},$$

then

$$(\text{e}-\liminf_{k\to\infty} \phi_k + \rho_k)(\bar{x}) \geq (\text{e}-\liminf_{k\to\infty} \phi_k)(\bar{x}) + (\text{e}-\liminf_{k\to\infty} \rho_k)(\bar{x}).$$

Proof. Take $x^k \to \bar{x}$ such that

$$\liminf_{k\to\infty} \phi_k(x^k) + \rho_k(x^k) = (\text{e}-\liminf_{k\to\infty} \phi_k + \rho_k)(\bar{x}).$$

It follows that

$$\begin{aligned} (\text{e}-\liminf_{k\to\infty} \phi_k + \rho_k)(\bar{x}) &= \liminf_{k\to\infty} \phi_k(x^k) + \rho_k(x^k) \\ &\geq \liminf_{k\to\infty} \phi_k(x^k) + \liminf_{k\to\infty} \rho_k(x^k) \\ &\geq (\text{e}-\liminf_{k\to\infty} \phi_k)(\bar{x}) + (\text{e}-\liminf_{k\to\infty} \rho_k)(\bar{x}). \end{aligned}$$

Here the equality is due to the choice of $\{x^k\}$, the first inequality follows by assumption and the second one from (5). □

The main contribution of this note now follows.

Theorem 3.2 ([4], Theorem 7.46 revisited)**.** *Let* $\phi, \phi_k, \rho, \rho_k : \mathbb{R}^n \to \overline{\mathbb{R}}$ $(k \in \mathbb{N})$ *and* $\bar{x} \in \mathbb{R}^n$ *such that*

$$\phi_k \xrightarrow{e} \phi \text{ at } \bar{x} \quad \text{and} \quad \rho_k \xrightarrow{e} \rho \text{ at } \bar{x}.$$

Then

$$(\text{e}-\liminf_{k\to\infty} \phi_k + \rho_k)(\bar{x}) \geq \phi(\bar{x}) + \rho(\bar{x})$$

if one of the following two conditions holds:

(I) $\phi(\bar{x}), \rho(\bar{x}) > -\infty$.
(II) $\bar{x} \in \operatorname{dom}\phi$ *and* $\rho(\bar{x}) = -\infty$ *(or* $\bar{x} \in \operatorname{dom}\rho$ *and* $\phi(\bar{x}) = -\infty$*)*

Moreover, we have $\phi_k + \rho_k \xrightarrow{e} \phi + \rho$ *at* $\bar{x}$ *if, in addition to (I) or (II) holding, one the following conditions holds:*

(i) $\phi_k \xrightarrow{c} \phi$ *at* $\bar{x}$ *or* $\rho_k \xrightarrow{c} \rho$ *at* $\bar{x}$.
(ii) *both* $\phi_k \xrightarrow{p} \phi$ *and* $\rho_k \xrightarrow{p} \rho$ *at* $\bar{x}$.

Proof. First note that the assumption (II) implies that $(\phi+\rho)(\bar{x}) = -\infty$, hence, trivially, $\liminf_{k\to\infty}(\phi_k+\rho_k)(x^k) \ge (\phi+\rho)(\bar{x})$ for any sequence $\{x^k\}\to\bar{x}$.

Now, assume that (I) holds, and let $\{x^k\}\to\bar{x}$ be given. Then, in view of the respective epi-convergence properties, we have

$$\liminf_{k\to\infty}\phi_k(x^k) \ge \phi(\bar{x}) \quad \text{and} \quad \liminf_{k\to\infty}\rho_k(x^k) \ge \rho(\bar{x}).$$

Due to assumption (I), we can apply Lemma 2.1 to infer that

$$\liminf_{k\to\infty}(\phi_k+\rho_k)(x^k) \ge \liminf_{k\to\infty}\phi_k(x^k) + \liminf_{k\to\infty}\rho_k(x^k) \ge (\phi+\rho)(\bar{x}).$$

In order to complete the proof, we follow the approach taken in [4, Theorem 7.46] where it is observed that it suffices to show that there exists $\{x^k\}\to\bar{x}$ such that

$$\limsup_{k\to\infty}(\phi_k+\rho_k)(x^k) \le (\phi+\rho)(\bar{x}) = \phi(\bar{x})+\rho(\bar{x}). \tag{7}$$

in either of the cases (i) or (ii).

For these purposes, first assume that assumption (i) holds. With no loss in generality, we only consider the case where $\phi_k \overset{c}{\to} \phi$ at $\bar{x}$. Moreover, let $\{x^k\}\to\bar{x}$ such that $\lim_{k\to\infty}\rho_k(x^k) = \rho(\bar{x})$, which exists as $\rho_k \overset{e}{\to} \rho$ at $\bar{x}$. Then it follows that

$$\limsup_{k\to\infty}\phi_k(x^k) = \phi(\bar{x}) \quad \text{and} \quad \limsup_{k\to\infty}\rho_k(x^k) = \rho(\bar{x}).$$

Under either (I) or (II), we can invoke Lemma 2.1a) to get (7).

If, in turn, assumption (ii) holds, we consider the constant sequence $\{x^k := \bar{x}\}$, and get

$$\limsup_{k\to\infty}\phi_k(x^k) = \phi(\bar{x}) \quad \text{and} \quad \limsup_{k\to\infty}\rho_k(x^k) = \rho(\bar{x}).$$

Again, under either (I) or (II), we can hence invoke Lemma 2.1 to get (7).

This concludes the proof. □

The following example illustrates the need for the conditions (I) and (II) in Theorem 3.2 and also servers as a counterexample to the current statement of [4, Theorem 7.46].

Example 3.3. *Consider the sequences of funtions* $f_k := k, g_k := -f_k$. *Then, obviously* $f_k \overset{c}{\to} +\infty$ *and* $g_k \overset{c}{\to} -\infty$, *hence, in particular* $\text{e-}\lim f_k + \text{e-}\lim g_k = +\infty - \infty = +\infty$. *On the other hand,* $\text{e-}\lim(f_k+g_k) = \text{e-}\lim 0 \equiv 0$.

4 DC functions

In this section we consider the application of Theorem 3.2 to the important class of functions known as *DC functions*, where the inf-addition rule (1) plays a fundamental role.

Definition 4.1 (DC Functions)**.** *Let*

$$\Gamma := \{\varphi : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\} \mid \varphi \ \ \textit{is closed proper and convex}\}.$$

Given $g, h \in \Gamma$, *define* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *by*

$$f(x) := \begin{cases} g(x) - h(x), & x \in \operatorname{dom} g \cup \operatorname{dom} h,\\ +\infty, & \textit{else}. \end{cases}$$

We call f *a DC function, where DC stands for difference of convex.*

Note that this definition for DC functions makes explicit use of the inf-addition rule (1). DC functions arise in numerous applications and have been extensively studied in the literature [2, 3, 5]. Smoothing for DC functions has played an important role in the development of numerical methods for the minimization of DC objective functions. It this setting the epi-convergence properties of these approximations is key to understanding the behavior of these methods. Here we apply Theorem 3.2 to give some insight into what kinds of epi-convergence results are possible.

Lemma 4.2. *Let $g, h \in \Gamma$. Then one the conditions (I) and (II) of Theorem 3.2 hold, with $\phi := g$ and $\rho := -h$, for all $\bar{x} \in \operatorname{dom} g \cup \operatorname{dom} h$. Therefore, every DC function $f := g - h$ satisfies one of conditions (I) and (II) on the set $\operatorname{dom} g \cup \operatorname{dom} h$.*

Proof. Let $\bar{x} \in \operatorname{dom} g \cup \operatorname{dom} h$. If $\bar{x} \in \operatorname{dom} h$, then (I) holds since $g, h \in \Gamma$. On the other hand, if $\bar{x} \notin \operatorname{dom} h$, then $\bar{x} \in \operatorname{dom} g$ and $-h(\bar{x}) = -\infty$ so that (II) holds. □

Proposition 4.3. *Let $g, h \in \Gamma$, $\{g_k\} \subset \Gamma$, $\{h_k\} \subset \Gamma$ and $\bar{x} \in \operatorname{dom} g \cup \operatorname{dom} h$ such that either*

(i) $g_k \overset{e}{\to} g$ at $\bar{x}$ and $h_k \overset{c}{\to} h$ at $\bar{x}$, *or*

(ii) $g_k \overset{e}{\to} g$ *and* $g_k \overset{p}{\to} g$ at $\bar{x}$ and $h_k \overset{e}{\to} h$ *and* $_k \overset{p}{\to} h$ at $\bar{x}$.

Then $g_k - h_k \overset{e}{\to} g - h$ at $\bar{x}$.

Proof. By Lemma 4.2, one of the conditons (I) and (II) holds at $\bar{x}$, so Theorem 3.2 applies. If (i) holds, then we also have $-h_k \overset{e}{\to} -h$ at $\bar{x}$ since $-h_k \overset{c}{\to} -h$ at $\bar{x}$, and the assertion follows from Theorem 3.2 applied to $\phi_{(k)} := g_{(k)}$ and $\rho_{(k)} := -h_{(k)}$, respectively. If (ii) holds, the result again follows from Theorem 3.2 applied to $\phi_{(k)} := g_{(k)}$ and $\rho_{(k)} := -h_{(k)}$, respectively. □

Corollary 4.4. *Let $g, h \in \Gamma$, $\{g_k\} \subset \Gamma$, $\{h_k\} \subset \Gamma$ and $\bar{x} \in \operatorname{dom} g \cup \operatorname{dom} h$ such that*

$$g_k \overset{e}{\to} g \text{ at } \bar{x} \quad \text{and} \quad h_k \overset{e}{\to} h \text{ at } \bar{x}.$$

If h is finite-valued in a neighborhood of $\bar{x}$, then $g_k - h_k \overset{e}{\to} g - h$ at $\bar{x}$.

Proof. By [4, Theorem 7.17 (c)], the hypotheses imply that $h_k \overset{c}{\to} h$ at $\bar{x}$, hence the result follows from Proposition 4.3. □

References

[1] L.D. Drager: *Notes onf sup's, inf's and sequences. Real Analysis Course Notes: Fall 2001*, Texas Tech University.

[2] J.-B Hiriart-Urruty: *How to regularize a difference of convex functions.* Journal of Mathematical Analysis and Applications 162, 1991, pp. 196–209.

[3] R. Horst and N.V. Thoai: *DC Programming: Overview.* Journal of Optimization Theory and Applications 103, 1999, pp. 1–43.

[4] R.T. Rockafellar and R.J.-B. Wets: *Variational Analysis.* A Series of Comprehensive Studies in Mathematics, Vol. 317, Springer, Berlin, Heidelberg, 1998.

[5] J.F. Toland: *A duality principle for non-convex optimisation and the calculus of variations* Archive for Rational Mechanics and Analysis 71(1), 1979, pp. 41-61,

University of Washington, Department of Mathematics, Box 354350 Seattle, Washington 98195-4350

E-mail address: burke@math.washington.edu

University of Würzburg, Institute of Mathematics, Campus Hubland Nord, Emil-Fischer-Strasse 30, 97074 Würzburg, Germany

E-mail address: hoheisel@mathematik.uni-wuerzburg.de