Spring School on Variational Analysis
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Topics in Convex Analysis in Matrix Space

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Lecture Notes

Abstract This set of notes constitutes a snapshot in time of some recent results by the author and his collaborators on different topics from convex analysis of functions of matrices. These topics are tied together by their common underlying themes, namely support functions, infimal convolution, and K-convexity. A complete exposition of the basic convex-analytic concepts employed makes the presentation self-contained.

Keywords Convex analysis, Fenchel conjugate, infimal convolution, K-convexity, generalized matrix-fractional function, Moore-Penrose pseudoinverse, support function, Hörmander's Theorem, variational Gram function.

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Contents

1 Introduction

This set of notes was created for a series of invited lectures at the *Spring School on Variational Analysis* in Paseky nad Jizerou, Czech Republic, May 19–25, 2019.

We start in Chapter 2 with some preliminaries about (finite dimensional) Euclidean spaces, extended arithmetic and lower semicontinuity which is used throughout.

Following the preliminary section there are the three main chapters, Chapter 3, 4 and 5: In Chapter 3 we present, in a self-contained way, the fundamental tools from convex analysis that are necessary to understand the subsequent study. An emphasis is put on the study of support and gauge functions, including *Hörmander's theorem* and, most importantly, on *infimal convolution*, upon which we build our presentation of many important results in convex analysis, and which is a main workhorse for large parts of our study. The approach centered around the conjugacy relations between infimal convolution and addition of functions is inspired by the excellent textbook [2] by Bauschke and Combettes. The rest of this section is an eclectic selection from other standard references such as [19] by Hiriart-Urruty and Lemaréchal, the classic and ever-current book by Rockafellar [24] and the monumental bible of variatonal analysis by Rockafellar and Wets [25], also reflecting some of the author's personal preferences.

In Chapter 4 we take a broader perspective on convexity through a generalized notion of convexity for vector-valued functions with respect to partial orderings induced by a (closed, convex) cone K, which is referred to as *K-convexity* in the literature. Important groundwork in this area was laid by Borwein in his thesis [4]. Another pivotal (and somewhat overlooked) contribution was made by Pennanen in [23]. Here, we focus on conjugacy results for the composition $g \circ F$ of a (closed, proper) convex function g with a nonlinear map F. Maps of this form are called *convex-composite functions*. We study the case where the generalized convexity properties of the nonlinear map F and the monotonicity properties of the convex function g align in such a way that the composite $g \circ F$ is still convex. Our main result, Theorem 4.7, provides sufficient conditions for the formula

$$(g \circ F)^*(p) = \min_{v \in -K^\circ} g^*(v) + \langle v,\, F\rangle^* (p)$$

to hold, thus complementing the work by Bot and Wanka on that subject, see e.g. [5]. The main ingredients for the proof of said result are K-convexity and the conjugacy calculus for infimal convolution. The material presented in Section 4 grew out of an ongoing collaboration with my postdoc Quang Van Nguyen at McGill.

Chapter 5 is based on a series of papers [7, 8, 9] by the author and James V. Burke as well as his PhD student Yuan Gao at the University of Washington, Seattle. The central object of study is the *generalized matrix-fractional function (GMF)*

$$\varphi_{A,B} : (X,V) \in \mathbb{R}^{n\times m} \times \mathbb{S}^n \mapsto \begin{cases} \frac{1}{2}\mathrm{tr}\left(\binom{X}{B}^T M(V)^\dagger \binom{X}{B}\right) & \text{if } \ \mathrm{rge}\binom{X}{B} \subset \mathrm{rge}\, M(V),\ V \in \mathcal{K}_A, \\ +\infty, & \text{else.} \end{cases}$$

Here, $A \in \mathbb{R}^{p\times n}$ and $B \in \mathbb{R}^{p\times m}$ are such that $\mathrm{rge}\, B \subset \mathrm{rge}\, A$, $\mathcal{K}_A \subset \mathbb{S}^n$ is the set of all symmetric matrices that are positive semidefinite on the kernel of A, and $M(V)^\dagger$ is the *Moore-Penrose pseudoinverse* of the bordered matrix $M(V) = \begin{pmatrix} V & A^T \\ A & 0 \end{pmatrix}$. The GMF occurs in different situations, and for different choices of A, B, in matrix optimization or machine and statistical learning, see Section 5.1 and 5.2. A main theoretical result, see Theorem 5.8, is the observation that the GMF is the support function of the set

$$\mathcal{D}(A,B) := \left\{ \left(Y, -\frac{1}{2}YY^T\right) \in \mathbb{R}^{n\times m} \times \mathbb{S}^n \;\middle|\; Y \in \mathbb{R}^{n\times m} :\; AY = B \right\}.$$

The second main result, Theorem 5.10, provides the formula

$$\overline{\operatorname{conv}}\,\mathcal{D}(A,B) = \left\{ (Y,W) \in \mathbb{R}^{n\times m} \times \mathbb{S}^n \;\middle|\; AY = B \text{ and } \frac{1}{2}YY^T + W \in \mathcal{K}_A^\circ \right\} \tag{1.1}$$

for the closed convex hull of the set $\mathcal{D}(A,B)$. Combining these two facts opens the door for a full convex-analytical study of the GMF, and for using it in various applications such as *nuclear norm smoothing*, see Section 5.5.2, or *variational Gram functions*, see Section 5.5.1. The description of the closed, convex hull of $\mathcal{D}(A,B)$ also establishes a tie between the GMF and our study of K-convexity in Chapter 4 (which is also explicitly used in the proof of Theorem 5.26): The special case $A = 0$ and $B = 0$ in (1.1) gives

$$\overline{\operatorname{conv}}\,\mathcal{D}(A,B) = \left\{ (Y,W) \in \mathbb{R}^{n\times m} \times \mathbb{S}^n \;\middle|\; \frac{1}{2}YY^T + W \preceq 0 \right\}.$$

In the language of K-convexity, this means that the closed, convex hull of $\mathcal{D}(0,0)$ is the negative K-epigraph of the map $Y \mapsto \frac{1}{2}YY^T$ with respect to the positive semidefinite matrices.

The notes end with an appendix on background from linear algebra and the bibliography.

2 Preliminaries

2.1 The Euclidean setting

In what follows, $\mathbb{E}$ will be an N-dimensional ($N \in \mathbb{N}$) Euclidean space, i.e. a finite-dimensional real vector space equipped with a *scalar product*, which we denote by $\langle \cdot, \cdot \rangle$. Recall that a scalar product on $\mathbb{E}$ is a mapping $\langle \cdot, \cdot \rangle : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ such that for all $x, y, z \in \mathbb{E}$ and $\lambda, \mu \in \mathbb{R}$ we have:

i) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$ *(linearity in first argument)*;

ii) $\langle x, y \rangle = \langle y, x \rangle$ (symmetry);

iii) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$ *(positive definiteness)*.

Altogether $\langle \cdot, \cdot \rangle$ is a *positive definite*, *symmetric bilinear form* on $\mathbb{E}$.

By $\|\cdot\|$ we label the norm[2] on $\mathbb{E}$ induced by the scalar product, i.e.

$$\|x\| := \sqrt{\langle x, x \rangle} \quad (x \in \mathbb{E}),$$

[2]Recall that a norm on $\mathbb{E}$ is a mapping $\|\cdot\|_* : \mathbb{E} \to \mathbb{R}$ such that for all $x, y \in \mathbb{E}$ and $\lambda \in \mathbb{R}$:

i) $\|x\|_* = 0 \iff x = 0$ (definiteness)

ii) $\|\lambda x\|_* = |\lambda| \cdot \|x\|_*$ (absolute homogeneity)

iii) $\|x + y\|_* \leq \|x\|_* + \|y\|_*$ (triangle inequality)

Scalar product and induced norm obey the famous *Cauchy-Schwarz inequality*

$$|\langle x,\, y\rangle| \leq \|x\| \cdot \|y\| \quad (x, y \in \mathbb{E}),$$

where equality holds if and only if x and y are linearly dependent.

The open ball with radius $\varepsilon > 0$ centered around $x \in \mathbb{E}$ is denoted by $B_\varepsilon(x)$. In particular, we put $\mathbb{B} := B_1(0)$ for the open unit ball.

If $(\mathbb{E}_i, \langle \cdot,\, \cdot\rangle_i)$ $(i = 1, \dots, m)$ is a Euclidean space, then $\bigtimes_{i=1}^m \mathbb{E}_i$ is also a Euclidean space equipped with the canonical scalar product

$$\langle \cdot,\, \cdot\rangle : \bigtimes_{i=1}^m \mathbb{E}_i \to \mathbb{R}, \quad \langle (x_1, \dots, x_m),\, (y_1, \dots, y_m)\rangle := \sum_{i=1}^m \langle x_i,\, y_i\rangle_i\,.$$

For $n = \dim \mathbb{E}_1, m = \dim \mathbb{E}_2 < \infty$, the set of all *linear (hence continuous) operators* is denoted by $\mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. Recall from Linear Algebra that $\mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ is isomorphic to $\mathbb{R}^{m \times n}$, the set of all real $m \times n$-matrices, and that $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ if and only if

1) $L(\lambda x) = \lambda L(x) \quad (x \in \mathbb{E}_1, \lambda \in \mathbb{R})$ (homogeneity);

2) $L(x + y) = L(x) + L(y) \quad (x, y \in \mathbb{E}_1)$ (additivity).

It is known from Linear Algebra that (since we restrict ourselves to finite dimensional Euclidean spaces alone) for $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ there exists[3] a unique mapping $L^* \in \mathcal{L}(\mathbb{E}_2, \mathbb{E}_1)$ such that

$$\langle Lx,\, y\rangle_{\mathbb{E}_2} = \langle x,\, L^*y\rangle_{\mathbb{E}_1} \quad (x, y \in \mathbb{E}_1).$$

The mapping L^* is called the *adjoint (mapping)* of L. If $\mathbb{E}_1 = \mathbb{E}_2$ and $L = L^*$, we call L *self-adjoint.*

With the well-known definitions for

$$\operatorname{rge} L := \{L(x) \in \mathbb{E}_2 \mid x \in \mathbb{E}_1\} \quad \text{(image of } L\text{)}$$

and

$$\ker L := \{x \in \mathbb{E}_1 \mid L(x) = 0\} \quad \text{(kernel of } L\text{)}$$

for $L : \mathbb{E}_1 \to \mathbb{E}_2$ linear, the following important relations are standard knowledge from Linear Algebra.

At this, recall that, for some nonempty subset $S \subset \mathbb{E}$ we define its *orthogonal complement* by

$$S^\perp := \{x \in \mathbb{E} \mid \langle s,\, x\rangle = 0 \quad (s \in S)\}\,.$$

Theorem 2.1 (Fundamental subspaces)**.** *Let $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. Then the following hold:*

a) $\ker L = (\operatorname{rge} L^*)^\perp$ *and* $(\ker L)^\perp = \operatorname{rge} L^*$*;*

b) $\ker L^* = (\operatorname{rge} L)^\perp$ *and* $(\ker L^*)^\perp = \operatorname{rge} L$.

[3] The unique existence of the adjoint mapping is already guaranteed if only the preimage space is finite dimensional.

Minkowski addition and multiplication

For two sets $A, B \subset \mathbb{E}$ we define their *Minkowski sum* by

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

If $B = \{b\}$ is a singleton, we put

$$A + B =: A + b.$$

Moreover, if $A = \emptyset$ (or $B = \emptyset$) then $A + B := \emptyset$.

In addition, for $\Lambda \subset \mathbb{R}$, we put

$$\Lambda \cdot A := \{\lambda a \mid a \in A, \lambda \in \Lambda\}.$$

The operation $(\Lambda, A) \subset 2^{\mathbb{R}} \times 2^{\mathbb{E}} \to 2^{\mathbb{E}}$ is called *(generalized) Minkowski multiplication*. For $\Lambda = \{\lambda\}$ we simply write

$$\lambda A := \{\lambda\} \cdot A.$$

Using the above notation, for instance, we have

$$B_\varepsilon(x) = x + \varepsilon\mathbb{B}$$

for all $x \in \mathbb{E}$ and $\varepsilon > 0$.

2.2 Extended arithmetic and extended real-valued functions

Let $\overline{\mathbb{R}} := [-\infty, \infty]$ be the *extended real line*. The following conventions for an extended arithmetic have become standard in the optimization and convex analysis community: The uncritical ones are

$$\alpha + \infty = +\infty = \infty + \alpha \quad \text{and} \quad \alpha - \infty = -\infty + \alpha = -\infty \qquad (\alpha \in \mathbb{R}),$$
$$\alpha \cdot \infty = \mathrm{sgnffl}(\alpha) \cdot \infty = \infty \cdot \alpha \quad \text{and} \quad \alpha \cdot (-\infty) = -\mathrm{sgnffl}(\alpha) \cdot \infty = -\infty \cdot \alpha \quad (\alpha \in \mathbb{R} \setminus 0).$$

Although we will try to avoid these cases whenever possible it is expedient for our purposes to also use the following conventions:

$$0 \cdot \infty = 0 = 0 \cdot (-\infty),$$
$$\infty - \infty = -\infty + \infty = \infty \quad \text{(inf-addition)}.$$

Every subset $S \subset \overline{\mathbb{R}}$ of the extended real line has a *supremum* (least upper bound) and an *infimum* (greatest lower bound), which could be infinite. We use the common conventions

$$\inf \emptyset = +\infty \quad \text{and} \quad \sup \emptyset = -\infty.$$

Functions of the type $f : \mathbb{E} \to \overline{\mathbb{R}}$ occur naturally in various areas of mathematics, in particular in optimization (e.g. optimal value functions) or measure theory, as soon as suprema or infima are involved.

The *domain* of a function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is given by

$$\operatorname{dom} f := \{x \mid f(x) < \infty\}.$$

We call f *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{E}$.

A central object of study for extended real-valued functions is the *epigraph*, which for $f : \mathbb{E} \to \overline{\mathbb{R}}$ is defined by

$$\operatorname{epi} f := \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid f(x) \leq \alpha\},$$

see Figure 1 for an illustration. The epigraph establishes a one-to-one correspondence of sets in $\mathbb{E} \times \mathbb{R}$ and functions $\mathbb{E} \to \overline{\mathbb{R}}$. In fact, all the important convex-analytical properties of an extended real-valued function (like lower semicontinuity, convexity, positive homogeneity or sublinearity) have their correspondence in the geometry and topology, respectively, of their epigraph. It is sometimes expedient to also consider the *strict epigraph*

$$\operatorname{epi}_< f := \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid f(x) < \alpha\}$$

of $f : \mathbb{E} \to \overline{\mathbb{R}}$.

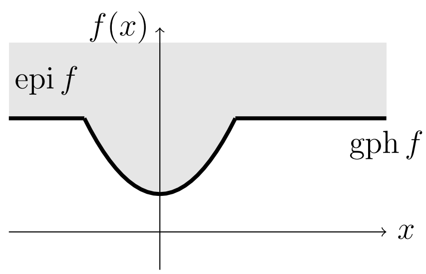

Figure 1: Epigraph of a function $f : \mathbb{R} \to \mathbb{R}$

Another useful tool are the *level sets* of f, which are defined by

$$\operatorname{lev}_{\leq \alpha} f := \{x \in \mathbb{E} \mid f(x) \leq \alpha\} \quad (\alpha \in \mathbb{R}).$$

We call f *level-bounded* if $\operatorname{lev}_{\leq \alpha} f$ is bounded for all $\alpha \in \mathbb{R}$. Level-boundedness is tremendously important for the existence of solutions of optimization problems, see Theorem 2.8.

The most prominent example of an extended real-valued function is as simple as it is important.

Definition 2.2 (Indicator function)**.** *For a set $S \subset \mathbb{E}$ the mapping $\delta_S : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ given by*

$$\delta_S(x) := \begin{cases} 0 & \text{if} \quad x \in S, \\ +\infty & \text{if} \quad x \notin S \end{cases}$$

is called the indicator (function) *of S.*

2.3 Lower semicontinuity

We now want to establish the notion of lower semicontinuity for extended real-valued functions. To this end, for $f : \mathbb{E} \to \overline{\mathbb{R}}$ and $\bar{x} \in \mathbb{E}$, we define

$$\liminf_{x \to \bar{x}} f(x) := \inf \left\{ \alpha \in \overline{\mathbb{R}} \;\middle|\; \exists \{x^k\} \to \bar{x} : \; f(x^k) \to \alpha \right\} \tag{2.1}$$

as the *lower limit* of f at $\bar{x}$.

Definition 2.3 (Continuity notions for extended-real valued functions)**.** *Let* $f : \mathbb{E} \to \overline{\mathbb{R}}$ *and* $\bar{x} \in \mathbb{E}$*. Then* f *is said to be* lower semicontinuous (lsc) *at* $\bar{x}$ *if*

$$\liminf_{x \to \bar{x}} f(x) \geq f(\bar{x}).$$

In addition, f *is called* continuous *at* $\bar{x}$ *if both* f *and* $-f$ *are lsc at* $\bar{x}$*, i.e. if*

$$\lim_{x \to \bar{x}} f(x) = \bar{x},$$

where the latter means that for any sequence $\{x_k\} \to \bar{x}$ *we have* $f(x_k) \to f(\bar{x})$*.*

Since the constant sequence $\{x_k = \bar{x}\}$ is admitted in (2.1), the inequality in the definition of lower semicontinuity can actually be substituted for and equality.

This fact that can also be phrased as follows: $f : \mathbb{E} \to \overline{\mathbb{R}}$ *is lower semicontinuous at* $\bar{x}$ *if and only if there does not exist a sequence* $\{x_k\} \to \bar{x}$ *such that* $\lim_{k\to\infty} f(x_k) < f(\bar{x})$*.* See also Figure 2 for an illustration: Here the lack of lower semicontinuity at $\bar{x}$ could be remedied by setting by assigning f the value $\liminf_{x \to \bar{x}} f(x) = \lim_{x \downarrow \bar{x}} f(x)$ at $\bar{x}$.

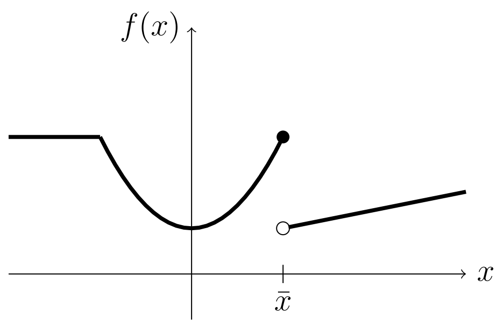

Figure 2: A function f *not* lsc at $\bar{x}$

We continue with an example of a function that is continuous in the extended real-valued sense.

Example 2.4 ((Negative) log-determinant)**.** *Consider the function*

$$f : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}, \quad f(X) := \begin{cases} -\log(\det X) & \text{if} \quad X \succ 0, \\ +\infty & \text{else} \end{cases} \tag{2.2}$$

which we call the (negative) log-determinant *or the* (negative) logdet *function, for short. Then f is proper and continuous: The properness is clear (as* $\mathrm{dom}\, f = \mathbb{S}^n_{++} \neq \emptyset$*). Recall that the determinant mapping $X \mapsto \det(X)$ is continuous (as it is a polynomial of the matrices' entries, cf.* Leibniz formula*), and hence f is continuous on its open domain* $\mathrm{dom}\, f = \mathbb{S}^n_{++}$*. Hence we only need to consider the critical cases of points on the boundary of the domain, i.e. in $\bar{X} \in \mathrm{bd}\,(\mathrm{dom}\, f) = \mathbb{S}^n_+ \setminus \mathbb{S}^n_{++}$ and sequences $\{X_k \in \mathbb{S}^n_{++}\} \to \bar{X}$. At this, Hence, for $\bar{X} \in \mathrm{bd}\,(\mathrm{dom}\, f)$ and $\{X_k \in \mathbb{S}^n_{++}\} \to \bar{X}$ we have $\det(X_k) \to 0$, and thus, we obtain*

$$\lim_{k\to\infty} f(X_k) = \lim_{k\to\infty} -\log(\det(X_k)) = +\infty = f(\bar{X}),$$

thus f is continuous.

Lower semicontinuity plays an important role in our study. In fact, we often find it useful to rectify the absence of lower semicontinuity of a function as follows:

We define the *lower semicontinuous hull* or *closure* of f to be the function $\mathrm{cl}\, f : \mathbb{E} \to \overline{\mathbb{R}}$,

$$(\mathrm{cl}\, f)(\bar{x}) := \liminf_{x\to\bar{x}} f(x).$$

As the constant sequence $\{x_k = \bar{x}\}$ is admitted in (2.1), we always have

$$\liminf_{x\to\bar{x}} f(x) \leq f(\bar{x}) \quad (\bar{x} \in \mathbb{E}),$$

hence

$$\mathrm{cl}\, f \leq f$$

for every function $f : \mathbb{E} \to \overline{\mathbb{R}}$. Moreover, we have

$$\mathrm{epi}\,(\mathrm{cl}\, f) = \mathrm{cl}\,(\mathrm{epi}\, f),$$

see Exercise 2.10 In particular, in view of the following result (which also clarifies why lsc functions are also called *closed*), this shows that $\mathrm{cl}\, f$ is always lsc.

Proposition 2.5 (Characterization of lower semicontinuity)**.** *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then the following are equivalent:*

i) f is lsc (on $\mathbb{E}$);

ii) $\mathrm{epi}\, f$ *is closed;*

iii) $\mathrm{lev}_{\leq\alpha} f$ *is closed for all $\alpha \in \mathbb{R}$.*

Proof. 'i)⇒ii)': Let $\{(x_k, \alpha_k) \in \mathrm{epi}\, f\} \to (x, \alpha)$. By lower semicontinuity we have

$$f(x) \leq \liminf_{k\to\infty} f(x_k) \leq \lim_{k\to\infty} \alpha_k = \alpha,$$

hence $(x, \alpha) \in \mathrm{epi}\, f$. This shows that $\mathrm{epi}\, f$ is closed.

'ii)⇒ iii)': Fix $\alpha \in \mathbb{R}$ and let $\{x_k \in \mathrm{lev}_{\leq\alpha} f\} \to x$. Then $\{(x_k, \alpha) \in \mathrm{epi}\, f\} \to (x, \alpha)$, and by closedness of $\mathrm{epi}\, f$, we have $(x, \alpha) \in \mathrm{epi}\, f$, i.e. $x \in \mathrm{lev}_{\leq\alpha} f$. Thus, $\mathrm{lev}_{\leq\alpha} f$ is closed, which proves the desired implication.

'iii)$\Rightarrow$ i):'(Contraposition) Suppose that f is *not* lsc. Then there exists $x \in \mathbb{E}$, $\{x_k\} \to x$ and such that

$$f(x_k) \to \alpha < f(x).$$

Now, pick $r \in (\alpha, f(x))$. Then we have

$$f(x_k) \le r < f(x) \quad \text{for all } k \text{ sufficiently large.}$$

But that means that $x \notin \mathrm{lev}_{\le r} f$, although almost every member of the sequence $\{x_k\}$ lies in $\mathrm{lev}_{\le r} f$. Hence, $\mathrm{lev}_{\le r} f$ cannot be closed.

$\square$

Due to the ubiquitiousness of the indicator function, we state a closedness result for it explicitly.

Corollary 2.6 (Lower semicontinuity of the indicator)**.** *For a set $C \subset \mathbb{E}$ its indicator δ_C is proper and lsc if and only if C is nonempty and closed.*

Lower semicontinuity plays a central role in minimization problems, as the following paragraph illustrates.

2.4 Optimization problems

Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ and $C \subset \mathbb{E}$. We define

$$\inf_C f := \inf_{x \in C} f(x) := \inf \{f(x) \mid x \in C\}$$

and

$$\sup_C f := \sup_{x \in C} f(x) := \sup \{f(x) \mid x \in C\}.$$

In this scenario, $\inf_C f$ and $\sup_C f$ describe a(n extended) real number. By slight abuse of notation (since a minimum/maximum does not need to exist) we write the optimization problems that come with f and C by

$$\min\ f(x) \quad \text{s.t.} \quad x \in C \quad \text{and} \quad \max_{x \in C}\ f(x) \quad \text{s.t.} \quad x \in C,$$

respectively. The function f is called *objective function* in this conext, where C is referred to as the *feasible set*. If $C = \mathbb{E}$, the respective minimization/maximization problems are called *unconstrained* and otherwise *constrained*.

Note that we always have

$$\inf_C f = -\sup_C -f \quad \text{and} \quad \sup_C f = -\inf_C -f,$$

hence there is no big loss in generality if we primarily focus on minimization problems.

We now define the notion of a local and global minimizers.

Definition 2.7 (Minimizers)**.** *Let* $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ *and* $C \subset \mathbb{E}$. *Then* $\bar{x} \in C \cap \operatorname{dom} f$ *is called*

i) local minimizer *of* f *over* C *if there exists* $\varepsilon > 0$ *such that*

$$f(\bar{x}) \leq f(x) \quad (x \in C \cap B_\varepsilon(\bar{x})).$$

ii) global minimizer *of* f *over* C *if*

$$f(\bar{x}) \leq f(x) \quad (x \in C).$$

A global (local) minimizer of f over C is also called a *(local) solution* of the optimization problem

$$\min \; f(x) \quad \text{s.t.} \quad x \in C.$$

Obviously, every global minimizer is a local minimizer, while the converse implication is, in general, not true. We will see, however, that in the convex setting, in turn, this does hold.

We define

$$\operatorname{argmin}_C f := \operatorname{argmin}_{x \in C} f(x) := \left\{ x \in C \;\middle|\; f(x) = \inf_C f \right\},$$

i.e. $\operatorname{argmin}_C f$ is the set of alll global minimizers of f over C. Using indicator functions, we have

$$\inf_C f = \inf_{\mathbb{E}} (f + \delta_C).$$

In fact, the minimization problems

$$\min \; f(x) \quad \text{s.t.} \quad x \in C$$

and

$$\min \; f(x) + \delta_C(x)$$

are fully equivalent, in that not only there optimal values but also there (global and local) solutions coincide (if they exist), in particular

$$\operatorname{argmin}_C f = \operatorname{argmin}_{\mathbb{E}} f + \delta_C.$$

We also define

$$\operatorname{argmax}_C f := \operatorname{argmax}_{x \in C} f(x) := \left\{ x \in C \;\middle|\; f(x) = \sup_C f \right\}.$$

We want to emphasize here that, when minimizing proper functions $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$, we always have the implication

$$\operatorname{argmin}_{\mathbb{E}} f \neq \emptyset \quad \Rightarrow \quad \inf_{\mathbb{E}} f \in \mathbb{R}.$$

However, the converse implication does *not* hold in general: Consider for example the function

$$f : x \in \mathbb{R} \mapsto \begin{cases} \frac{1}{x} & \text{if} \quad x > 0, \\ +\infty & \text{else.} \end{cases}$$

Then clearly, $\inf_{\mathbb{R}} f = 0$ but $\mathrm{argmin}_f = \emptyset$.

The significance of lower semicontinuity for minimization problems is highlighted by the following famous result along the lines of the foregoing remark.

Theorem 2.8 (Existence of minima). *Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be proper, lsc and level-bounded. Then*

$$\mathrm{argmin}_{\mathbb{E}} f \neq \emptyset \quad \textit{and} \quad \inf_{\mathbb{E}} f \in \mathbb{R}.$$

Proof. Let $f^* := \inf_{\mathbb{E}} f < \infty$. There exists a sequence $\{x_k\}$ such that $f(x_k) \to f^*$. Choosing $\alpha \in (f^*, +\infty)$ we have $x_k \in \mathrm{lev}_\alpha f$ for all $k \in \mathbb{N}$ sufficiently large. Since f is lsc, $\mathrm{lev}_\alpha f$ is closed by Proposition 2.5, and since f is level-bounded by assumption, lev_α is compact. Hence, by the Bolzano-Weierstrass Theorem, there exists $\bar{x} \in \mathrm{lev}_\alpha$ and an infinite subset $K \subset \mathbb{N}$ such that $x_k \to_K \bar{x}$.

Lower semicontinuity of f then implies

$$f(\bar{x}) \leq \liminf_{x \to \bar{x}} f(x) \leq \lim_{k \in K} f(x_k) = f^*,$$

hence $f(\bar{x}) = f^*$, which proves the assertion.

□

The above theorem is the blueprint for what is called the *direct method of the calculus of variations.*

Exercises to Chapter 2

2.1 **(Openness of $\mathbb{S}^n_{++}$)** Argue that $\mathbb{S}^n_{++}$ is open in $\mathbb{S}^n$.

2.2 **(Riesz representation theorem - finite dimensional version)** Let $(\mathbb{E}, \langle \cdot, \cdot \rangle)$ be a (finite dimensional) Euclidean space and $L \in \mathcal{L}(\mathbb{E}, \mathbb{R})$. Show that there exists a unique $b \in \mathbb{E}$ such that

$$L(x) = \langle b, x \rangle \quad (x \in \mathbb{E}).$$

2.3 **(Orthogonal matrices)** Show that $\mathrm{O}(n)$ is a compact subset of $\mathbb{R}^{n \times n}$.

2.4 **(Logdet and trace inequality)** Let $A \in S^n_{++}$. Show that

$$\log(\det A) + n \leq \mathrm{tr}\,(A).$$

2.5 **(Matrix-fractional function)** For $\Omega := \mathbb{R}^n \times \mathbb{S}^n_{++}$ consider the function

$$f : \Omega \to \mathbb{R}, \quad f(x, V) = \frac{1}{2} x^T V^{-1} x.$$

Prove that f is differentiable by showing that

$$\nabla f(x,V) = \left[V^{-1}x, -\frac{1}{2}V^{-1}xx^TV^{-1}\right] \quad (x, V \in \mathbb{R}^n \times \mathbb{S}^n)$$

(w.r.t the standard scalar product on $\mathbb{R}^n \times \mathbb{S}^n$).

Hint: Use the fact that for $T \in \mathbb{R}^{n\times n}$ with $\|T\| < 1$ we have

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k \quad \textit{(Neumann series)}.$$

2.6 **(Minimizing a linear function over the unit ball)** Let $g \in \mathbb{R}^n \setminus \{0\}$. Compute the solution of the optimization problem

$$\min\ \langle g,\, d\rangle \quad \text{s.t.} \quad \|d\| \le 1.$$

2.7 **(Minimizing a quadratic function)** For $A \in \mathbb{S}^n$ and $b \in \mathbb{R}^n$ consider the quadratic function

$$q : \mathbb{R}^n \to \mathbb{R}, \quad q(x) = \frac{1}{2}x^TAx + b^Tx.$$

Prove (without using first-order optimality conditions) that the following are equivalent:

i) $\inf_{\mathbb{R}^n} q > -\infty$;

ii) $A \succeq 0$ and $b \in \operatorname{rge} A$;

iii) $\operatorname{argmin}_{\mathbb{R}^n} q \neq \emptyset$.

2.8 (**Topology of Minkowski sum)** Let $A, B \subset \mathbb{E}$ nonempty. Prove:

a) $A + B$ is open if A or B is open.

b) $A + B$ is closed if both A and B are closed and at least one of them is bounded. Illustrate by a counterexample that the boundedness assumption cannot be omitted in general.

2.9 **(Closures and interiors of epigraphs)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then the following hold:

a) $(\bar{x}, \bar{\alpha}) \in \operatorname{cl}(\operatorname{epi} f) \quad \Longleftrightarrow \quad \bar{\alpha} \ge \liminf_{x\to\bar{x}} f(x)$.

b) $(\bar{x}, \bar{\alpha}) \in \operatorname{int}(\operatorname{epi} f) \quad \Longleftrightarrow \quad \bar{\alpha} > \limsup_{x\to\bar{x}} f(x)$.

2.10 **(Lower semicontinuous hull)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Show that

$$\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f),$$

i.e. $\operatorname{cl} f$ is the function whose epigraph is the closure of $\operatorname{epi} f$.

2.11 **(Domain of an lsc function)** Is the domain of an lsc function closed?

2.12 **(Closedness of a positive combination)** For $p \in \mathbb{N}$ let $f_i : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be lsc and $\alpha_i \geq 0$. Show that $f := \sum_{i=1}^p \alpha_i f_i$ is lsc.

2.13 **(Closedness preserving compositions)** Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be lsc.

a) Show that $f \circ g$ is lsc if $g : \mathbb{E}' \to \mathbb{E}$ is continuous.

b) Show that $\phi \circ f$ is lsc if $\phi : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and we use the convention $\phi(+\infty) = \sup_{t \in \mathbb{R}} \phi(t)$.

3 Fundamentals from Convex Analysis

3.1 Convex sets

Definition 3.1 (Convex sets)**.** *A set $C \subset \mathbb{E}$ is called* convex *if*

$$\lambda x + (1-\lambda) y \in C \quad (x, y \in C,\ \lambda \in [0,1]). \tag{3.1}$$

In other words, a convex set is simply a set which contains all connecting lines of points from the set, see Figure 3 for examples.

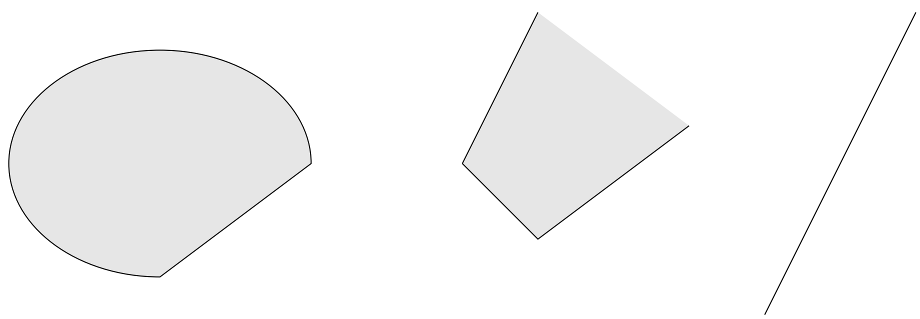

Figure 3: Convex sets in $\mathbb{R}^2$

A vector of the form

$$\sum_{i=1}^r \lambda_i x_i, \quad \sum_{i=1}^r \lambda_i = 1,\ \lambda_i \geq 0\ (i = 1, \dots, r)$$

is called a *convex combination* of the points $x_1, \dots, x_r \in \mathbb{E}$. It is easily seen that a set $C \subset \mathbb{R}^n$ is convex if and only if it contains all convex combinations of its elements.

Below is a list of important classes of convex sets as well as operations that preserve convexity.

Example 3.2 (Convex sets)**.**

a) (Subspaces) Every subspace of $\mathbb{E}$ (in particular $\mathbb{E}$ itself) is convex, as convex combinations are special cases of linear combinations.

b) *(Minkowski sum) The* Minkowski sum

$$A + B := \{a + b \mid a \in A, b \in B\}$$

of two convex sets $A, B \subset \mathbb{E}$ *is convex: For* $x, y \in A + B$ *there exist* $a, a' \in A$ *and* $b, b' \in B$ *such that* $x = a + b$ *and* $y = a' + b'$*. Then for* $\lambda \in [0, 1]$ *we have*

$$\begin{aligned} \lambda x + (1 - \lambda)y &= \lambda(a + b) + (1 - \lambda)(a' + b') \\ &= \lambda a + (1 - \lambda)a' + \lambda b + (1 - \lambda)b' \end{aligned}$$

By convexity of A *and* B*, respectively, we see that* $\lambda a + (1 - \lambda)a' \in A$ *and* $\lambda b + (1 - \lambda)b' \in B$*, hence* $\lambda x + (1 - \lambda)y \in A + B$*.*

c) *(Affine sets) Any set* $S \subset \mathbb{E}$ *which has a representation* $S = x + U$*, where* $x \in \mathbb{E}$ *and* $U \subset \mathbb{E}$ *is a subspace is called an* affine set *and is, due to a) and b), in particular, convex. It can be seen that the subspace* U *is uniquely determined and given by* $S - S$*. Moreover,* $S \subset \mathbb{E}$ *is affine if and only if*

$$\alpha x + (1 - \alpha)y \in S \quad (x, y \in S, \alpha \in \mathbb{R}).$$

For these details see, e.g. [24, Section 1].

d) *(Intersection) Arbitrary intersections of convex sets are convex, see Exercise 3.1.1a).*

e) *(Linear images and pre-images) For convex sets* $C \subset \mathbb{E}_1$ *and* $D \subset \mathbb{E}_2$ *and a linear mapping* $F : \mathbb{E}_1 \to \mathbb{E}_2$ *the sets* $F(C)$ *and* $F^{-1}(D)$ *are convex, see Exercise 3.1.1 b).*

f) *(Hyperplanes) For* $s \in \mathbb{E} \setminus \{0\}$ *and* $\gamma \in \mathbb{R}$ *the set*

$$\{x \in \mathbb{E} \mid \langle s,\, x\rangle = \gamma\}$$

is called a hyperplane*. It is a convex set, which is easily verified elementary or as a special case part e) with* $F : x \mapsto \langle s,\, x\rangle$ *and* $D = \{\gamma\}$*.*

g) *(Half-spaces) Sets of the form*

$$\{x \in \mathbb{E} \mid \langle s,\, x\rangle \geq \gamma\}, \quad \{x \in \mathbb{E} \mid \langle s,\, x\rangle > \gamma\}$$

interval

h) *(Intervals) The intervals (closed, open, half-open) are exactly the convex sets in* $\mathbb{R}$*.*

3.1.1 The convex hull

Definition 3.3 (Convex hull)**.** *Let* $M \subset \mathbb{E}$ *be nonempty. The* convex hull *of* M *is the set*

$$\operatorname{conv} M := \bigcap_{\substack{M \subset C, \\ C \text{ convex}}} C,$$

i.e. the convex hull of M *is the smallest convex set containing* M*.*

We can obtain an intrinsic characterization of the convex hull of a set by means of *convex combinations* of their elements. At this, for $x_1, \dots, x_p \in \mathbb{E}$ and $\lambda \in \Delta_p$ we call $\sum_{i=1}^p \lambda_i x_i$ a convex combination.

Proposition 3.4 (Characterization of the convex hull)**.** *Let $M \subset \mathbb{E}$ be nonempty. The we have*

$$\operatorname{conv} M = \left\{ \sum_{i=1}^r \lambda_i x_i \mid r \in \mathbb{N},\ \lambda \in \Delta_r,\ x_i \in M\ (i = 1, \dots, r) \right\}$$

Proof. Exercise 3.1.2. □

Proposition 3.4 tells us that the convex hull of a set can be seen as the set of all convex combinations of elements from the set in question.

In an N-dimensional ($N \in \mathbb{N}$) space $\mathbb{E}$ this can be sharpened as follows.

Theorem 3.5 (Carathéodory's Theorem)**.** *Let $M \subset \mathbb{E}$ be nonempty. Then we have*

$$\operatorname{conv} M = \left\{ \sum_{i=1}^{N+1} \lambda_i x_i \mid \lambda \in \Delta_{N+1},\ x_i \in M\ (i = 1, \dots, N+1) \right\},$$

i.e. every vector in $\operatorname{conv} M$ *can be written as a convex combination of at most* $N + 1$ *elements from M.*

Proof. Let $x \in \operatorname{conv} M$. By Proposition 3.4 there exists $r \in \mathbb{N}$, $\lambda_1, \dots, \lambda_r > 0$ and $x_1, \dots, x_r \in M$ such that

$$\sum_{i=1}^r \lambda_i = 1 \quad \text{and} \quad \sum_{i=1}^r \lambda_i x_i = x. \tag{3.2}$$

If $r \le N + 1$ there is nothing to do.

Hence, let $r > N + 1$. We are going to show that x can already be written as a convex combination of $r - 1$ elements from M, which then (inductively) gives the assertion.

As $r > N + 1$, the vectors

$$(x_1, 1), \dots, (x_r, 1) \in \mathbb{E} \times \mathbb{R}$$

are linearly dependent. Hence, there exist $\alpha_1, \dots, \alpha_r \in \mathbb{R}$ such that

$$0 = \sum_{i=1}^r \alpha_i (x_i, 1) \tag{3.3}$$

and $\alpha_i \neq 0$ for at least one index i.

W.l.o.g. we can assume that

$$\left| \frac{\alpha_i}{\lambda_i} \right| \le \left| \frac{\alpha_r}{\lambda_r} \right| \quad \forall i = 1, \dots, r - 1, \tag{3.4}$$

(i.e. $r = \operatorname{argmax}\left\{\left|\frac{\alpha_i}{\lambda_i}\right| \mid i = 1, \dots, r\right\}$) hence, in particular $\alpha_r \neq 0$. Putting $\beta_i := -\frac{\alpha_i}{\alpha_r}$ $(i = 1, \dots, r-1)$, equation (3.3) yields

$$x_r = \sum_{i=1}^{r-1} \beta_i x_i \quad \text{and} \quad \sum_{i=1}^{r-1} \beta_i = 1. \tag{3.5}$$

Then (3.2) and (3.5) imply

$$x = \sum_{i=1}^{r} \lambda_i x_i = \sum_{i=1}^{r-1} \lambda_i x_i + \lambda_r \sum_{i=1}^{r-1} \beta_i x_i = \sum_{i=1}^{r-1} (\lambda_i + \lambda_r \beta_i) x_i,$$

Setting $\tilde{\lambda}_i := \lambda_i + \lambda_r \beta_i$ $(i = 1, \dots, r-1)$ we thus obtain

$$x = \sum_{i=1}^{r-1} \tilde{\lambda}_i x_i,$$

and

$$\sum_{i=1}^{r-1} \tilde{\lambda}_i = \sum_{i=1}^{r-1} \lambda_i + \lambda_r \sum_{i=1}^{r-1} \beta_i = 1 - \lambda_r + \lambda_r = 1.$$

Due to (3.4) we also have

$$\tilde{\lambda}_i = \lambda_i - \lambda_r \frac{\alpha_i}{\alpha_r} > 0.$$

Thus, x is already a convex combination of the $r-1$ vectors $x_1, \dots, x_{r-1}$, which concludes the proof. $\square$

3.1.2 Topological properties of convex sets

We start by reviewing the fundamental concept of the (topological) closure, interior and boundary of an arbitray set $M \subset \mathbb{E}$. The *(topological) closure* $\operatorname{cl} M$ of M, which is the intersection all closed sets containing M, can also be written as

$$\operatorname{cl} M = \{x \mid \forall \varepsilon > 0 \; \exists y \in B_\varepsilon(x) \cap M\},$$

in particular, $\operatorname{cl} M$ is the set of all cluster (or, equivalently, limit) points of sequences in M. It is easy to see that $\operatorname{cl} M$ can also be written as

$$\operatorname{cl} M = \bigcap_{\varepsilon > 0} M + \varepsilon \mathbb{B}. \tag{3.6}$$

The *interior* $\operatorname{int} M$ of M is defined by

$$\operatorname{int} M := \{x \in M \mid \exists \varepsilon > 0 : B_\varepsilon(x) \subset M\}.$$

The *boundary* $\operatorname{bd} M$ of M is given by

$$\operatorname{bd} M := \operatorname{cl} M \setminus \operatorname{int} M.$$

The first topological result shows that convexity of a set is inherited to its interior and its closure.

Proposition 3.6 (Closure and interior of a convex set)**.** *Let $S \subset \mathbb{E}$ be convex. Then* $\operatorname{cl} S$ *and* $\operatorname{int} S$ *(which could well be empty, even if S is not) are convex, too.*

Proof. Since
$$\operatorname{cl} S = \bigcap_{\varepsilon > 0} (S + \varepsilon \mathbb{B}),$$
and $S + \varepsilon \mathbb{E}$ is convex (see Example 3.2 b)), the convexity of $\operatorname{cl} S$ follows from Example 3.2 d).

Now, let $x, y \in \operatorname{int} S$, hence there exist open neighborhoods $U_1, U_2 \subset S$ of x and y, respectively. It follows for $\lambda \in [0,1]$ that
$$\lambda x + (1-\lambda) y \subset \lambda U_1 + (1-\lambda) U_2 \subset \lambda S + (1-\lambda) S \subset S,$$
which proves the result as $\lambda U_1 + (1-\lambda) U_2$ is open by Exercise 2.8 a). □

Boundedness, *closedness* and *compactness* are fundamental topological properties of subsets in $\mathbb{E}$. At this point, we want to study in how far they are preserved under the convex hull operation.

Recall that, in (the finite-dimensional space) $\mathbb{E}$, a subset is compact if and only if it is bounded and closed.
The following result shows that compactness is preserved under the convex hull operator.

Proposition 3.7. *Let $M \subset \mathbb{E}$ be compact. Then* $\operatorname{conv} M$ *is compact.*

Proof. Exercise 3.1.3 a). □

As an immediate consequence, we obtain that boundedness, too, is preserved under the affine hull operator.

Corollary 3.8. *Let $M \subset \mathbb{E}$ be bounded. Then* $\operatorname{conv} M$ *is bounded.*

Proof. Exercise 3.1.3 b). □

Unfortunately, closedness is, in general, not preserved under the convex hull operation, which we will illustrate by an example below. In the face of Proposition 3.7 the set to choose here, necessarily needs to be unbounded.

Example 3.9. *Consider $M := \{\binom{0}{0}\} \cup \{\binom{a}{1} \mid a \geq 0\} \subset \mathbb{R}^2$. Then M is closed and*
$$\begin{pmatrix} 1 \\ \frac{1}{k} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} k \\ 1 \end{pmatrix} + \left(1 - \frac{1}{k}\right)\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \operatorname{conv} M \quad (k \in \mathbb{N}).$$
On the other hand, as one can easily verify, $\lim_{k\to\infty} \binom{1}{\frac{1}{k}} = \binom{1}{0} \notin \operatorname{conv} M$, and hence $\operatorname{conv} M$ *is not closed, see Figure 4.*

This justifies the following definition.

Definition 3.10 (Closed convex hull)**.** *Let $S \subset \mathbb{E}$ be nonempty. Then its* closed convex hull *is the intersection of all closed convex sets containing it, we denote it by $\overline{\operatorname{conv}}\, S$.*

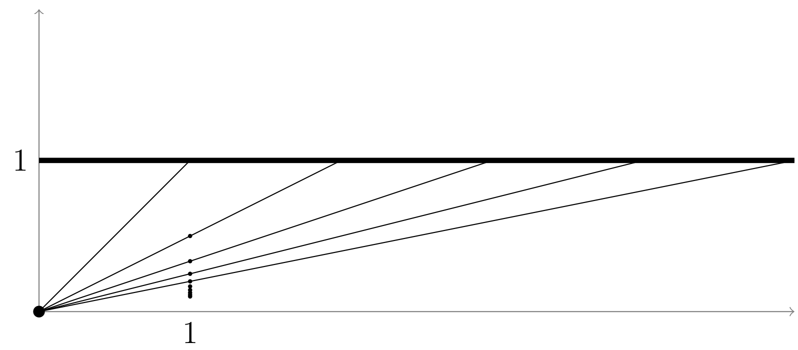

Figure 4: Closed set, non-closed convex hull

It is not surprising that the closed convex hull equals the closure of the convex hull of a set.

Proposition 3.11. *Let $S \subset \mathbb{E}$ be nonempty. Then* $\overline{\mathrm{conv}}\, S = \mathrm{cl}\,(\mathrm{conv}\, S)$.

Proof. '$\subset$': $\mathrm{cl}\,(\mathrm{conv}\, S)$ is a closed convex set containing S, hence $\overline{\mathrm{conv}}\, S \subset \mathrm{cl}\,(\mathrm{conv}\, S)$, as $\overline{\mathrm{conv}}\, S$ is the smallest closed and convex set containing S.
'$\supset$': We have $S \subset \overline{\mathrm{conv}}\, S$, hence $\mathrm{conv}\, S \subset \overline{\mathrm{conv}}\, S$, thus $\mathrm{cl}\,(\mathrm{conv}\, S) \subset \overline{\mathrm{conv}}\, S$, as $\overline{\mathrm{conv}}\, S$ is closed and convex.

□

3.1.3 Projection on convex sets and a separation theorem

For a set $S \subset \mathbb{R}^n$ and a given point in $x \in \mathbb{E}$ we want to assign to x the subset of points in S which have the shortest distance to it. We formalize this in the following definition.

Definition 3.12 (Projection on a set)**.** *Let $S \subset \mathbb{E}$ be nonempty and $x \in \mathbb{E}$. Then we define the* projection *of x on S by*

$$P_S(x) := \mathrm{argmin}_{y \in S} \|x - y\| \subset S.$$

Observe that no changes occur if we substitute $y \mapsto \|y - x\|$ for $y \mapsto \frac{1}{2}\|y - x\|^2$ in the above definition.

In general, the projection P_S is a set-valued map $\mathbb{E} \rightrightarrows S$. It is well known, however, that it is single-valued for nonempty, closed, convex sets. We record this fact in the next result.

Lemma 3.13 (Projection on closed convex sets)**.** *Let $C \subset \mathbb{E}$ be nonempty, closed and convex. Then P_C is a mapping $\mathbb{E} \to C$ with $x = P_C(x)$ if and only if $x \in C$.*

Figure 5 illustrates this fact.
The following theorem gives an important characterization of the projection on a closed convex set in terms of a variational inequality.

For its proof, observe that by the definition of the Euclidean norm and the canonical scalar product $\langle \cdot, \cdot \rangle$ we have

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\,\langle x,\, y \rangle + \|y\|^2 \quad (x, y \in \mathbb{E}).$$

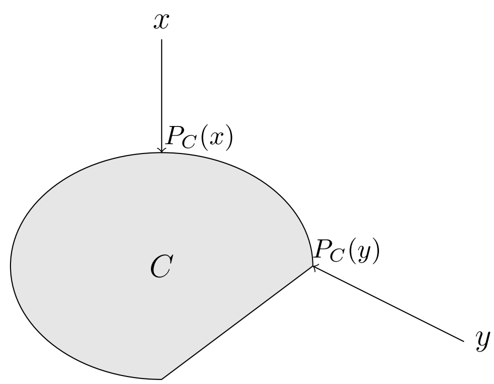

Figure 5: Projection on a closed convex set

Theorem 3.14 (Projection Theorem)**.** *Let $C \subset \mathbb{E}$ be nonempty, closed and convex and let $x \in \mathbb{E}$. Then $\bar{v} = P_C(x)$ if and only if*

$$\bar{v} \in C \quad \textit{and} \quad \langle \bar{v} - x,\, v - \bar{v} \rangle \geq 0 \quad (v \in C). \tag{3.7}$$

Proof. First, assume that $\bar{v} = P_C(x) \in C$ and define $f : \mathbb{E} \to \mathbb{R},\ f(v) = \frac{1}{2}\|v - x\|^2$. By convexity of C, we have $\bar{v} + \lambda(v - \bar{v}) \in C$ for all $v \in C$ and $\lambda \in (0, 1)$. This implies

$$\frac{1}{2}\|\bar{v} - x\|^2 = f(\bar{v}) \leq f(\bar{v} + \lambda(v - \bar{v})) = \frac{1}{2}\|(\bar{v} - x) + \lambda(v - \bar{v})\|^2 \quad (v \in C,\ \lambda \in (0,1)),$$

which, in turn, gives

$$\begin{aligned} 0 &\leq \frac{1}{2}\|(\bar{v} - x) + \lambda(v - \bar{v})\|^2 - \frac{1}{2}\|\bar{v} - x\|^2 \\ &= \lambda \langle \bar{v} - x,\, v - \bar{v} \rangle + \frac{\lambda^2}{2}\|v - \bar{v}\|^2 \end{aligned}$$

for all $v \in C$ and $\lambda \in (0, 1)$. Dividing by λ yields

$$0 \leq \langle \bar{v} - x,\, v - \bar{v} \rangle + \frac{\lambda}{2}\|y - \bar{v}\|^2.$$

Letting $\lambda \downarrow 0$ gives the desired inequality in (3.7).

In order to see the converse implication, let $\bar{v} \in \mathbb{E}$ such that (3.7) holds. For $v \in C$ we hence obtain

$$\begin{aligned} 0 &\geq \langle x - \bar{v},\, v - \bar{v} \rangle \\ &= \langle x - \bar{v},\, v - x + x - \bar{v} \rangle \\ &= \|x - \bar{v}\|^2 + \langle x - \bar{v},\, v - x \rangle \\ &\geq \|x - \bar{v}\|^2 - \|x - \bar{v}\| \cdot \|v - x\|, \end{aligned}$$

where the last inequality is due to the Cauchy-Schwarz inequality. If $x \neq \bar{v}$ we can divide by $\|x - \bar{v}\| > 0$ and infer

$$\|x - \bar{v}\| \leq \|x - v\| \quad (v \in C),$$

i.e. $\bar{v} = P_C(x)$. If, in turn, □

A geometrical interpretation of the projection theorem is as follows: The angle between $x - P_C(x)$ and $v - P_C(x)$ must be at least $90°$ for all $v \in C$.

Remark 3.15. *A nonempty set $S \subset \mathcal{H}$ in an arbitrary Hilbert space is called* Chebyshev set *if P_S is single-valued (i.e. for every $x \in \mathcal{H}$ the projection $P_S(x)$ is a singleton).*

In finite dimension the (nonempty) closed and convex sets are exactly the Chebyshev sets, see [2, Corollary 21.11]. In infinite dimension this is still an open problem known as the Chebyshev problem.

The projection theorem in the form of Theorem 3.14 remains valid (with literally the same proof) if the assumption that $\mathbb{E}$ be finite dimensional is dropped and one works in Hilbert space.

The following separation theorem is an immediate consequence of the projection theorem.

Theorem 3.16 (Separation Theorem)**.** *Let $C \subset \mathbb{E}$ be nonempty, closed and convex, and let $x \notin C$. Then there exists $s \in \mathbb{E} \setminus \{0\}$ with*

$$\langle s, x\rangle > \sup_{v \in C} \langle s, v\rangle .$$

Proof. Put $s := x - P_C(x) \neq 0$. Then the projection theorem yields

$$0 \geq \langle x - P_C(x), v - P_C(x)\rangle = \langle s, v - x + s\rangle = \langle s, v\rangle - \langle s, x\rangle + \|s\|^2 \quad (v \in C).$$

Thus,

$$\langle s, x\rangle - \|s\|^2 \geq \langle s, v\rangle \quad (v \in C),$$

hence, s fulfills the requirements of the theorem. □

We would like to note some technicalities about the former theorem.

Remark 3.17. *Under the assumptions of Theorem 3.16 the following hold:*

a) The vector s can always be substituted for $-s$ and thus, there exists $s \in \mathbb{E} \setminus \{0\}$ such that $\langle s, x\rangle < \inf_{v \in C} \langle s, v\rangle$.

b) By positive homogeneity, we can assume w.l.o.g. that $\|s\| = 1$.

c) The statement of the separation theorem can also be formulated as follows: For $C \subset \mathbb{E}$ nonempty, closed and convex and $x \notin C$ there exist $s \in \mathbb{E} \setminus \{0\}$ and $\beta \in \mathbb{R}$ such that

$$\langle s, x\rangle > \beta \geq \langle s, v\rangle \quad (v \in C).$$

It is not quite clear yet why the above theorem was labeled *separation* theorem. In the situation of the theorem, define $\gamma := \frac{1}{2}(\langle s, x\rangle + \sup_{y \in C} \langle s, y\rangle)$. Then

$$x \in \{z \mid \langle s, z\rangle > \gamma\} \quad \text{and} \quad C \subset \{z \mid \langle s, z\rangle < \gamma\},$$

i.e. $\{x\}$ and C lie in two distinct open half-spaces induced by the hyperplane $H = \left\{z \mid s^T z = \gamma\right\}$. We say that H separates the set C from the point $x \notin C$. This situation is illustrated in Figure 9.

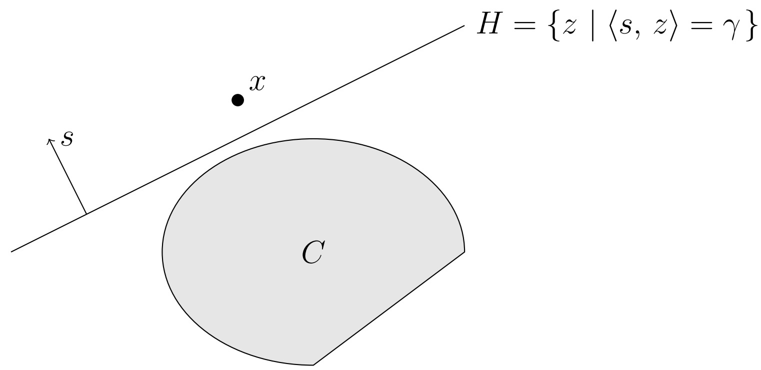

Figure 6: Separation of a point from a closed convex set

Remark 3.18. *In view of Remark 3.15 it is clear that the separation theorem in the form of Theorem 3.16 remains valid if for an arbitrary real Hilbert space instead of* $\mathbb{E}$ *without even changing the proof.*

There is an extension of this result to arbitrary Banach spaces, but the existing proofs rely on the axiom of choice, usually in the form of Zorn's Lemma.

3.1.4 The relative interior

For a nonempty set $M \subset \mathbb{E}$, its affine hull is given by

$$\operatorname{aff} M := \bigcap \{S \in \mathbb{E} \mid M \subset S,\ S \text{ affine}\},$$

cf. Example 3.2 c). One can easily see (like in the case of the linear hull) that the intrinsic characterization

$$\operatorname{aff} M = \left\{ \sum_{i=1}^{r} \lambda_i x_i \;\middle|\; \sum_{i=1}^{r} \lambda_i = 1,\ x_i \in M\ (i = 1, \dots, r) \right\}$$

holds, i.e. $\operatorname{aff} M$ is the set of all *affine combinations of* M, see Exercise 3.1.6.

For convex sets the (topological) interior is too restrictive a concept as in many cases it is going to be empty. For instance, consider the line segment $C := [x, y]$ for some $x, y \in \mathbb{R}^n$. We have $\operatorname{int} C = \emptyset$, but on the other hand it would be nice to declare the set $(x, y)(:= C \setminus \{x, y\})$ as the 'interior of C in some sense'. Since $\operatorname{aff} C = \{\lambda x + (1 - \lambda) y \mid \lambda \in \mathbb{R}\}$, it is easily verified that

$$(x, y) = \{z \mid \exists \varepsilon > 0 : B_\varepsilon(z) \cap \operatorname{aff} C \subset C\},$$

i.e. (x, y) is the *interior* of C with respect to the *relative topology* induced by the affine hull of C. We are going to make a general principle out of that.

Definition 3.19 (Relative interior/boundary and convex dimension). *Let* $C \subset \mathbb{E}$ *be convex. Then the* relative interior $\operatorname{ri} C$ *of* C *is its interior with respect to the relative topology induced by* $\operatorname{aff} C$*, i.e.*

$$\operatorname{ri} C := \{x \in C \mid \exists \varepsilon > 0 :\ B_\varepsilon(x) \cap \operatorname{aff} C \subset C\}.$$

C	$\operatorname{aff} C$	$\dim C$	$\operatorname{ri} C$
$\{x\}$	$\{x\}$	0	$\{x\}$
$[x, x']$	$\{\lambda x + (1-\lambda)x' \mid \lambda \in \mathbb{R}\}$	1	(x, x')
Δ_n	$\{a \mid e_n^T a = 1\}$	$n-1$	$\{a \in \Delta_n \mid a_i > 0\}$
$\overline{B}_\varepsilon(x)$	$\mathbb{E}$	N	$B_\varepsilon(x)$

Table 1: Examples for relative interiors

The relative boundary *of C is given by*

$$\operatorname{rbd} C := \operatorname{cl} C \setminus \operatorname{ri} C,$$

which is closed. In addition, we define $\dim C := \dim(\operatorname{aff} C)$ *to be the* (convex) dimension *of C.*

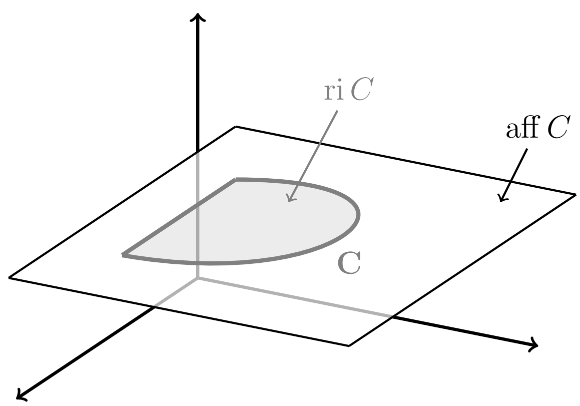

Figure 7: Illustration of relative interior

Figure 3.1.4 illustrates the relative interior and affine hull.

By definition, for any convex set $C \subset \mathbb{E}$, we have

$$\operatorname{int} C \subset \operatorname{ri} C \subset C \subset \operatorname{cl} C. \tag{3.8}$$

One might ask the question why we did not define a relative closure along with the relative interior and boundary, respectively. The reason for this is that there is no difference between the closure in the standard topology and in the topology relative to the affine hull. This holds since, briefly speaking, $\operatorname{aff} C$ is closed and hence $\operatorname{cl} C \subset \operatorname{aff} C$.

Table 3.1.4 contains a list of examples of frequently used sets and their relative interiors. We urge the reader to verify them. Note that the first two show that the relative interior (as opposed to the interior) is non-monotonic in the sense that $C_1 \subset C_2$ does not necessarily imply $\operatorname{ri} C_1 \subset \operatorname{ri} C_2$. However, Corollary 3.27 points shows that, in most situations, monotonicity is valid.

Remark 3.20. *The relative interior of a convex set $C \subset \mathbb{E}$ coincides with the interior if (and only if) C has full dimension, i.e. when* $\operatorname{aff} C = \mathbb{E}$, *in which case all topological questions reduce to the standard topology.*

If, on the other hand, $\dim C = m < N (= \dim \mathbb{E})$, *and we choose any m-dimensional subspace $U \subset \mathbb{E}$, there exists an invertible affine mapping $F : \mathbb{E} \to \mathbb{E}$ such that $F(\operatorname{aff} C) = U$, see [24, Corollary 1.6.1]. That is F maps* $\operatorname{aff} C$ *to U in a homeomorphic (even diffeomorphic) way. Thus, it holds that*

$$\operatorname{aff} F(C) = F(\operatorname{aff} C) = U,$$

and therefore, it is often possible to reduce topological questions about arbitrary (lower dimensional) convex sets to the full dimensional case by just working with the affine, diffeomorphic image $F(A)$ as a full dimensional convex set in some (sub-)space U.

Remark 3.20 already comes into play in the proof of the next result.

Proposition 3.21 (Line segment principle)**.** *Let $C \subset \mathbb{E}$ be convex as well as $x \in \operatorname{ri} C$ and $y \in \operatorname{cl} C$. Then we have $[x, y) \in \operatorname{ri} C$, i.e.*

$$(1-\lambda)x + \lambda y \in \operatorname{ri} C \quad (\lambda \in [0,1))$$

Proof. In view of Remark 3.20 we may assume w.l.o.g that $\dim C = N$, i.e. $\operatorname{ri} C = \operatorname{int} C$. Now, let $\lambda \in [0,1)$. Since $y \in \operatorname{cl} C$, we have

$$y \in C + \varepsilon\mathbb{B} \quad (\varepsilon > 0). \tag{3.9}$$

Hence, using Minkowski addition, we get

$$\begin{aligned} B_\varepsilon((1-\lambda)x + \lambda y) &= (1-\lambda)x + \lambda y + \varepsilon\mathbb{B} \\ &\overset{(3.9)}{\subset} (1-\lambda)x + \lambda(C + \varepsilon\mathbb{B}) + \varepsilon\mathbb{B} \\ &= (1-\lambda)\Big[\underbrace{x + \frac{1+\lambda}{1-\lambda}\varepsilon\mathbb{B}}_{=B_{\varepsilon\frac{1+\lambda}{1-\lambda}}(x)}\Big] + \lambda C \end{aligned}$$

for all $\varepsilon > 0$. Since $x \in \operatorname{int} C$, we have $B_{\varepsilon\frac{1+\lambda}{1-\lambda}}(x) \subset C$ for all $\varepsilon > 0$ sufficiently small. Hence, for these $\varepsilon > 0$, we get

$$B_\varepsilon((1-\lambda)x + \lambda y) \subset (1-\lambda)C + \lambda C = C,$$

which shows that $(1-\lambda)x + \lambda y \in \operatorname{int} C$, and hence concludes the proof. $\square$

We encourage the reader to emulate the proof of Proposition 3.21 without the assumption that C has nonempty interior to see that nothing changes other than the necessity of intersecting with $\operatorname{aff} C$.

An immediate consequence of the line segment principle is the convexity of the relative interior of a convex set (just take $y \in \operatorname{ri} C$ in Proposition 3.21).

We now show that for $C \subset \mathbb{E}$ convex, the three convex sets $C, \operatorname{ri} C$ and $\operatorname{cl} C$ have the same affine hull, hence the same convex dimension.

Theorem 3.22. *Let $C \subset \mathbb{E}$ convex. Then* $\operatorname{aff}(\operatorname{cl} C) = \operatorname{aff} C = \operatorname{aff}(\operatorname{ri} C)$. *In particular, we have* $\dim C = \dim \operatorname{cl} C = \dim \operatorname{ri} C$ *and* $\operatorname{ri} C \neq \emptyset$ *if* $C \neq \emptyset$.

Proof. Since $\operatorname{aff} C$ is closed we have $\operatorname{cl} C \subset \operatorname{aff} C$. Using again the properties of the respective hull operators 'cl' and 'aff' we hence obtain

$$\operatorname{aff} C \subset \operatorname{aff}(\operatorname{cl} C) \subset \operatorname{aff}(\operatorname{aff} C) = \operatorname{aff} C,$$

therefore, in particular, we have $\operatorname{aff} C = \operatorname{aff}(\operatorname{cl} C)$.

We now show that $\operatorname{aff} C = \operatorname{aff}(\operatorname{ri} C)$: In view of Remark 3.20 we can assume that $\operatorname{aff} C = \mathbb{E}$, i.e. $\dim C = N$. Hence, it suffices to show that $\operatorname{int} C \neq \emptyset$ under this assumption, since then also $\operatorname{aff}(\operatorname{int} C) = \mathbb{E}$. For these purposes, let $x_0, x_1, \dots, x_N \in C$ be affinely independent and define

$$S := \operatorname{conv}\{x_0, x_1, \dots, x_N\} \subset C.$$

We show that

$$\bar{x} := \frac{1}{N+1} \sum_{i=0}^{N} x_i \in S$$

is an interior point of S hence also of C.

Notice that $\mathbb{E} = \operatorname{aff} S = \operatorname{aff}\{x_0, x_1, \dots, x_N\}$, and $x_0, x_1, \dots, x_N$ are affinely independent. Hence, for every $y \in \mathbb{E}$, we have unique scalars $\beta_0(y), \beta_1(y), \dots, \beta_N(y) \in \mathbb{R}$ with $\sum_{i=0}^{N} \beta_i(y) = 1$ such that

$$\sum_{i=0}^{N} \beta_i(y) x_i = \bar{x} + y = \frac{1}{N+1} \sum_{i=0}^{N} x_i + y.$$

Thus, putting $\alpha_i(y) := \beta_i(y) - \frac{1}{N+1}$ $(i = 0, 1, \dots, N)$, the vector $\alpha(y) \in \mathbb{R}^{N+1}$ is the unique solution of the linear system

$$y = \sum_{i=0}^{N} \alpha_i x_i, \quad 0 = \sum_{i=0}^{N} \alpha_i.$$

The thus induced mapping $y \in \mathbb{E} \mapsto \alpha(y)$ is linear, hence continuous. Thus, we can find $\delta > 0$ such that for all $y \in \delta\mathbb{B}$, we have $|\alpha_i(y)| \leq \frac{1}{N+1}$. But then $\beta_i(y) = \alpha_i(y) + \frac{1}{N+1} \geq 0$ and hence $\bar{x} + y = \sum_{i=0}^{N} \beta_i(y) x_i \in \operatorname{conv}\{x_0, \dots, x_N\} = S$ for all $y \in \delta\mathbb{E}$. Thus, $\bar{x} + \delta\mathbb{B} \subset S$, which gives the assertion.

□

We continue by providing a list of useful properties of the relative interior.

Proposition 3.23. *Let $C \subset \mathbb{E}$ convex. Then the following hold:*

a) $\operatorname{ri}(\operatorname{ri} C) = \operatorname{ri} C = \operatorname{ri}(\operatorname{cl} C)$;

b) $\operatorname{cl} C = \operatorname{cl}(\operatorname{ri} C)$;

c) $\operatorname{rbd} C = \operatorname{rbd}(\operatorname{ri} C) = \operatorname{rbd}(\operatorname{cl} C)$.

Proof. By Remark 3.20 we can again assume that $\operatorname{aff} C = \mathbb{E}$, i.e. the standard and the relative topology coincide. The statements then follow from the well-known facts for the interior, boundary and closure of sets with nonempty interior.

□

The fact that two convex set with the same closure have the same relative interior is an immediate consequence, which we want to state explicitly because it is so frequently used.

Corollary 3.24. *Let $C_1, C_2 \subset \mathbb{E}$ such that $\operatorname{cl} C_1 = \operatorname{cl} C_2$. Then $\operatorname{ri} C_1 = \operatorname{ri} C_2$.*

Proof. From Proposition 3.23 we infer that $\operatorname{ri} C_1 = \operatorname{ri}(\operatorname{cl} C_1) = \operatorname{ri}(\operatorname{cl} C_2) = \operatorname{ri} C_2$. □

We next present another useful principle for the relative interior of a convex set that we call the stretching principle.

Proposition 3.25 (Stretching principle)**.** *Let $C \subset \mathbb{E}$ be a nonempty convex set . Then it holds that*

$$z \in \operatorname{ri} C \quad \Longleftrightarrow \quad \forall x \in C\ \exists \mu > 1 : \mu z + (1-\mu)x \in C.$$

Proof. First, let $z \in \operatorname{ri} C$. By definition, there exists $\varepsilon > 0$ such that $B_\varepsilon(z) \cap \operatorname{aff} C \in C$. Moreover, for every $x \in C$ and $\mu \in \mathbb{R}$ we have $\mu z + (1-\mu)x \in \operatorname{aff} C$. For every μ sufficiently close to 1, $\mu z + (1-\mu)x \in B_\varepsilon(z)$. Hence, $\mu z + (1-\mu)x \in \operatorname{aff} C \cap B_\varepsilon(z) \subset C$.

Now, suppose that z satisfies the condition on the right-hand side of the equivalence: As $\operatorname{ri} C \neq \emptyset$ by Theorem 3.22, there exists $x \in \operatorname{ri} C$. By assumption we then have $y := \mu z + (1-\mu)x \in C$ for some $\mu > 1$. Then $z = \lambda y + (1-\lambda)x$, where $\lambda := \mu^{-1}$. By the line segment principle $z \in \operatorname{ri} C$. □

Proposition 3.25 basically says that every line segment in C having $z \in \operatorname{ri} C$ as one endpoint can be, to some extent, *stretched* beyond z without leaving C.

In the following result we want to investigate how relative interiors and closures behave with regard to intersection of sets. For these purposes, recall that for an intersection of a family of sets $A_i \in \mathbb{E}$ $(i \in I)$, it always holds that

$$\operatorname{cl} \bigcap_{i \in I} A_i \subset \bigcap \operatorname{cl} A_i$$

due to the monotonicity and idempotence of the closure operator and the fact that an arbitrary intersection of closed sets is closed.

Proposition 3.26. *Let $C_i \subset \mathbb{E}$ be convex for $i \in I$ (an index set) such that $\bigcap_{i\in I} \operatorname{ri} C_i \neq \emptyset$. Then the following hold:*

a) $\operatorname{cl} \bigcap_{i\in I} C_i = \bigcap_{i\in I} \operatorname{cl} C_i$*;*

b) $\operatorname{ri} \bigcap_{i\in I} C_i = \bigcap_{i\in I} \operatorname{ri} C_i$ *if I is finite.*

Proof. Fix $x \in \bigcap_{i\in I} \operatorname{ri} C_i$. By the line segment principle, given any $y \in \bigcap_{i\in I} \operatorname{cl} C_i$, we have $(1-\lambda)x + \lambda y \in \bigcap_{i\in I} \operatorname{ri} C_i$ for all $\lambda \in [0,1)$. Moreover, we have $y = \lim_{\lambda\to 1}(1-\lambda)x + \lambda y$. As y was chosen arbitrarily in $\bigcap_{i\in I} \operatorname{cl} C_i$, we obtain

$$\bigcap_{i\in I} \operatorname{cl} C_i \subset \operatorname{cl} \bigcap_{i\in I} \operatorname{ri} C_i \subset \operatorname{cl} \bigcap_{i\in I} C_i \subset \bigcap_{i\in I} \operatorname{cl} C_i.$$

This proves a) and shows at the same time that $\bigcap_{i\in I} \operatorname{ri} C_i$ and $\bigcap_{i\in I} C_i$ have the same closure, hence, by Corollary 3.24, the same relative interior. Thus,

$$\operatorname{ri} \bigcap_{i\in I} C_i \subset \bigcap_{i\in I} \operatorname{ri} C_i.$$

In order to prove the opposite inclusion we assume that I is finite. Fix $z \in \bigcap_{i\in I} \operatorname{ri} C_i$. Then by the stretching principle (applied to every C_i), for every $x \in \bigcap C_i$ and every $i \in I$ there exists $\mu_i > 0$ such that $\mu_i z + (1-\mu_i) \in C_i$. Putting $\mu := \min_{i\in I} \mu_i > 1$ (I finite!), we see that $\mu z + (1-\mu)x \in \bigcap C_i$. Hence, again by the stretching principle, $z \in \operatorname{ri} \bigcap C_i$, which completes the proof.

□

As was pointed out above, the relative interior operator is not necessarily monotone. The following result tells us, however, that in many situations, it actually is.

Corollary 3.27. *Let $C_1, C_2 \subset \mathbb{E}$ be convex sets such that $C_2 \subset \operatorname{cl} C_1$ but $C_2 \not\subset \operatorname{rbd} C_1$. Then $\operatorname{ri} C_2 \subset \operatorname{ri} C_2$.*

Proof. Since by assumption $C_2 \subset \operatorname{cl} C_1$, by Proposition 3.23 b) and the monotonicity of the closure operator we have

$$\operatorname{cl}(\operatorname{ri} C_2) = \operatorname{cl} C_2 \subset \operatorname{cl} C_1 \tag{3.10}$$

We claim that $\operatorname{ri} C_1 \cap \operatorname{ri} C_2 \neq \emptyset$: Otherwise (3.10) would yield $\operatorname{ri} C_2 \subset \operatorname{cl} C_1 \setminus \operatorname{ri} C_1 = \operatorname{rbd} C_1$ which is closed, hence $C_2 \subset \operatorname{cl}(\operatorname{ri} C_2) \subset \operatorname{rbd} C_1$, which contradicts our assumption. Hence $\operatorname{ri} C_1 \cap \operatorname{ri} C_2 \neq \emptyset$, and we may apply Proposition 3.26 to obtain

$$\operatorname{ri} C_1 \cap \operatorname{ri} C_2 = \operatorname{ri} C_1 \cap \operatorname{ri}(\operatorname{cl} C_2) = \operatorname{ri}(C_1 \cap \operatorname{cl} C_2) = \operatorname{ri} C_2,$$

hence $\operatorname{ri} C_2 \subset \operatorname{ri} C_2$. □

Next, we want to show that affine mappings preserve relative interiors (as was already foreshadowed in Remark 3.20). To this end, recall that for a continuous function $f : \mathbb{E}_1 \to \mathbb{E}_2$ and $A \in \mathbb{E}_1$, we have

$$f(\operatorname{cl} A) \subset \operatorname{cl}(f(A)).$$

Proposition 3.28 (Relative interior under affine mappings)**.** *Let $F : \mathbb{E}_1 \to \mathbb{E}_2$ be affine and $C \subset \mathbb{E}_1$ convex. Then*

$$\operatorname{ri} F(C) = F(\operatorname{ri} C).$$

Proof. First note that by Example 3.2 e) the set $F(C)$ is convex, so that we can even talk about its relative interior.

By the continuity of F, the monotonicity of the closure operation and Proposition 3.23 b) we obtain

$$F(C) \subset F(\operatorname{cl} C) = F(\operatorname{cl}(\operatorname{ri} C)) \subset \operatorname{cl} F(\operatorname{ri} C) \subset \operatorname{cl} F(C).$$

Hence, taking the (idempotent) closure again on both sides, we get $\operatorname{cl} F(C) = \operatorname{cl} F(\operatorname{ri} C)$. Therefore, see Corollary 3.24, $\operatorname{ri} F(C) = \operatorname{ri} F(\operatorname{ri} C) \subset F(\operatorname{ri} C)$.

In order to prove the converse inclusion take $z \in F(\operatorname{ri} C)$. Moreover, let $x \in F(C)$. In addition, choose $z' \in F^{-1}(z) \subset \operatorname{ri} C$ and $x' \in F^{-1}(x) \subset C$. By the stretching principle, there exists $\mu > 1$ such that $\mu z' + (1-\mu)x' \in C$, and thus

$$F(\mu z' + (1-\mu)x') = \mu z + (1-\mu)x \in F(C).$$

As $x \in F(C)$ was chosen arbitrarily, we can apply the stretching principle (in the opposite direction to before) to z and infer that $z \in \operatorname{ri} F(C)$, which concludes the proof. □

We close out this paragraph on the relative interior with a result that will be very useful for our study in section 5.3. It is usually referenced to [24, Theorem 6.8].

Theorem 3.29. *Let $C \subset \mathbb{E}_1 \times \mathbb{E}_2$. For each $y \in \mathbb{E}_1$ we define $C_y := \{z \in \mathbb{E}_2 \mid (y,z) \in C\}$ and $D := \{y \mid C_y \neq \emptyset\}$. Then $\operatorname{ri} C = \{(y,z) \mid y \in \operatorname{ri} D,\ z \in \operatorname{ri} C_y\}$.*

Proof. The projection $L : (y,z) \mapsto y$ has $L(C) = D$, hence $L(\operatorname{ri} C) = \operatorname{ri} D$, by Proposition 3.28. Given $y \in \operatorname{ri} D$ and the affine set $M := \{(y,z) \mid z \in \mathbb{E}_2\}$, Proposition 3.26 ($M$ affine!) and Exercise 3.1.8 we have

$$M \cap \operatorname{ri} C = \operatorname{ri}(M \cap C) = \operatorname{ri}(\{y\} \times C_y) = \operatorname{ri}\{y\} \times \operatorname{ri} C_y = \{(y,z) \mid z \in \operatorname{ri} C_y\},$$

which is exactly the set of points in $\operatorname{ri} C$ mapped to y by L. This shows the desired statement (clear?). □

Exercises for Section 3.1

3.1.1 **(Convexity preserving operations on sets)**

a) (Intersection) Let I be an arbitrary index set (possibly uncountable) and let $C_i \subset \mathbb{E}$ $(i \in I)$ be a family of convex sets. Show that $\bigcap_{i \in I} C_i$ is convex.

b) (Linear images and preimages) Let $F \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ and let $C \subset \mathbb{E}_1$, $D \subset \mathbb{E}_2$ be convex. Show that

$$F(C) := \{Ax \mid x \in C\} \quad \text{and} \quad F^{-1}(D) = \{x \mid Ax \in D\}$$

are convex.

3.1.2 **(Convex hull of a set)** Let $M \subset \mathbb{E}$. Show that

$$\operatorname{conv} M = \left\{ \sum_{i=1}^{r} \lambda_i x_i \mid r \in \mathbb{N},\ \lambda \in \Delta_r,\ x_i \in M\ (i = 1, \dots, r) \right\},$$

i.e. $\operatorname{conv} M$ is the set of all convex combinations of vectors in M.

Hint: You may use the comment in the notes that a convex set contains all convex combinations of its elements.

3.1.3 **(Topology of convex hulls)** Let $M \subset \mathbb{E}$. Show the following:

a) If M is compact then $\operatorname{conv} M$ is compact.
b) If M is bounded then $\operatorname{conv} M$ is bounded.

3.1.4 **(Convex hulls)** Let $F : \mathbb{E} \to \mathbb{E}'$ be affine and $A, C \subset \mathbb{E}$ and $B \subset \mathbb{E}'$ nonempty. Prove the following:

a) $\operatorname{conv} F(A) = F(\operatorname{conv} A)$;
b) $\operatorname{conv} (A \times B) = \operatorname{conv} A \times \operatorname{conv} B$;
c) $\operatorname{conv} (A + C) = \operatorname{conv} A + \operatorname{conv} C$.

3.1.5 **(Spectahedron)** For $n \in \mathbb{N}$ compute $\operatorname{conv} \left\{ uu^T \in \mathbb{S}^n \mid u \in \mathbb{R}^n : \|u\| = 1 \right\}$.

3.1.6 **(Affine hulls)** Let $M \subset \mathbb{E}$ be nonempty and let $F : \mathbb{E} \to \mathbb{E}'$ be linear. Show the following:

a) $\operatorname{aff} M = \{\sum_{i=1}^{r} \lambda_i x_i \mid \sum_{i=1}^{r} \lambda_i = 1,\ x_i \in M\ (i = 1, \dots, r)\}$.
b) $\operatorname{aff} (F(M)) = F(\operatorname{aff} M)$.

3.1.7 **(Characterization of relative interior points)** Let $C \subset \mathbb{E}$ be nonempty and convex and $x \in C$. Show that the following are equivalent:

i) $x \in \operatorname{ri} C$;
ii) $\mathbb{R}_+(C - x)$ is a subspace of $\mathbb{E}$.

3.1.8 **(Relative interiors and cartesian products)** For $i = 1, \dots, p$ let $C_i \subset \mathbb{E}_i$. Show that

$$\operatorname{ri} \left(\bigtimes_{i=1}^{p} C_i \right) = \bigtimes_{i=1}^{p} \operatorname{ri} C_i.$$

3.1.9 **(Conex hull and relative boundary)** Let $C \subset \mathbb{E}$ be nonempty, convex and compact. Show that $\operatorname{conv} (\operatorname{rbd} C) = C$.

3.1.10 **(Open mapping theorem - finite dimensional version)** Let $A \subset \mathbb{E}$ be open and $L \in \mathcal{L}(\mathbb{E}, \mathbb{E}')$ surjective. Then $L(A)$ is open.

3.2 Convex cones

3.2.1 Convex cones and conical hulls

Definition 3.30 (Cones)**.** *A nonempty set $K \subset \mathbb{E}$ is said to be a* cone *if*

$$\lambda K \subset K \quad (\lambda \geq 0),$$

i.e. K is a cone if and only if it is closed under multiplication with nonnegative scalars. We call K pointed *if for any $p \in \mathbb{N}$ the implication*

$$x_1 + \cdots + x_p = 0 \quad \Rightarrow \quad x_i = 0 \; (i = 1, \dots, p)$$

holds as soon as $x_i \in K$.

Note that, in our definition, a cone always contains the origin. In the literature this is not necessarily the case.

For obvious reasons, *convex cones* are of particular interest to our study. We have the following handy characterization of convexity of a cone.

Proposition 3.31 (Convex cones)**.** *Let $K \subset \mathbb{E}$ be a cone. Then K is convex if and only if $K + K \subset K$.*

Proof. Let K be convex. Then

$$x + y = 2 \cdot \underbrace{\frac{1}{2}(x + y)}_{\in K} \in K \quad (x, y \in K)$$

hence $K + K \subset K$.

If, in turn, $K + K \subset K$, then

$$\underbrace{\lambda x}_{\in K} + \underbrace{(1 - \lambda)y}_{\in K} \in K + K \subset K \quad (x, y \in K, \lambda \in [0, 1]),$$

i.e. K is convex. $\square$

Pointedness of convex cones can be handily characterized.

Proposition 3.32 (Pointedness of convex cones)**.** *Let $K \subset \mathbb{E}$ be a convex cone. Then K is pointed if and only if $K \cap (-K) = \{0\}$.*

Proof. Exercise 3.2.1 $\square$

We proceed with a list of prominent examples of cones.

Example 3.33 (Cones)**.**

a) *(Nonnegative Orthant) For all $n \in \mathbb{N}$, the nonnegative orthant $\mathbb{R}^n_+$ is a pointed, convex cone, which is also a polyhedron as*

$$\mathbb{R}^n_+ = \left\{ x \in \mathbb{R}^n \;\middle|\; (-e_i)^T x \leq 0 \; (i = 1, \dots, n) \right\}.$$

b) (Cone complementarity constraints) Let $K \subset \mathbb{E}$ be a cone. Then the set

$$\Lambda := \{(x, y) \in \mathbb{E} \times \mathbb{E} \mid x, y \in K, \quad \langle x, y\rangle = 0\}$$

is a cone. A prominent realization is $K = \mathbb{R}^n$, in which case Λ is called the complementarity constraint set.

c) (Positive semidefinite matrices) For $n \in \mathbb{N}$, the set $\mathbb{S}^n_+$ of positive semidefinite $n \times n$ matrices is a pointed, convex cone.

The next example is important enough to merit its own definition.

Definition 3.34 (Polar cone)**.** *Let $K \in \mathbb{E}$ be a cone. Then the* polar (cone) *of K is defined by*

$$K^\circ := \{d \in \mathbb{E} \mid \langle d, x\rangle \leq 0 \ (x \in K)\}.$$

respectively. Moreover, $K^{\circ\circ} := (K^\circ)^\circ$ is called the bipolar cone *of K.*

The cone $K^* := -K^\circ$ is sometimes referred to as the dual (cone) *of K, and K is called* self-dual *if $K^* = K$.*

In order to visualize the normal cone, we think of $\mathbb{E}$ as $\mathbb{R}^n$: Then the normal cone of the cone K is set of all vectors, which have an angle $\geq 90°$ to every vector in K.

Clearly, for an arbitrary cone $K \subset \mathbb{E}$, its polar K° is always a closed, convex cone. Hence, for K to be self-dual, it must necessarily be a closed, convex cone. Moreover, polarization is order-reversing, i.e. for $K_1 \subset K_2 \subset \mathbb{E}$, we have $K_2^\circ \subset K_1^\circ$.

We continue with some elementary examples of polar cones.

Example 3.35 (Polar cones)**.**

a) It holds that $\{0\}^\circ = \mathbb{E}$ and $\mathbb{E}^\circ = \{0\}$, which is a special case of part b).

b) If S is a subspace, $S^\circ = S^\perp$ (cf. Exercise 3.2.4).

c) For $0 \neq w \in \mathbb{E}$, the polar of the ray $\{tw \mid t \geq 0\}$ is the half-space $\{w \in \mathbb{E} \mid \langle w, x\rangle \leq 0\}$.

d) The negative orthant $\mathbb{R}^n_+$ and the positive semidefinite $n \times n$ matrices $\mathbb{S}^n_+$ are self-dual.

Since the intersection of convex cones is again a convex cone, using our usual routine, we can also build up *(convex) conical hulls* of arbitrary sets.

Definition 3.36 ((Convex) Conical hull)**.** *Let $S \subset \mathbb{E}$ be nonempty. Then the* (convex) conical hull *of S is the set*

$$\operatorname{cone} S := \bigcap_{\substack{S \subset M, \\ M \text{ convex cone}}} M.$$

Moreover, we define the closed (convex) conical hull of S to be

$$\overline{\operatorname{cone}}\, S := \operatorname{cl}(\operatorname{cone} S).$$

We notice, without proof, that cone S is the intersection of all closed, convex cones that contain S.

3.2.2 The horizon cone

The next conical approximation of a set is, loosely speaking, comprised of the *directions* in which one can go (starting at least one point in the set) without ever leaving the set, and thus takes account of the unboundedness of the set.

Definition 3.37 (Horizon cone)**.** *For a nonempty set $S \subset \mathbb{E}$ the set*

$$S^\infty := \{v \in \mathbb{E} \mid \exists \{x_k \in S\}, \{t_k\} \downarrow 0 : \ t_k x_k \to v\}$$

is called the horizon cone *of S. We put $\emptyset^\infty := \{0\}$.*

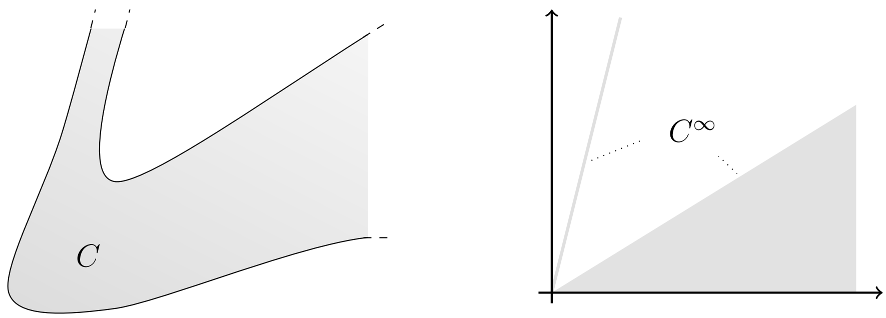

Figure 8: The horizon cone of an unbounded set

The following result shows that the horizon cone is indeed a closed cone.

Lemma 3.38. *The horizon cone of a set $C \subset \mathbb{E}$ is a closed cone.*

Proof. The fact that C^∞ is a cone is trivial. For the closedness of the horizon cone we can invoke a diagonal sequence arguments as it is usually used to show closedness of the tangent cone. ☐

The horizon cone can be used to very handily express boundedness of an arbitrary set in $\mathbb{E}$.

Proposition 3.39 (Horizon criterion for boundedness)**.** *A set $S \subset \mathbb{E}$ is bounded if and only if $S^\infty = \{0\}$.*

Proof. If S is bounded, then for all sequences $\{x_k \in S\}, \{t_k\} \downarrow 0$, we have $t_k x_k \to 0$, hence $S^\infty = \{0\}$.

If, in turn, S is unbounded, then there exists $\{x_k \in S\}$ with $\|x_k\| \to \infty$. Putting $t_k = \frac{1}{\|x_k\|}$, we have $\{t_k\} \downarrow 0$ and $t_k x_k = \frac{x_k}{\|x_k\|}$ converges (at least on a subsequence) to some $v \neq 0$ and, hence $v \in S^\infty \supsetneq \{0\}$. ☐

Figure 8 nicely illustrates the foregoing result. It shows that the horizon cone of an unbounded nonconvex set, which is a non-trivial closed cone, which is not necessarily convex.

We now want to show that the horizon cone is always going to be convex if the said in question is convex itself. For these purposes, we introduce another conical approximation for convex sets.

Definition 3.40 (Recession cone)**.** *For nonempty convex sets $C \subset \mathbb{E}$, the convex cone*

$$0^+(C) := \{v \mid \forall x \in C, \lambda \geq 0 : \ x + \lambda v \in C\},$$

is called the recession cone *of C,*

The recession cone is very closely related to the horizon cone as we will now see.

Proposition 3.41 (Horizon vs. recession cone)**.** *Let $C \subset \mathbb{E}$ be nonempty and convex. Then*

$$C^\infty = 0^+(\mathrm{cl}\, C).$$

In particular, C^∞ is (a closed and) convex (cone) if C is convex.

Proof. Let $v \in C^\infty$. Then there exist $\{x_k \in C\}, \{t_k\} \downarrow 0$ such that $t_k x_k \to v$. Now, let $\lambda \geq 0$ and $x \in C$ be given. As C is convex, we have

$$(1 - \lambda t_k)x + \lambda t_k x_k \in C$$

for all $k \in \mathbb{N}$ sufficiently large. Hence,

$$x + \lambda v = \lim_{k \to \infty} (1 - \lambda t_k)x + \lambda t_k x_k \in \mathrm{cl}\,(C),$$

which shows that $v \in 0^+(\mathrm{cl}\, C)$.

Now let $v \in 0^+(\mathrm{cl}\, C)$. Fixing $x \in C$, we get $x + \lambda v \in \mathrm{cl}\, C$ for all $\lambda \geq 0$. Hence, we can find a sequence $\{x_k \in C\}$ such that $\|x + kv - x_k\| \leq \frac{1}{k}$ for all $k \in \mathbb{N}$. Putting $\lambda_k := \frac{1}{k}$ $(k \in \mathbb{N})$, we obtain

$$\|v - \lambda_k x_k\| \leq \frac{1}{k}(\|x\| + \|x + kv - x_k\|) \to 0,$$

hence $\lambda_k x_k \to v$, i.e. $v \in C^\infty$. □

An obvious consequence of the foregoing proposition is the fact that horizon and recession cone coincide for nonempty, closed and convex sets.

Corollary 3.42. *Let $C \subset \mathbb{E}$ be nonempty, closed and convex. Then*

$$C^\infty = 0^+(C).$$

Exercises to Section 3.2

3.2.1 **(Pointedness of convex cones)** Let $K \subset \mathbb{E}$ be a convex cone. Show that K is pointed if and only if $K \cap (-K) = \{0\}$.

3.2.2 **(Convex cones vs. subspaces)** Let $K \subset \mathbb{E}$ be a convex cone. Show that:

a) $K \cap (-K)$ is the largest subspace that is contained in K;

b) $K - K$ is the smallest subspace containing K, i.e. $K = K = \mathrm{span}\, K$.

c) K is a subspace if and only if $K = -K$.

3.2.3 **(Characterizing the conical hull)** Let $S \subset \mathbb{E}$ be nonempty. Prove that

$$\operatorname{cone} S = \left\{ \sum_{i=1}^{r} \lambda_i x_i \mid r \in \mathbb{N}, x_i \in S, \lambda_i \geq 0 \ (i = 1, \dots, r) \right\} = \mathbb{R}_+(\operatorname{conv} S) = \operatorname{conv}(\mathbb{R}_+ S).$$

3.2.4 **(Polar of subspace)** Let $U \subset \mathbb{E}$ be a subspace. Show that $U^\circ = U^\perp$.

3.2.5 **(Pointedness and polarity)** Let K be a convex cone. Show that

$$w \in \operatorname{int} K \quad \Longleftrightarrow \quad \langle v, w \rangle < 0 \quad (v \in K^\circ \setminus \{0\}).$$

3.2.6 **(Polar cone and normal vectors)** Let $C \subset \mathbb{E}$ be nonempty. Then it holds that

$$(\overline{\operatorname{cone}} \, C)^\circ = \{ w \in \mathbb{E} \mid \langle w, x \rangle \leq 0 \ (x \in C) \}.$$

3.3 Convex functions

We start the section with the basic definition of a convex function.

Definition 3.43 (Convex function). *A function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is said to be* convex *if* $\operatorname{epi} f$ *is a convex set.*

Note that in the above definition we could have substitued the epigraph for the *strict epigraph* $\operatorname{epi}_< f := \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid f(x) < \alpha\}$ of f, see Exercise 3.3.3. Moreover, note that convex functions have convex level sets, see Exercise 3.3.6.

Recall that the domain of a function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is defined by $\operatorname{dom} f := \{x \in \mathbb{E} \mid f(x) < \infty\}$. Using the linear mapping

$$L : (x, \alpha) \in \mathbb{E} \times \mathbb{R} \mapsto x \in \mathbb{E}, \tag{3.11}$$

we have $\operatorname{dom} f = L(\operatorname{epi} f)$, and hence Example 3.2 e) yields the following immediate but important result.

Proposition 3.44 (Domain of a convex function). *The domain of a convex function is convex.*

Recall that a (convex) function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is proper if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{E}$.

Improper convex functions are somewhat pathological (cf. Exercise 3.3.4), but they do occur; rather as by-products then as primary objects of study. For example the function

$$f : x \in \mathbb{R} \mapsto \begin{cases} -\infty & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| = 1, \\ +\infty & \text{if} \quad |x| > 1. \end{cases}$$

is improper and convex.

Convex functions have an important interpolation property, which we summarize in the next result for the case that f does not take the value $-\infty$.

Proposition 3.45 (Characterizing convexity)**.** *A function $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if for all $x, y \in \mathbb{E}$ we have*

$$f(\lambda x + (1-\lambda)y) \leq (1-\lambda)f(x) + \lambda f(y) \quad (\lambda \in [0,1]). \tag{3.12}$$

Proof. First, let f be convex. Take $x, y \in \mathbb{E}$ and $\lambda \in [0,1]$. If $x \notin \operatorname{dom} f$ or $y \notin \operatorname{dom} f$ the inequality (3.12) holds trivially, since the right-hand side is going to be $+\infty$. If, on the other hand, $x, y \in \operatorname{dom} f$, then $(x, f(x)), (y, f(y)) \in \operatorname{epi} f$, hence by convexity

$$(\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \in \operatorname{epi} f,$$

i.e. $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, which proves the first implication.

In turn, let (3.12) hold for all $x, y \in \mathbb{E}$. Now, take $(x, \alpha), (y, \beta) \in \operatorname{epi} f$ and let $\lambda \in [0,1]$. Due to (3.12) we obtain

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda\alpha + \lambda\beta,$$

i.e. $\lambda(x, \alpha) + (1-\lambda)(y, \beta) \in \operatorname{epi} f$, which shows the converse implication. □

We move the analogous characterization of convexity for functions $\mathbb{E} \to \overline{\mathbb{R}}$ to Exercise 3.3.3, because these kinds of functions are not our primary object of study.

Definition 3.46 (Convexity on a set)**.** *Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ and $C \subset \operatorname{dom} f$ nonempty, convex. Then we call f* convex on C *if* (3.12) *holds for all $x, y \in C$.*

With this terminology we can formulate the following useful result.

Corollary 3.47. *Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. Then the following are equivalent.*

i) f is convex.

ii) f is convex on its domain.

Proof. The implication' i)⇒ii)' is obvious from the characterization of convexity in Proposition 3.45

For the converse implication note that (3.12) always holds for any pair of points x, y if one of them is not in the domain.

This completes the proof. □

As an immediate consequence of Corollary 3.47, we can make the following statement about proper, convex functions:

> *"The proper and convex functions $\mathbb{E} \to \overline{\mathbb{R}}$ are exactly those proper functions $f : \mathbb{E} \to \overline{\mathbb{R}}$ for which there exists a nonempty, convex set $C \subset \mathbb{E}$ such that* (3.12) *holds on C and f takes the value $+\infty$ outside of C."*

We are mainly interested in proper, convex (even lsc) functions $\mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. Hence, we introduce the abbreviations

$$\Gamma := \Gamma(\mathbb{E}) := \{f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\} \mid f \text{ proper and convex}\}$$

and

$$\Gamma_0 := \Gamma_0(\mathbb{E}) := \{f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\} \mid f \text{ proper, lsc and convex}\}$$

which we will use frequently in the remainder.

Although we are not primarily interested in continuity properties of convex functions we still want to mention the following to standard result which can be substantially refined, see the analysis in [24, §10].

Theorem 3.48 (Lipschitz continuity of convex functions; [3, Th. 4.1.3])**.** *A proper convex function is (locally Lipschitz) continuous on the interior of its domain.*

In particular, every finite-valued convex function is (locally Lipschitz) continuous.

Exercises for Section 3.3

3.3.1 **(Univariate convex functions)** Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ and $I \subset \operatorname{dom} f$ be an open interval. Show the following :

a) f is convex on I if and only if the *slope-function*

$$x \mapsto \frac{f(x) - f(x_0)}{x - x_0}$$

is nondecreasing on $I \setminus \{x_0\}$.

b) Let f is differentiable on I: Then f is convex on I if f' is nondecreasing on I, i.e.

$$f'(s) \leq f'(t) \quad (s, t \in I : s \leq t).$$

c) Let f is twice differentiable on I. Then f is convex on I if and only if $f''(x) \geq 0$ for all $x \in I$.

3.3.2 **(Convexity of maximum eigenvalue)** Let $f : \mathbb{S}^n \to \mathbb{R}$ be given by

$$f(A) = \lambda_{\max}(A),$$

where $\lambda_{\max}(A)$ denotes the largest eigenvalue of A. Show that f is convex.
Hint: Rayleigh quotient.

3.3.3 (**Characterization of convexity)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Show the equivalence of:

i) f is convex;

ii) The *strict epigraph* $\operatorname{epi}_{<} f := \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid f(x) < \alpha\}$ of f is convex;

iii) For all $\lambda \in (0, 1)$ we have $f(\lambda x + (1-\lambda)y) < \lambda\alpha + (1-\lambda)\beta$ whenever $f(x) < \alpha$ and $f(y) < \beta$.

3.3.4 **(Properness and closedness of convex functions)** Prove the following:

a) An improper convex function $f : \mathbb{E} \to \overline{\mathbb{R}}$ must have $f(x) = -\infty$ for all $x \in \mathrm{ri}\,(\mathrm{dom}\, f)$.

b) An improper convex function which is lsc, can only have infinite values.

c) If f is convex then $\mathrm{cl}\, f$ is proper if and only if f is proper.

3.3.5 **(Jensen's Inequality)** Show that $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if

$$f\left(\sum_{i=1}^{p} \lambda_i x_i\right) \leq \sum_{i=1}^{p} \lambda_i f(x_i) \quad \forall x_i \in \mathbb{E}\ (i = 1, \dots, p),\ \lambda \in \Delta_p.$$

3.3.6 **(Quasiconvex functions)** A function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is called *quasiconvex* if the level sets $\mathrm{lev}_{\leq\alpha} f$ are convex for every $\alpha \in \mathbb{R}$. Show:

a) Every convex function is quasiconvex.

b) $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is quasiconvex if an only if

$$f(\lambda x + (1 - \lambda) y) \leq \max\{f(x), f(y)\} \quad (x, y \in \mathrm{dom}\, f,\ \lambda \in [0, 1]).$$

c) If $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is quasiconvex then $\mathrm{argmin} f$ is a convex set.

3.4 Functional operations preserving convexity

Proposition 3.49 (Positive combinations of convex functions)**.** *For $p \in \mathbb{N}$ let $f_i : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be convex (and lsc) and $\alpha_i \geq 0$ for $i = 1, \dots, p$. Then*

$$\sum_{i=1}^{p} \alpha_i f_i$$

is convex (and lsc). If, in addtion, $\cap_{i=1}^{p} \mathrm{dom}\, f_i \neq \emptyset$, then f is also proper.

Proof. The convexity assertion is an immediate consequence of the characterization in (3.12). For the additional closedness see Exercise 2.12. The properness statement is obvious. □

Note that the latter result tells us that Γ and Γ_0 are convex cones.

Proposition 3.50 (Pointwise supremum of convex functions)**.** *For an arbitrary index set I let f_i be convex (and lsc) for all $i \in I$. Then the function $f = \sup_{i \in I} f_i$, i.e.*

$$f(x) = \sup_{i \in I} f_i(x) \quad (x \in \mathbb{E})$$

is convex (and lsc).

Proof. It holds that

$$\operatorname{epi} f = \left\{ (x,\alpha) \;\middle|\; \sup_{i\in I} f_i(x) \le \alpha \right\} = \{(x,\alpha) \mid \forall i \in I : f_i(x) \le \alpha\} = \bigcap_{i\in I} \operatorname{epi} f_i.$$

Since the intersection of (closed) convex sets it (closed) convex, this gives the assertion. □

Proposition 3.51 (Pre-composition with and affine mapping)**.** *Let $H : \mathbb{E}_1 \to \mathbb{E}_2$ be affine and $g : \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ (lsc and) convex. Then the function $f := g \circ H$ is (lsc and) convex.*

Proof. Let $x, y \in \mathbb{E}_1$ and $\lambda \in (0,1)$. Then we have

$$f(\lambda x+(1-\lambda x)) = g(\lambda H(x)+(1-\lambda)y) \le \lambda g(H(x))+(1-\lambda)g(H(y)) = \lambda f(x)+(1-\lambda)f(y),$$

which gives the convexity of f. The closedness of f, under the closedness of g, follows from the continuity (as a consequence of affineness) of H, cf. Exercise 2.13 □

Proposition 3.52 (Convexity under epi-composition)**.** *Let $f \in \Gamma$ and $L \in \mathcal{L}(\mathbb{E}, \mathbb{E}')$. Then the function $Lf : \mathbb{E}' \to \overline{\mathbb{R}}$ defined by*

$$(Lf)(y) := \inf \{f(x) \mid L(x) = y\}$$

is convex.

Proof. We first show that , with $T : (x,\alpha) \mapsto (Lx, \alpha)$, we have

$$\operatorname{epi}_< Lf = T(\operatorname{epi}_< f). \tag{3.13}$$

To this end, recall that

$$\operatorname{epi}_< Lf = \{(y,\alpha) \mid Lf(y) < \alpha\} \quad \text{and} \quad \operatorname{epi}_< f = \{(x,\alpha) \mid f(x) < \alpha\}.$$

First, let $(x,\alpha) \in \operatorname{epi}_< f$. Then $T(x,\alpha) = (L(x),\alpha)$ and

$$(Lf)(L(x)) = \inf_z \{f(z) \mid L(z) = L(x)\} \le f(x) < \alpha,$$

thus, $T(x,\alpha) \in \operatorname{epi}_< Lf$.

In turn, if $(y,\alpha) \in \operatorname{epi}_< Lf$, i.e. $\inf\{f(z) \mid L(z) = y\} < \alpha$, then $L^{-1}(y) \ne \emptyset$, hence, there exists $x \in L^{-1}(y)$ with $f(x) < \alpha$. Thus, we have $T(x,\alpha) = (y,\alpha)$ and $(x,\alpha) \in \operatorname{epi}_< f$. This proves (3.13).

Now, as f is convex, $\operatorname{epi}_< f$ is convex (see Exercise 3.3.3). But, since T is linear, from (3.13) it follows that also $\operatorname{epi}_< Lf$ is convex, which proves the convexity of Lf.

□

Theorem 3.53 (Infimal projection)**.** *Let $\psi : \mathbb{E}_1 \times \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ be convex. Then the* optimal value function

$$p : \mathbb{E}_1 \to \overline{\mathbb{R}},\ p(x) := \inf_{y \in \mathbb{E}_2} \psi(x,y)$$

is convex. Moreover, the set-valued mapping

$$x \mapsto \operatorname{argmin}_{y\in\mathbb{E}_2} \psi(x,y) \subset \mathbb{E}_2.$$

is convex-valued.

Proof. It can easily be shown that $\operatorname{epi}_{<}p = L(\operatorname{epi}_{<}\psi)$ under the linear mapping $L : (x, y, \alpha) \mapsto (x, \alpha)$. This immediately gives the convexity of p.

The remaining assertion follows immediately from the fact that the solution set of a convex minimization problem is convex (Exercise), and $y \mapsto \psi(x, y)$ is convex for all $x \in \mathbb{E}_1$. $\square$

3.5 Conjugacy of convex functions

3.5.1 The closed convex hull of a function

We start this section with the fundamental result that every closed, proper, convex function has an affine minorant.

Proposition 3.54 (Affine minorization principle)**.** *For $f \in \Gamma_0$ there exists an affine minorant, i.e. there exist $a \in \mathbb{E}$ and $\beta \in \mathbb{R}$ such that*

$$f(x) \geq \langle a,\, x\rangle + \beta \quad (x \in \mathbb{E}).$$

Proof. By assumption $\operatorname{epi} f$ is nonempty, closed and convex and there exists $\bar{x} \in \operatorname{dom} f$. Choosing $\gamma < f(\bar{x})$, we have $(\bar{x}, \gamma) \notin \operatorname{epi} f$. By the separation theorem (Theorem 3.16) there exists $0 \neq (a, \lambda) \in \mathbb{E} \times \mathbb{R}$ and $\beta \in \mathbb{R}$ such that

$$\langle a,\, x\rangle + \lambda\alpha \geq \beta > \langle a,\, \bar{x}\rangle + \lambda\gamma \quad ((x, \alpha) \in \operatorname{epi} f). \tag{3.14}$$

In particular, for $(\bar{x}, f(\bar{x})) \in \operatorname{epi} f$ this implies

$$\lambda\big(f(\bar{x}) - \gamma\big) > 0.$$

Since $f(\bar{x}) > \gamma$, this implies $\lambda > 0$. Dividing (3.14) by λ this gives

$$\alpha \geq -\left\langle \frac{a}{\lambda},\, x\right\rangle + \frac{\beta}{\lambda} \quad ((x, \alpha) \in \operatorname{epi} f).$$

Since $(x, f(x)) \in \operatorname{epi} f$ this yields

$$f(x) \geq -\left\langle \frac{a}{\lambda},\, x\right\rangle + \frac{\beta}{\lambda} \quad (x \in \operatorname{dom} f).$$

Since this inequality holds trivially for $x \notin \operatorname{dom} f$, the function $x \mapsto -\left\langle \frac{a}{\lambda},\, x\right\rangle + \frac{\beta}{\lambda}$ has the desired properties. $\square$

We continue with the main result of this section which says that every closed, proper, convex function is the pointwise supremum of its affine minorants. It will be the main workhorse for this section and Section 3.5.2 on Fenchel conjugates.

Theorem 3.55 (Envelope representation in Γ_0)**.** *Let $f \in \Gamma_0$. Then f is the pointwise supremum of all affine functions minorizing it, i.e.*

$$f(x) = \sup\left\{h(x) \mid h \leq f,\ h \text{ affine}\right\}.$$

Proof. The inequality

$$f(x) \geq \sup\{h(x) \mid h \leq f,\ h \text{ affine}\}$$

is clear.

In order to establish the reverse inequality we are going to show that for all $(\bar{x}, \gamma)$ such that $f(\bar{x}) > \gamma$ there exists an affine function $g : \mathbb{E} \to \mathbb{R}$ $g \leq f$ and $\gamma \leq g(\bar{x})$. As $\gamma < f(\bar{x})$ can be chosen arbitrarily close to $f(\bar{x})$ this will yield the assertion:

To this end, let $(\bar{x}, \gamma)$ such that $\gamma < f(\bar{x})$ be given. Then $(\bar{x}, \gamma) \notin \operatorname{epi} f$, and by separation there exists $(a, \lambda) \neq (0, 0)$ and $\beta \in \mathbb{R}$ such that

$$\langle a,\, x\rangle + \lambda\alpha \geq \beta > \langle a,\, \bar{x}\rangle + \lambda\gamma \quad ((x, \alpha) \in \operatorname{epi} f). \tag{3.15}$$

Note that we do not necessarily have $\bar{x} \in \operatorname{dom} f$. However, from (3.15) we infer at least $\lambda \geq 0$ (clear?). Hence, we consider two cases:
Case 1: $\lambda > 0$. Dividing (3.15) by $\lambda > 0$ yields

$$\left\langle \frac{a}{\lambda},\, x\right\rangle + \alpha \geq \frac{\beta}{\lambda} \quad \Longleftrightarrow \quad \alpha \geq -\left\langle \frac{a}{\lambda},\, x\right\rangle + \frac{\beta}{\lambda} \quad ((x, \alpha) \in \operatorname{epi} f).$$

In particular, for $(x, f(x)) \in \operatorname{epi} f$ we infer that

$$f(x) \geq g(x) := -\left\langle \frac{a}{\lambda},\, x\right\rangle + \frac{\beta}{\lambda} \quad (x \in \operatorname{dom} f).$$

For $x \notin \operatorname{dom} f$ this inequality is satisfied anyway. Due to (3.15) we also have

$$g(\bar{x}) = -\left(\frac{a}{\lambda}\right)^T \bar{x} + \frac{\beta}{\lambda} > \gamma,$$

so that g has the desired properties.

Case 2: $\lambda = 0$. In this case (3.15) reduces to

$$\langle a,\, x\rangle \geq \beta > \langle a,\, \bar{x}\rangle \quad ((x, \alpha) \in \operatorname{epi} f),$$

which is obviously equivalent to

$$\langle a,\, x\rangle \geq \beta > \langle a,\, \bar{x}\rangle \quad (x \in \operatorname{dom} f). \tag{3.16}$$

In particular, we have $\bar{x} \notin \operatorname{dom} f$ here, hence $f(\bar{x}) = \infty$. We now define

$$\hat{g} : \mathbb{E} \to \mathbb{R}, \quad \hat{g}(x) := -\langle a,\, x\rangle + \beta.$$

Then from (3.16) we immediately infer that

$$\hat{g}(x) \leq 0 \quad (x \in \operatorname{dom} f) \quad \text{and} \quad \hat{g}(\bar{x}) > 0. \tag{3.17}$$

Invoking Proposition 3.54 yields another affine mapping $\tilde{g} : \mathbb{E} \to \mathbb{R}$ such that

$$\tilde{g}(x) \leq f(x) \quad (x \in \mathbb{E}). \tag{3.18}$$

By means of $\hat{g}$ and $\tilde{g}$ we now define another affine function

$$g : \mathbb{E} \to \mathbb{R}, \quad g(x) := \tilde{g}(x) + \rho \hat{g}(x).$$

where $\rho > 0$ will be specified shortly. From (3.17) and (3.18) we infer that

$$g(x) = \tilde{g}(x) + \rho \hat{g}(x) \leq f(x) + \rho \underbrace{\hat{g}(x)}_{\leq 0} \leq f(x) \quad (x \in \operatorname{dom} f),$$

which, in particular, implies $g \leq f$ as well as

$$g(\bar{x}) = \tilde{g}(\bar{x}) + \rho \underbrace{\hat{g}(\bar{x})}_{>0} > \gamma$$

for all $\rho > 0$ sufficiently large. Hence g has the the desired properties in this case, which concludes the proof. $\square$

We will now establish the notion of the *convex* and *closed convex hull* of a function, and we will see that for functions that are minorized by an affine function, the latter coincides with the supremum over all its affine minorants.

For these purposes, let $f : \mathbb{E} \to \overline{\mathbb{R}}$ and observe that from our contruction of the lower semicontinuous hull $\operatorname{cl} f$ of f in Section 2.3 it follows that

$$(\operatorname{cl} f)(x) = \sup \{h(x) \mid h \leq f,\ h \text{ lsc}\},$$

i.e. $\operatorname{cl} f$ is the largest lsc function that minorizes f. We take this and Theorem 3.55 as a guide to build of the (closed) convex hull of f.

Definition 3.56 (Convex hull of a function)**.** *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then the pointwise supremum of all convex functions minorizing f, i.e.*

$$(\operatorname{conv} f)(x) := \sup \{h(x) \mid h \leq f,\ h \text{ convex}\}$$

is called the convex hull *of f. Moreover, we define the* closed convex hull *of f to be*

$$(\overline{\operatorname{conv}} f)(x) := \sup \{h(x) \mid h \leq f,\ h \text{ lsc and convex}\}.$$

Since both convexity and lower semicontinuity are preserved under pointwise suprema (cf. Proposition 3.50), we find that $\operatorname{conv} f$ is the largest convex function that minorizes f, and analogously, $\overline{\operatorname{conv}} f$ is the largest lsc and convex function that minorizes f. Note that we always have

$$\overline{\operatorname{conv}} f \leq \operatorname{conv} f \leq f$$

and

$$\overline{\operatorname{conv}} f = \operatorname{cl} (\operatorname{conv} f).$$

Moreover, there is an epigraphical characterization of the closed convex hull for proper functions that have an affine minorant, cf. Exercise 3.5.2. An analogous statement does not hold for the convex hull, see the discussion in [19].

We close this section with a result which tells us that for a proper function which has an affine minorant, its closed convex hull is the pointwise supremum of its affine minorants. Hence the set of functions over which the supremum in the definition of the close convex hull is taken over can be substantially reduced.

Here, notice that since every affine function $\mathbb{E} \to \mathbb{R}$ is convex and lsc, by the definition of the respective hulls, the functions f, $\operatorname{cl} f$, $\operatorname{conv} f$ and $\overline{\operatorname{conv}}\, f$ have the same affine minorants (if any).

Corollary 3.57 (Affine envelope representation of $\overline{\operatorname{conv}}$)**.** *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be proper and minorized by an affine function. Then $\overline{\operatorname{conv}}\, f$ is the pointwise supremum of all affine functions minorizing f, i.e.*

$$(\overline{\operatorname{conv}}\, f)(x) = \sup \{h(x) \mid h \leq f,\ h \ \text{ affine}\} .$$

Proof. Recall that f and $\overline{\operatorname{conv}}\, f$ have the same affine minorants. Since, by assumption f has one, so does $\overline{\operatorname{conv}}\, f$, and thus f does not take the value $-\infty$. Moreover, f is not constantly $+\infty$ and hence so neither is $\overline{\operatorname{conv}}\, f \leq f$. Therefore, $\overline{\operatorname{conv}}\, f$ is proper and has an affine minorant hence, by Theorem 3.55, $\overline{\operatorname{conv}}\, f$ is the pointwise supremum of its affine minorants. But, again, since these conincide with those of f, the result follows. □

3.5.2 The Fenchel conjugate

We start with the central definition of this section.

Definition 3.58 (Fenchel conjugate)**.** *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then its* Fenchel conjugate *(or simply* conjugate*) is the function $f^* : \mathbb{E} \to \overline{\mathbb{R}}$ defined by*

$$f^*(y) := \sup_{x \in \mathbb{E}} \{\langle x,\, y\rangle - f(x)\}.$$

The function $f^{**} := (f^*)^*$ is called the (Fenchel) biconjugate *of f.*

Note that, clearly, we can restrict the supremum in the above definition of the conjugate to the domain of the underlying function f, i.e.

$$f^*(y) = \sup_{x \in \operatorname{dom} f} \{\langle x,\, y\rangle - f(x)\}.$$

Moreover, by definition, we always have

$$f(x) + f^*(y) \geq \langle x,\, y\rangle \quad (x, y \in \mathbb{E}), \tag{3.19}$$

which is known as the *Fenchel-Young inequality.*

The mapping $f \mapsto f^*$ from the space of extended real-valued functions to itself is called the *Legendre-Fenchel transform.*

We always have

$$f \leq g \quad \Longrightarrow \quad f^* \geq g^*,$$

i.e. the Legendre-Fenchel transform is order-reversing. Moreover, this implies that for any function f we have

$$f \geq f^{**}.$$

We will study the case when equality occurs in depth below.

Before we start analyzing the conjugate function in-depth, we want to motivate why we would be interested in studying it: Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. We notice that

$$\operatorname{epi} f^* = \{(y, \beta) \mid \langle x,\, y\rangle - f(x) \leq \beta \quad (x \in \mathbb{E})\}\,. \tag{3.20}$$

This means that the conjugate of f is the function whose epigraph is the set of all (y, β) defining affine functions $x \mapsto \langle y,\, x\rangle - \beta$ that minorize f. In view of Corollary 3.57, if f is proper and has an affine minorant , the pointwise supremum of these affine mappings is the closed convex hull of f, i.e., through its epigraph, f^* encodes information about f and its closed convex hull.

Since

$$f^*(y) = \sup_{x\in\mathbb{E}}\{\langle x,\, y\rangle - f(x)\} = \sup_{(x,\alpha)\in\operatorname{epi} f}\{\langle y,\, x\rangle - \alpha\} \quad (y \in \mathbb{E}), \tag{3.21}$$

we also have

$$\operatorname{epi} f^* = \{(y, \beta) \mid \langle x,\, y\rangle - \alpha \leq \beta \quad ((x, \alpha) \in \operatorname{epi} f)\}$$

We use our recent findings to establish our main result on conjugates and biconjugates known in the literature as the Fenchel-Moreau theorem, crediting the founding fathers of convex analysis.

Theorem 3.59 (Fenchel-Moreau Theorem). *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be proper and have an affine minorant. Then the following hold:*

a) f^* and f^{**} are closed, proper and convex;

b) $f^{**} = \overline{\operatorname{conv}}\, f$;

c) $f^* = (\operatorname{conv} f)^* = (\operatorname{cl} f)^* = (\overline{\operatorname{conv}}\, f)^*$.

Proof. a) Applying Proposition 3.50 to (3.21), we see that f^* is lsc and convex. If f^* attained the value $-\infty$, f would be constantly $+\infty$, which is false. On the other hand, f^* is not identically $+\infty$, since that would imply that its epigraph $\operatorname{epi} f^*$ which encodes all minorizing affine mappings of f, were empty, which is false by assumption . Hence, f^* is proper.

Applying the same arguments to $f^{**} = (f^*)^*$ (and observing that f^* is proper and has an affine minorant) gives that f^{**} is closed, proper and convex, too.

b) Applying (3.21) to f^{**}, for $x \in \mathbb{E}$, we have

$$f^{**}(x) = \sup_{(y,\beta)\in\operatorname{epi} f^*}\{\langle y,\, x\rangle - \beta\}.$$

Hence, in view of (3.20), f^{**} is the pointwise supremum of all affine minorants of f. Therefore, by Corollary 3.57, we see that $f^{**} = \overline{\operatorname{conv}}\, f$.

c) Since the affine minorants of $f, \operatorname{conv} f, \operatorname{cl} f$ and $\overline{\operatorname{conv}} f$ coincide, their conjugates have the same epigraph and hence are equal.

$\square$

$f \geq f^{**}$ for any function $f : \mathbb{E} \to \overline{\mathbb{R}}$ that has an affine minorant, and it holds that $f^{**} = f$ if and only if f is closed and convex. Thus, the Legendre-Fenchel transform induces a one-to-one correspondence on Γ_0: For $f, g \in \Gamma_0$, f is conjugate to g if and only if g is conjugate to f and we write $f \overset{*}{\longleftrightarrow} g$ in this case. This is called the *conjugacy correspondence*. A property on one side is reflected by a dual property on the other as we will see in the course of our study.

A list of some elementary conjugacy operations is given below

Proposition 3.60 (Elementary cases of conjugacy). *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then the following hold:*

a) $(f - \langle a, \cdot \rangle)^* = f^*((\cdot) + a) \quad (a \in \mathbb{E})$*;*

b) $(f + \gamma)^* = f^* - \gamma \quad (\gamma \in \mathbb{R})$*;*

c) $(\lambda f)^* = \lambda f^* \left(\frac{(\cdot)}{\lambda} \right) \quad (\lambda > 0)$*.*

Proof. Exercise 3.5.1. $\square$

3.5.3 Special cases of conjugacy

Convex quadratic functions For $Q \in \mathbb{S}^n, b \in \mathbb{R}^n$ we consider the *quadratic function* $q : \mathbb{R}^n \to \mathbb{R}$ defined by

$$q(x) := \frac{1}{2} x^T Q x + b^T x. \tag{3.22}$$

By the well known characterization of convexity in the smooth case, we know that q is convex if and only if Q is (symmetric) positive semidefinite. Hence, for the remainder we assume that $Q \succeq 0$.

Proposition 3.61 (Conjugate of convex quadratic functions). *For q from* (3.22) *with $Q \in \mathbb{S}^n_+$ we have*

$$q^*(y) = \begin{cases} \frac{1}{2}(y-b)^T Q^\dagger (y-b) & \text{if} \quad y - b \in \operatorname{rge} Q, \\ +\infty & \text{else.} \end{cases}$$

In particular, if $Q \succ 0$, we have

$$q^*(y) = \frac{1}{2}(y-b)^T Q^{-1} (y-b)$$

Proof. By definition, we have

$$q^*(y) = \sup_{x \in \mathbb{R}^n} \left\{ x^T y - \frac{1}{2} x^T Q x - b^T x \right\} = - \inf_{x \in \mathbb{E}} \left\{ \frac{1}{2} x^T Q x - (b-y)^T x \right\}. \tag{3.23}$$

The necessary and sufficient optimality conditions of $\bar{x}$ to be a minimizer of the convex function $x \mapsto \frac{1}{2}x^T Qx - (b-y)^T x$ read

$$Q\bar{x} = y - b. \tag{3.24}$$

Hence, if $y - b \notin \operatorname{rge} Q$, from Exercise 2.7, we know that $\inf f = -\infty$, hence $q^*(y) = +\infty$ in that case.

Otherwise, we have $y - b \in \operatorname{rge} Q$, hence, in view of Theorem A.4, (3.24) is equivalent to

$$\bar{x} = Q^\dagger(y-b) + z, \quad z \in \ker A.$$

Inserting $\bar{x} = Q^\dagger(y-b)$ (we can choose $z = 0$) in (3.23) yields

$$\begin{aligned} q^*(y) &= (Q^\dagger(y-b))^T y - \frac{1}{2}(Q^\dagger(y-b))^T QQ^\dagger(y-b) - b^T Q^\dagger(y-b) \\ &= (y-b)Q^\dagger(y-b) - \frac{1}{2}(y-b)Q^\dagger QQ^\dagger(y-b) \\ &= \frac{1}{2}(y-b)Q^\dagger(y-b), \end{aligned}$$

where we make use of Theorem A.4 a) and c). Part d) of the latter result gives the remaining assertion.

□

We point out that, by the foregoing result, the function $f = \frac{1}{2}\|\cdot\|^2$ is self-conjugated in the sense that $f^* = f$. Exercise 3.5.5 shows that this is the only function on $\mathbb{R}^n$ that has this property. Clearly, by an isomorphy argument, the same holds for the respective function on an arbitrary Euclidean space.

Support functions

Definition 3.62 (Positive homogeneity, subadditivity, and sublinearity)**.** *Let* $f : \mathbb{E} \to \overline{\mathbb{R}}$. *Then we call* f *with* $0 \in \operatorname{dom} f$

i) positively homogeneous *if*

$$f(\lambda x) = \lambda f(x) \quad (\lambda > 0, x \in \mathbb{E});$$

b) subadditive *if*

$$f(x+y) \le f(x) + f(y) \quad (x, y \in \mathbb{E});$$

c) sublinear *if*

$$f(\lambda x + \mu y) \le \lambda f(x) + \mu f(y) \quad (x, y \in \mathbb{E},\ \lambda, \mu > 0).$$

Note that in the definition of positive homogeneity we could have also just demanded an inequality, since $f(\lambda x) \le \lambda f(x)$ for all $\lambda > 0$ implies that

$$f(x) = f(\lambda^{-1}\lambda x) \le \frac{1}{\lambda} f(\lambda x).$$

We note that norms are sublinear.

Example 3.63. *Every norm $\|\cdot\|$ is sublinear.*

We next proivide a usful list of characerizations of positive homogeneneity and sublinearity, respectively.

Proposition 3.64. (Positive homogeneity, sublinearity and subadditivity) *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then the following hold:*

a) f is positively homogeneous if and only if $\operatorname{epi} f$ is a cone. In this case $f(0) \in \{0, -\infty\}$.

b) If f is lsc and positively homogeneous with $f(0) = 0$ it must be proper.

c) The following are equivalent:

   i) f is sublinear;

   ii) f is positively homogeneous and convex;

   iii) f is positively homogeneous and subadditive;

   iv) $\operatorname{epi} f$ is a convex cone.

Proof. Exercise 3.5.9 □

We continue with the prototype of a sublinear functions, so-called *support functions*, which will from now on occur ubiquitiously.

Definition 3.65 (Support functions). *Let $C \in \mathbb{E}$ nonempty. The* support function *of C is defined by*

$$\sigma_C : x \in \mathbb{E} \mapsto \sup_{s \in C} \langle s,\, x\rangle .$$

We start our investigation of support functions with a list of elementary properties.

Proposition 3.66 (Support functions). *Let $C \subset \mathbb{E}$ be nonempty. Then*

a) $\sigma_C = \sigma_{\operatorname{cl} C} = \sigma_{\operatorname{conv} C} = \sigma_{\overline{\operatorname{conv}}\, C}$.

b) σ_C is proper, lsc and sublinear.

c) $\delta_C^* = \sigma_C$ and $\sigma_C^* = \delta_{\overline{\operatorname{conv}}\, C}$.

d) If C is closed and convex then $\sigma_C \overset{*}{\longleftrightarrow} \delta_C$.

Proof. a) Obviously, closures do not make a difference. On the other hand, we have

$$\left\langle \sum_{i=1}^{N+1} \lambda_i s_i,\, x \right\rangle = \sum_{i=1}^{N+1} \lambda_i \langle s_i,\, x\rangle \le \max_{i=1,\dots,r} \langle s_i,\, x\rangle$$

for all $s_i \in C, \lambda \in \Delta_{N+1}$, which shows that convex hulls also do not change anything.

b) By Proposition 3.50 σ_C is lsc and convex, and as $0 \in \operatorname{dom}\sigma_C$ and since $\lambda\sigma_C(x) = \sigma_C(\lambda x)$ for all $x \in \mathbb{E}$ and $\lambda > 0$ this shows properness and positive homogeneity, which gives the assertion in view of Proposition 3.64 c).

c) Clearly, $\delta_C^* = \sigma_C$. Hence, $\sigma_C^* = \delta_C^{**} = \overline{\operatorname{conv}}\,\delta_C = \delta_{\overline{\operatorname{conv}}\,C}$, since

$$\overline{\operatorname{conv}}\,(\operatorname{epi}\delta_C) = \overline{\operatorname{conv}}\,(C \times \mathbb{R}_+) = \overline{\operatorname{conv}}\,C \times \mathbb{R}_+ = \operatorname{epi}(\delta_{\overline{\operatorname{conv}}\,C}).$$

d) Follows immediately from c).

□

One of our main goals in this paragraph is to show that, in fact, part b) of Propostion 3.66 can be reversed in the sense that every proper, lsc and sublinear function is a support function. As a preparation we need the following result.

Proposition 3.67. *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be closed, proper and convex. Then the following are equivalent:*

i) f only takes the values 0 and $+\infty$;

ii) f^* is positively homogeneous (i.e. sublinear, since convex).

Proof. 'i)⇒ii):' In this case $f = \delta_C$ for some closed convex set $C \subset \mathbb{E}$. Hence, $f^* = \sigma_C$, which is sublinear, cf. Proposition 3.66.

In turn, let f^* be positively homogeneous (hence sublinear). Then, for $\lambda > 0$ and $y \in \mathbb{E}$, we have

$$\begin{aligned} f^*(y) &= \lambda f^*(\lambda^{-1}y) \\ &= \lambda \sup_{x\in\mathbb{E}} \left\{\left\langle x,\, \lambda^{-1}y\right\rangle - f(x)\right\} \\ &= \sup_{x\in\mathbb{E}} \{\langle x,\, y\rangle - \lambda f(x)\} \\ &= (\lambda f)^*(y). \end{aligned}$$

Thus, $(\lambda f)^* = f^*$ for all $\lambda > 0$ and hence, by the Fenchel-Moreau Theorem, we have

$$\lambda f = (\lambda f)^{**} = f^{**} = f \quad (\lambda > 0).$$

But as f is proper, hence does not takte the value $-\infty$, this immediately implies that f only takes the values $+\infty$ and 0. □

Theorem 3.68 (Hörmander's Theorem)**.** *A function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is proper, lsc and sublinear if and only if it is a support function.*

Proof. By Proposition 3.66 b), every support function is proper, lsc and sublinear.

In turn, if f is proper, lsc and sublinear (hence $f = f^{**}$), by Proposition 3.67, f^* is the indicator of some set $C \subset \mathbb{E}$, which necessary needs to be nonempty, closed and convex, as $f^* \in \Gamma_0$. Hence, $f^{**} = \delta_C^* = \sigma_C$. □

We now want to give a slight refinement of Hörmander's Theorem, in that we describe the set that a proper, lsc sublinear function supports.

Corollary 3.69. *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be proper and sublinear. Then* $\mathrm{cl}\, f$ *is the support function of the closed convex set*

$$\{s \in \mathbb{E} \mid \langle s,\, x\rangle \le f(x) \ (x \in \mathbb{E})\}\,.$$

Proof. Since $\mathrm{cl}\, f$ is proper (cf. Exercise 3.3.4) closed and sublinear it is a support function of a closed convex set C. Therefore, we have $\mathrm{cl}\, f = \delta_C^*$ and thus $f^* = (\mathrm{cl}\, f)^* = \delta_C$. Hence, $C = \{s \in \mathbb{E} \mid f^*(s) \le 0\}$. But $f^*(s) \le 0$ if and only $\langle s,\, x\rangle - f(x) \le 0$ for all $x \in \mathbb{E}$. □

3.5.4 Gauges, polar sets and dual norms

We now present a class of functions that makes a connection between support functions and norms.

Definition 3.70 (Gauge function)**.** *Let $C \subset \mathbb{E}$. The* gauge (function) *of C is defined by*

$$\gamma_C : x \in \mathbb{E} \mapsto \inf\{\lambda \ge 0 \mid x \in \lambda C\}\,.$$

For a closed convex set that contains the origin, its gauge has very desirable convex-analytical properties.

Proposition 3.71. *Let $C \subset \mathbb{E}$ be nonempty, closed and convex with $0 \in C$. Then γ_C is proper, lsc and sublinear.*

Proof. γ_C is obviously proper as $\gamma_C(0) = 0$. Moreover, for $t > 0$ and $x \in \mathbb{E}$, we have

$$\begin{aligned}
\gamma_C(tx) &= \inf\{\lambda \ge 0 \mid tx \in \lambda C\} \\
&= \inf\left\{\lambda \ge 0 \,\middle|\, x \in \frac{\lambda}{t} C\right\} \\
&= \inf\{t\mu \ge 0 \mid x \in \mu C\} \\
&= t \inf\{\mu \ge 0 \mid x \in \mu C\} \\
&= t\gamma_C(x),
\end{aligned}$$

i.e. γ_C is positively homogeneous (since also $0 \in \mathrm{dom}\,\gamma_C$). We now show that it is also subadditive, hence altogether, sublinear: To this end, take $x, y \in \mathrm{dom}\,\gamma_C$ (otherwise there is nothing to prove). Due to the identity

$$\frac{x+y}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}\frac{x}{\lambda} + \frac{\mu}{\lambda+\mu}\frac{y}{\mu} \quad (\lambda+\mu \ne 0),$$

we realize, by convexity of C, that $x+y \in (\lambda+\mu)C$ if $x \in \lambda C$ and $y \in \mu C$ for all $\lambda, \mu \ge 0$. This implies that $\gamma_C(x+y) \le \gamma_C(x) + \gamma_C(y)$.

In order to prove lower semicontinuity of γ_C notice that (by Exercise 3.5.12 and positive homogeneity) we have $\mathrm{lev}_{\le\alpha}\gamma_C = \alpha C$ for $\alpha > 0$, $\mathrm{lev}_{\le\alpha}\gamma_C = \emptyset$ for $\alpha < 0$ and $\mathrm{lev}_{\le 0}\gamma_C = C^\infty$ (again by Exercise 3.5.12), hence all level sets of γ_C are closed, i.e. γ_C is lsc.

This concludes the proof.

□

Note that in the proof of Proposition 3.71, we do not need the assumption that C contains the origin to prove sublinearity. We do need it, though, to get lower semicontinuity, cf. Exercise 3.5.12

Since the gauge of a closed convex set that contains 0 is proper, lsc and sublinear we know, in view of Hörmander's Theorem (see Theorem 3.68), that it is the support function of some closed convex set. It can be described beautifully using the concept of *polar sets* which generalizes the notion of polar cones, cf. Definition 3.34.

Definition 3.72 (Polar sets)**.** *Let $C \subset \mathbb{E}$. Then its* polar set *is defined by*

$$C^\circ := \{v \in \mathbb{E} \mid \langle v, x \rangle \leq 1 \ (x \in C)\}.$$

Moreover, we put $C^{\circ\circ} := (C^\circ)^\circ$ and call it the bipolar set *of C.*

Note that there is no ambiguity in notation, since the polar cone and the polar set of a cone coincide, see Exercise 3.5.11 Moreover, as an intersection of closed half-spaces, C° is a closed, convex set containing 0. In addition, like we would expect, we have

$$C \subset D \quad \Rightarrow D^\circ \subset C^\circ,$$

and

$$C \subset C^{\circ\circ}.$$

Before we continue to pursue our question for the support function representation of gauges, we provide the famous *bipolar theorem*. Its proof is based on separation.

Theorem 3.73 (Bipolar Theorem)**.** *Let $C \subset \mathbb{E}$. Then $C^{\circ\circ} = \overline{\mathrm{conv}}\,(C \cup \{0\})$.*

Proof. Since $C \cup \{0\} \subset C^{\circ\circ}$ and $C^{\circ\circ}$ is closed and convex, we clearly have $\overline{\mathrm{conv}}\,(C \cup \{0\}) \subset C^{\circ\circ}$. Now assume there were $\bar{x} \in C^{\circ\circ} \setminus \overline{\mathrm{conv}}\,(C \cup \{0\})$. By strong separation, there exists $s \in \mathbb{E} \setminus \{0\}$ such that

$$\langle s, \bar{x} \rangle > \sigma_{\overline{\mathrm{conv}}\,(C \cup \{0\})}(s) \geq \max\{\sigma_C(s), 0\}.$$

After rescaling s accordingly (cf. Remark 3.17) we can assume that

$$\langle s, \bar{x} \rangle > 1 \geq \sigma_C(s),$$

in particular, $s \in C^\circ$. On the other hand $\langle s, \bar{x} \rangle > 1$ and $\bar{x} \in C^{\circ\circ}$, which is a contradiction. □

As a consequence of the bipolar theorem we see that every closed convex set $C \subset \mathbb{E}$ containing 0 satisfies $C = C^{\circ\circ}$. Hence, the mapping $C \mapsto C^\circ$ establishes a one-to-one correspondence on the closed convex sets that contain the origin. This is connected to conjugacy through gauge functions as is highlighted by the next result.

Proposition 3.74. *Let $C \subset \mathbb{E}$ be closed and convex with $0 \in C$. Then*

$$\gamma_C = \sigma_{C^\circ} \ \stackrel{*}{\longleftrightarrow} \ \delta_{C^\circ} \quad \textit{and} \quad \gamma_{C^\circ} = \sigma_C \ \stackrel{*}{\longleftrightarrow} \ \delta_C.$$

Proof. Since, by Proposition 3.71, γ_C is proper, lsc and sublinear we have

$$\gamma_C = \sigma_D, \quad D = \{v \in \mathbb{E} \mid \langle v, x\rangle \le \gamma_C(x)\ (x \in \mathbb{E})\}$$

in view of Corollary 3.69. To prove that $\gamma_C = \sigma_{C^\circ}$, we need to show that $D = C^\circ$. Since $\gamma_C(x) \le 1$ if (and only if; see Exercise 3.5.12) $x \in C$, the inclusion $D \subset C^\circ$ is clear. In turn, let $v \in C^\circ$, i.e. $\langle v, x\rangle \le 1$ for all $x \in C$. Now let $x \in \mathbb{E}$. By the definition of γ_C, there exists $\lambda_k \to \gamma_C(x)$ and $c_k \in C$ such that $x = \lambda_k c_k$ for all $k \in \mathbb{N}$. But then

$$\langle v, x\rangle = \lambda_k \langle v, c_k\rangle \le \lambda_k \to \gamma_C(x),$$

hence $v \in D$, which proves $\gamma_C = \sigma_{C^\circ}$. Since $C^{\circ\circ} = C$, this implies $\gamma_{C^\circ} = \sigma_C$. The conjugacy relations are due to Proposition 3.66. $\square$

Exercise 3.5.12 tells us that the gauge of a symmetric, compact convex set with nonempty interior is a norm. This justifies the following definition.

Definition 3.75 (Dual norm)**.** *Let $\|\cdot\|_*$ be a norm on $\mathbb{E}$ with closed unit ball B_*. Then we call*

$$\|\cdot\|_*^\circ := \gamma_{B_*^\circ}$$

its dual norm*.*

Corollary 3.76 (Dual norms)**.** *For any norm $\|\cdot\|_*$ with (closed) unit ball B its dual norm is σ_B, the support of its unit ball. In particular, we have $\|\cdot\|^\circ = \|\cdot\|$, i.e. the Euclidean norm is self-dual.*

Exercises for Section 3.5

3.5.1 **(Elementary conjugacy operations)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Prove the following:

a) $(f - \langle a, \cdot\rangle)^* = f^*((\cdot) + a) \quad (a \in \mathbb{E})$;

b) $(f + \gamma)^* = f^* - \gamma \quad (\gamma \in \mathbb{R})$;

c) $(\lambda f)^* = \lambda f^*\left(\frac{(\cdot)}{\lambda}\right) \quad (\lambda > 0)$.

3.5.2 **(Closed convex hulls)** Let $C \subset \mathbb{E}$ be nonempty. Show the following:

a) $\overline{\mathrm{conv}}\, C = \mathrm{cl}\,(\mathrm{conv}\, C)$.

b) Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be proper and have an affine minorant. Then $\mathrm{epi}\,(\overline{\mathrm{conv}}\, f) = \overline{\mathrm{conv}}\,(\mathrm{epi}\, f)$.

3.5.3 **(Convex quadratic functions)** Let $A \in \mathbb{S}^n$ and define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \frac{1}{2} x^T A x.$$

a) Show that f is convex if and only if $A \in \mathbb{S}^n_+$.

b) For $A \in \mathbb{S}^n_{++}$ compute f^* and f^{**}.

3.5.4 **(Conjugate of separable sum)** For $f_i : \mathbb{E}_i \to \overline{\mathbb{R}}$ $(i = 1, 2)$ the *separable sum of* f_1 *and* f_2 is defined by

$$f_1 \oplus f_2 : (x_1, x_2) \in \mathbb{E}_1 \times \mathbb{E}_2 \mapsto f_1(x_1) + f_2(x_2).$$

Show that $(f_1 \oplus f_2)^* = f_1^* \oplus f_2^*$.

3.5.5 **(Self-conjugacy)** Show that $\frac{1}{2}\|\cdot\|^2$ is the only function $f : \mathbb{E} \to \overline{\mathbb{R}}$ with $f^* = f$.

3.5.6 **(Projections and conjugate functions)** Let $\phi : \mathbb{E} \to \overline{\mathbb{R}}$ proper with and affine minorant, let $V \subset \mathbb{E}$ be a subspace containing $\mathrm{aff}\,(\mathrm{dom}\,\phi)$, and set $U := V^\perp$. Show that for any $z \in \mathrm{aff}\,(\mathrm{dom}\,\phi)$ and $s \in \mathbb{E}$ we have

$$\phi^*(s) = \langle P_U(s),\, z\rangle + \phi^*(P_V(s)).$$

3.5.7 **(Conjugate of max-function)** Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = \max_{i=1,\dots,n} x_i$. Show that $f^* = \delta_{\Delta_n}$ where $\Delta_n := \left\{\lambda \in \mathbb{R}^n_+ \mid \sum_{i=1}^n \lambda_i = 1\right\}$.

3.5.8 **(Conjugate of negative logdet)** Compute f^* for

$$f : X \in \mathbb{S}^n \mapsto \begin{cases} -\log(\det X) & \text{if} \quad X \succ 0, \\ +\infty & \text{else.} \end{cases}$$

3.5.9 **(Positive homogeneity, sublinearity and subadditivity)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Show the following:

a) f is positively homogeneous if and only if $\mathrm{epi}\, f$ is a cone. In this case $f(0) \in \{0, -\infty\}$.

b) If f is lsc and positively homogeneous with $f(0) = 0$ it must be proper.

c) The following are equivalent:

   i) f is sublinear;

   ii) f is positively homogeneous and convex;

   iii) f is positively homogeneous and subadditive;

   iv) $\mathrm{epi}\, f$ is a convex cone.

3.5.10 **(Finiteness of support functions)** Let $S \subset \mathbb{E}$ be nonempty. Then σ_S is finite if and only if S is bounded.

3.5.11 **(Polar sets)** Show the following:

a) If $C \in \mathbb{E}$ is a cone, we have $\{v \mid \langle v,\, x\rangle \le 0\ (x \in C)\} = \{v \mid \langle v,\, x\rangle \le 1\ (x \in C)\}$.

b) $C \subset \mathbb{E}$ is bounded if and only if $0 \in \mathrm{int}\, C^\circ$.

c) For any closed half-space H containing 0 we have $H^{\circ\circ} = H$.

3.5.12 **(Gauge functions)** Let $C \subset \mathbb{E}$ be nonempty, closed and convex with $0 \in C$. Prove:

a) $C = \mathrm{lev}_{\leq 1}\gamma_C, \quad C^\infty = \gamma_C^{-1}(\{0\}), \quad \mathrm{dom}\,\gamma_C = \mathbb{R}_+ C$

b) The following are equivalent:

i) γ_C is a norm (with C as its unit ball);

ii) C is bounded, symmetric ($C = -C$) with nonempty interior.

3.5.13 **(Cone polarity and conjugacy)** Let $K \subset \mathbb{E}$ be a convex cone. Then $\delta_K \overset{*}{\longleftrightarrow} \delta_{K^\circ}$.

3.5.14 **(Directional derivative of a convex function)** Let $f \in \Gamma, x \in \mathrm{dom}\, f$ and $d \in \mathbb{E}$. Show that the following hold:

a) The difference quotient

$$t > 0 \mapsto q(t) := \frac{f(x+td) - f(x)}{t}$$

is nondecreasing.

b) $f'(x;d)$ exists (in $\overline{\mathbb{R}}$) with

$$f'(x;d) = \inf_{t>0} q(t),$$

c) $f'(x;\cdot)$ is sublinear with $\mathrm{dom}\, f'(x;\cdot) = \mathbb{R}_+(\mathrm{dom}\, f - x)$.

d) $f'(x;\cdot)$ is proper and lsc for $x \in \mathrm{ri}\,(\mathrm{dom}\, f)$.

3.6 Horizon functions

There is a functional correspondence of the horizon/recession calculus for (convex) cones which is obtained through the epigraphical perspective.

Definition 3.77 (Horizon functions)**.** *For any function $f : \mathbb{E} \to \overline{\mathbb{R}}$ the associated* horizon function $f^\infty : \mathbb{E} \to \overline{\mathbb{R}}$ *is the function determined through*

$$\mathrm{epi}\, f^\infty = (\mathrm{epi}\, f)^\infty \quad (f \not\equiv +\infty) \quad \textit{and} \quad f^\infty = \delta_{\{0\}} \quad (f \equiv +\infty).$$

As the horizon cone of a set is, by Lemma 3.38, always a closed cone, we find that the horizon function of is always lsc and positively homogeneous. If f is convex, then, by Proposition 3.41, f^∞ is, in even sublinear (and lsc), cf. Proposition 3.64.

We have the following analytic description of the horizon function of a closed, proper, convex functions.

Proposition 3.78. *Let $f \in \Gamma$. Then f^∞ is lsc and sublinear. If f is also lsc (i.e. $f \in \Gamma_0$) we have*

$$f^\infty(w) = \lim_{t \downarrow 0} \frac{f(\bar{x} + tw) - f(\bar{x})}{t} \quad (\bar{x} \in \mathrm{dom}\, f).$$

Proof. By Proposition 3.41 we have

$$\begin{aligned}(w,\beta)\in \operatorname{epi} f^{\infty} &\iff (w,\beta)\in(\operatorname{epi} f)^{\infty}\\ &\iff \forall (x,\alpha)\in \operatorname{epi} f, t>0:\ (x,\alpha)+t(w,\beta)\in \operatorname{epi} f\\ &\iff \forall x\in \operatorname{dom} f, t>0: \frac{f(x+tw)-f(x)}{t}\le\beta\\ &\iff \forall x\in \operatorname{dom} f: \sup_{t>0}\frac{f(x+tw)-f(x)}{t}\le\beta.\end{aligned}$$

The fact that the supremum is a limit is due to Exercise 3.5.14 a).

□

Proposition 3.79. *Let $f\in\Gamma_0$. Then for all $\alpha\in\mathbb{R}$ such that* $\operatorname{lev}_{\le\alpha}f\neq\emptyset$ *we have*

$$\{x \mid f(x)\le\alpha\}^{\infty}=\{x \mid f^{\infty}(x)\le 0\}. \tag{3.25}$$

Proof. This follows from Proposition 3.41 and Proposition 3.78. □

Proposition 3.79 motivates the following definition of the horizon cone of a closed, proper, convex function.

Definition 3.80 (Horizon cone of closed proper convex function)**.** *Let $f\in\Gamma_0$. Then we define the* horizon cone of f *by* $\operatorname{hzn} f:=\operatorname{lev}_{\le 0}f^{\infty}$.

We have the following duality correspondences.

Theorem 3.81. *Let $f\in\Gamma$. Then* $(\overline{\operatorname{cone}}(\operatorname{dom} f))^{\circ}=\operatorname{hzn} f^*$. *Dually, if f is also lsc (i.e. $f\in\Gamma_0$), then* $(\operatorname{hzn} f)^{\circ}=\overline{\operatorname{cone}}(\operatorname{dom} f)$.

Proof. We first note that $f^*\in\Gamma_0$ by Theorem 3.59, hence $\operatorname{hzn} f^*$ is well-defined and, by Propositon 3.79, for any $\alpha>\inf f^*$, equals the horizon cone of

$$\begin{aligned}\operatorname{lev}_{\le\alpha}f^* &= \{y \mid f^*(y)\le\alpha\}\\ &= \{y \mid \langle y,\, x\rangle - f(x)\le\alpha\ (x\in\operatorname{dom} f)\}\\ &= \{y \mid \langle y,\, x\rangle \le f(x)+\alpha\ (x\in\operatorname{dom} f)\}.\end{aligned}$$

From the latter representation we find that

$$\begin{aligned}w\in\operatorname{hzn} f^* &\iff \forall y\in\operatorname{lev}_{\le\alpha}f^*, \lambda\ge 0: y+\lambda w\in\operatorname{lev}_{\le\alpha}f^*\\ &\iff \forall y\in\operatorname{lev}_{\le\alpha}f^*, \lambda\ge 0, x\in\operatorname{dom} f: \lambda\langle x,\, w\rangle+\langle y,\, x\rangle\le f(x)+\alpha\\ &\iff \forall x\in\operatorname{dom} f:\ \langle w,\, x\rangle\le 0.\end{aligned}$$

Therefore

$$\operatorname{hzn} f^*=\{w \mid \langle x,\, w\rangle\le 0\ (x\in\operatorname{dom} f)\},$$

which proves the first statement, see e.g. Exercise 3.2.6. The second statement follows as $f=f^{**}$ when $f\in\Gamma_0$, see Theorem 3.59. □

3.7 The convex subdifferential

For motivational purposes we consider the following well known result for characterizing convexity for smooth functions which is a consequence of the mean-value theorem.

Proposition 3.82. *Let $C \subset \mathbb{E}$ be open and convex and let $f : \mathbb{E} \to \mathbb{R}$ be continuously differentiable on C. Then f is convex on C if and only if*

$$f(x) \geq f(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x} \rangle \quad (x, \bar{x} \in C). \tag{3.26}$$

In view of the latter result the following notion of a surrogate for the derivative for an arbitrary convex function appears natural.

Definition 3.83 (The convex subdifferential)**.** *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ and $\bar{x} \in \mathbb{E}$. Then $g \in \mathbb{E}$ is called a* subgradient *of f at $\bar{x}$ if*

$$f(x) \geq f(\bar{x}) + \langle g, x - \bar{x} \rangle \quad (x \in \mathbb{E}). \tag{3.27}$$

The set

$$\partial f(\bar{x}) := \{ v \in \mathbb{E} \mid f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle \quad (x \in \mathbb{E}) \}$$

of all subgradients is called the (convex) subdifferential *of f at $\bar{x}$.*

Note that we did not restrict ourselves to convex functions in the above definition, but we point out that its the class Γ where the subdifferential is most meaningful.

Moreover, note that, clearly, in (3.27) (which is called the *subgradient inequality*), we can restrict ourselves to points $x \in \operatorname{dom} f$, since the inequality holds trivially outside of $\operatorname{dom} f$.

We point out that the subdifferential $\partial f(\bar{x})$ of $f : \mathbb{E} \to \overline{\mathbb{R}}$ at $\bar{x} \in \mathbb{E}$ contains exactly the slopes of all affine minorants of f that coincide with f at $\bar{x}$, see Figure 9.

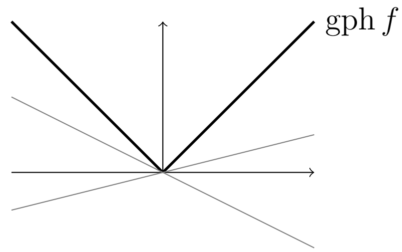

Figure 9: Affine minorants at a point of nondifferentiability

The subdifferential of a convex function might well be empty, contain only a single point, be bounded (hence compact as it is always closed, as we wil see shortly) or unbounded as the following examples illustrate.

Example 3.84.

a) *(Subdifferential of indicator function) Let $C \subset \mathbb{E}$ be convex and $\bar{x} \in C$. Then*

$$\begin{aligned} g \in \partial \delta_C(\bar{x}) \iff & \delta_C(x) \geq \delta_C(\bar{x}) + \langle g, x - \bar{x} \rangle \quad (x \in \mathbb{E}) \\ \iff & 0 \geq \langle g, x - \bar{x} \rangle \quad (x \in C), \end{aligned}$$

i.e. $\partial \delta_C(\bar{x}) = \{v \in \mathbb{E} \mid \langle v, x - \bar{x} \rangle \leq 0 \ (x \in C)\}$. The latter set is also called the normal cone *of C at $\bar{x}$, and is denoted by $N_C(\bar{x})$. It plays a central role in the derivation of optimality conditions of convex optimization problems.*

b) *(Subdifferential of Euclidean norm) We have*

$$\partial \| \cdot \| (\bar{x}) = \begin{cases} \frac{\bar{x}}{\|\bar{x}\|} & \text{if } \bar{x} \neq 0, \\ \operatorname{cl} \mathbb{B} & \text{if } \bar{x} = 0 \end{cases}$$

as can be verified by elementary considerations, see Exercise 3.7.1.

c) *(Empty subdifferential) Consider*

$$f : x \in \mathbb{R} \mapsto \begin{cases} -\sqrt{x} & \text{if} \quad x \geq 0, \\ +\infty & \text{else.} \end{cases}$$

Then $\partial f(x) = \emptyset$ for all $x \leq 0$, see Exercise 3.7.2.

We start our conceptual study of the subdifferential with some elementary properties.

Proposition 3.85 (Elementary properties of the subdifferential)**.** *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then the following hold:*

a) *$\partial f(\bar{x})$ is closed and convex for all $\bar{x} \in \operatorname{dom} f$.*

b) *If f is proper then $\partial f(x) = \emptyset$ for $x \notin \operatorname{dom} f$.*

c) *We have $0 \in \partial f(\bar{x})$ if and only if $\bar{x} \in \operatorname{argmin}_{\mathbb{E}} f$ (Generalized Fermat's rule)*

d) *If f is convex then $\partial f(\bar{x}) = \{v \in \mathbb{E} \mid (v, -1) \in N_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))\}$ $(\bar{x} \in \operatorname{dom} f)$.*

Proof. a) We have

$$\partial f(\bar{x}) = \bigcap_{x \in \mathbb{E}} \{v \mid \langle x - \bar{x}, v \rangle \leq f(\bar{x}) - f(x)\},$$

and intersection preserves closedness and convexity.

b) Obvious.

c) By definition we have

$$0 \in \partial f(\bar{x}) \iff f(x) \geq f(\bar{x}) \quad (x \in \mathbb{E}).$$

d) Notice that

$$
\begin{aligned}
v \in \partial f(\bar{x}) &\iff f(x) \geq f(\bar{x}) + \langle v,\, x - \bar{x}\rangle \quad (x \in \operatorname{dom} f)\\
&\iff \alpha \geq f(\bar{x}) + \langle v,\, x - \bar{x}\rangle \quad ((x,\alpha) \in \operatorname{epi} f)\\
&\iff 0 \geq \langle (v,-1),\, (x-\bar{x}, \alpha - f(\bar{x}))\rangle \quad ((x,\alpha) \in \operatorname{epi} f)\\
&\iff (v,-1) \in N_{\operatorname{epi} f}(\bar{x}, f(\bar{x})).
\end{aligned}
$$

□

There is a tight connection between subdifferentiation and conjugation of convex functions.

Theorem 3.86 (Subdifferential and conjugate function)**.** *Let $f : \mathbb{E} \to \mathbb{R}$. Then the following are equivalent:*

i) $y \in \partial f(x)$*;*

ii) $x \in \operatorname{argmax}_z \{\langle z,\, y\rangle - f(z)\}$*;*

iii) $f(x) + f^*(y) = \langle x,\, y\rangle$*;*

If $f \in \Gamma_0$ these are also equivalent to:

iv) $x \in \partial f^*(y)$*;*

v) $y \in \operatorname{argmax}_w \{\langle x,\, w\rangle - f^*(w)\}$*.*

Proof. Notice that

$$
\begin{aligned}
y \in \partial f(x) &\iff f(z) \geq f(x) + \langle y,\, z - x\rangle \quad (z \in \mathbb{E})\\
&\iff \langle y,\, x\rangle - f(x) \geq \sup_z \{\langle y,\, z\rangle - f(z)\}\\
&\iff f(x) + f^*(y) \leq \langle x,\, y\rangle\\
&\iff f(x) + f^*(y) = \langle x,\, y\rangle\,,
\end{aligned}
$$

where the last equality makes use of the Fenchel-Young inequality (3.19). This establishes the equivalences between i), ii) and iii). Applying the same reasoning to f^* and noticing that $f^{**} = f$ if $f \in \Gamma_0$, gives the missing equivalences. □

One consequence of Theorem 3.86 is that the set-valued mappings ∂f and ∂f^* are inverse to each other. We notice some other interesting implications of the latter theorem combined with Proposition 3.66.

Corollary 3.87. *Let $C \subset \mathbb{E}$. Then the following hold:*

a) For $x \in \operatorname{dom} \sigma_C$, we have $\partial \sigma_C(x) = \{v \in C \mid x \in N_{\overline{\operatorname{conv}} C}(v)\}$.

b) If C is a closed, convex cone the following are equivalent:

i) $y \in \partial \delta_C(x)$*;*

ii) $x \in \partial \delta_{C^\circ}(y)$*;*

iii) $x \in C, \quad y \in C^\circ \quad$ *and* $\quad \langle x,\, y\rangle = 0.$

Smoothness properties of convex functions Although we are not primarily interested in studying the smoothness properties of convex functions we still want to mention the following important facts which can be gathered from the extensive study in [24, §25], and which will be sporadically used later on.

Theorem 3.88 (Differentiability of convex functions)**.** *Let $f \in \Gamma$ and let $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$. Then the following are equivalent:*

i) $\partial f(\bar{x})$ is a singleton;

ii) f is differentiable at $\bar{x}$;

iii) f is continuously differentiable at $\bar{x}$.

In either case, $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$.

Exercises for Section 3.7

3.7.1 **(Subdifferential of Euclidean norm)** Show that

$$\partial\|\cdot\|(\bar{x}) = \begin{cases} \frac{\bar{x}}{\|\bar{x}\|} & \text{if} \quad \bar{x} \neq 0, \\ \operatorname{cl}\mathbb{B} & \text{if} \quad \bar{x} = 0. \end{cases}$$

3.7.2 **(Empty subdifferential)** Consider

$$f : x \in \mathbb{R} \mapsto \begin{cases} -\sqrt{x} & \text{if} \quad x \geq 0, \\ +\infty & \text{else.} \end{cases}$$

Show that $\partial f(x) = \emptyset$ for all $x \leq 0$.

3.8 Infimal convolution and the Attouch-Brézis Theorem

In this section we study a special instance of infimal projection in the spirit of Theorem 3.53 in the following form.

Definition 3.89 (Infimal convolution)**.** *Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. Then the function*

$$f\#g : \mathbb{E} \to \overline{\mathbb{R}}, \quad (f\#g)(x) := \inf_{u \in \mathbb{E}} \{f(u) + g(x-u)\}$$

is called the infimal convolution *of f and g. We call the infimal convolution $f\#g$* exact at $x \in \mathbb{E}$ *if*

$$\operatorname{argmin}_{u\in\mathbb{E}}\{f(u) + g(x-u)\} \neq \emptyset.$$

We simply call $f\#g$ exact if it is exact at every $x \in \operatorname{dom} f\#g$.

Observe that we have the representation

$$(f\#g)(x) = \inf_{u_1,u_2 : u_1+u_2=x} \{f(u_1) + g(u_2)\}. \tag{3.28}$$

This has some obvious, yet useful consequences.

Lemma 3.90. *Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. Then the following hold:*

a) $\operatorname{dom} f \# g = \operatorname{dom} f + \operatorname{dom} g$*;*

b) $f \# g = g \# f$*.*

Moreover, observe the trivial inequality

$$(f \# g)(x) \leq f(u) + g(x - u) \quad (u \in \mathbb{E}). \tag{3.29}$$

Infimal convolution preserves convexity, as can be seen in the next result.

Proposition 3.91 (Infimal convolution of convex functions)**.** *Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be convex. Then $f \# g$ is convex.*

Proof. Defining

$$\psi : \mathbb{E} \times \mathbb{E} \to \mathbb{R} \cup \{+\infty\}, \quad \psi(x, y) := f(y) + g(x - y),$$

we see that ψ is convex (jointly in (x, y)) as a sum of the convex functions $(x, y) \mapsto f(y)$ and $(x, y) \mapsto g(x - y)$, the latter being convex by Proposition 3.51. By definition of the infimal convolution, we have

$$(f \# g)(x) = \inf_{y \in \mathbb{E}} \psi(x, y),$$

hence, Theorem 3.53 yields the assertion.

□

We continue with an important class of functions that can be constructed using infimal convolution, and that is intimately tied to projection mappings.

Example 3.92 (Distance functions)**.** *Let $C \subset \mathbb{E}$. Then the function $d_C := \delta_C \# \|\cdot\|$ is called the* distance (function) *to the set C. It holds that*

$$d_C(x) = \inf_{u \in C} \|x - u\|.$$

Hence, it is clear that, if $C \subset \mathbb{E}$ is (nonempty) closed and convex, we have

$$d_C(x) = \|x - P_C(x)\|.$$

Moreover, Proposition 3.91 tells us that the distance function of a closed set is convex.

Proposition 3.93 (Conjugacy of inf-convolution)**.** *Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. Then the following hold:*

a) $(f \# g)^* = f^* + g^*$*;*

b) If $f, g \in \Gamma_0$ such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ *then* $(f + g)^* = \operatorname{cl}(f^* \# g^*)$*.*

Proof. a) By definition, for all $y \in \mathbb{E}$, we have

$$\begin{aligned}(f\#g)^*(y) &= \sup_x \left\{\langle x,\, y\rangle - \inf_u \{f(u) + g(x-u)\}\right\}\\ &= \sup_{x,u} \{\langle x,\, y\rangle - f(u) - g(x-u)\}\\ &= \sup_{x,u} \{(\langle u,\, y\rangle - f(u)) + (\langle x-u,\, y\rangle - g(x-u))\}\\ &= f^*(y) + g^*(y).\end{aligned}$$

b) From a) and the fact that f, g are closed, proper convex, we have

$$(f^*\#g^*)^* = f^{**} + g^{**} = f + g,$$

which is proper, as $\operatorname{dom} f$ meets $\operatorname{dom} g$, closed and convex. Thus,

$$\overline{\operatorname{conv}}\,(f^*\#g^*) = (f^*\#g^*)^{**} = (f+g)^*.$$

By Proposition 3.91 the convex hull on the left can be omitted, hence $\operatorname{cl}(f^*\#g^*) = (f+g)^*$. □

The closure operation in Proposition 3.93 can be omitted under the qualification condition

$$\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) \neq 0. \tag{3.30}$$

This in fact is the statement of the following prominent theorem which in the infinite dimensional setting is attributed to Attouch and Brézis [1], and we reflect that in calling it the *Attouch-Brézis Theorem* even though in the finite dimensional setting it was well established before.

Theorem 3.94 (Attouch-Brézis)**.** *Let $f, g \in \Gamma_0$ such that* (3.30) *holds. Then $(f+g)^* = f^*\#g^*$, and the infimal convolution is exact, i.e. the infimum in the infimal convolution is attained on* $\operatorname{dom} f^*\#g^*$.

Proof. By Proposition 2.5 it suffices to show that $f^*\#g^*$ has closed level to sets. To this end let $r \in \mathbb{R}$ and $\{z_k \in \operatorname{lev}_{\leq r} f^*\#g^*\} \to \bar{z}$. Given $\{\delta_k\} \downarrow 0$ there hence exists $\{(x_k, y_k) \in \mathbb{E} \times \mathbb{E}\}$ such that

$$f^*(x_k) + g^*(y_k) \leq r + \delta_k \quad \text{and} \quad L^*(x_k, y_k) = z_k \quad (k \in \mathbb{N}),$$

where $L : x \in \mathbb{E} \mapsto (x, x) \in \mathbb{E}\times\mathbb{E}$. Now let (p_k, q_k) be the orthogonal projection of (x_k, y_k) onto the linear space $V := \operatorname{span}(\operatorname{dom} f \times \operatorname{dom} g) - \operatorname{rge} L$. Then by Exercise 3.5.6 (applied to $\phi = f \oplus g$, see Exercise 3.5.4), we have $f^*(p_k) + g^*(q_k) = f^*(x_k) + g^*(y_k)$. Moreover, $V^\perp = \operatorname{span}(\operatorname{dom} f \times \operatorname{dom} g)^\perp \cap \ker L^* \subset \ker L^*$, hence $L^*(p_k, q_k) = z_k$ for all $k \in \mathbb{N}$. All in all, we find that

$$f^*(p_k) + g^*(q_k) \leq r + \delta_k \quad \text{and} \quad p_k + q_k = z_k \quad (k \in \mathbb{N}). \tag{3.31}$$

If $\{(p_k, q_k)\}$ is bounded we can assume w.l.o.g. that $\{(p_k, q_k)\} \to (\bar{p}, \bar{q})$. Passing to the limit in (3.31) hence yields

$$f^*(\bar{p}) + g^*(\bar{q}) \leq \liminf_{k\to\infty} f^*(p_k) + g^*(q_k) \leq r \quad \text{and} \quad \bar{p} + \bar{q} = \bar{z}.$$

Therefore $\bar{z} \in \mathrm{lev}_{\leq r} f^* \# g^*$, which shows that the latter is closed, and hence $f^* \# g^*$ is lsc. Moreover with $r := (f^* \# g^*)(\bar{z})$ it holds that

$$f^*(\bar{p}) + g^*(\bar{z} - \bar{p}) = \inf_{p\in\mathbb{E}} \{f^*(p) + g^*(\bar{z} - p)\} = (f^* \# g^*)(\bar{z}),$$

which gives exactness.

We now show that $\{(p_k, q_k)\}$ is, in fact, bounded: By our assumption that $\mathrm{ri}\,(\mathrm{dom}\, f) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset$ and that $L : x \mapsto (x, x)$ we have that $(0, 0) \in \mathrm{ri}\,(\mathrm{dom}\, f \times \mathrm{dom}\, g) - \mathrm{rge}\, L = \mathrm{ri}\,(\mathrm{dom}\, f \times \mathrm{dom}\, g - \mathrm{rge}\, L)$. Hence $V = \mathrm{aff}\,(\mathrm{dom}\, f \times \mathrm{dom}\, g - \mathrm{rge}\, L)$ and so there exists $\varepsilon > 0$ such that $B_\varepsilon(0, 0) \cap V \subset \mathrm{dom}\, f \times \mathrm{dom}\, g - \mathrm{rge}\, L$. Hence for any $(r, s) \in B_\varepsilon(0, 0) \cap V$ there exist $(u, v) \in \mathrm{dom}\, f \times \mathrm{dom}\, g$ and $x \in \mathbb{E}$ such that $(r, s) = (u - x, v - x)$. Therefore, for all $k \in \mathbb{N}$, we have

$$\begin{aligned}
\langle (p_k, q_k),\, (r, s)\rangle &= \langle (p_k, q_k),\, (u - x, v - x)\rangle \\
&= \langle (p_k, q_k),\, (u, v)\rangle - \langle (p_k, q_k),\, (x, x)\rangle \\
&\leq f^*(p_k) + g^*(q_k) + f(u) + g(v) - \langle p_k + q_k,\, x\rangle \\
&\leq r + \delta_k + f(u) + g(v) + \langle z_k,\, x\rangle\,,
\end{aligned}$$

where the first inequality is simply the due to the Fenchel-Young inequality (3.19), and the second one follows from (3.31). Since $\{\delta_k\} \downarrow 0$, $(u, v) \in \mathrm{dom}\, f \times \mathrm{dom}\, g$ and $\{z_k\}$ converges, and is hence bounded, the last expression is bounded from above. Therefore

$$\sup_{k\in\mathbb{N}} \langle (p_k, q_k),\, (r, s)\rangle < +\infty \quad ((r, s) \in V).$$

by scaling (r, s) accordingly. This readily implies that $\{(p_k, q_k) \in V\}$ is bounded. □

As a consequence we can study the conjugacy of a composition $g \circ L$ where g is closed (proper, convex). We also bring to mind Proposition 3.52 in this context.

Proposition 3.95. *Let $g : \mathbb{E} \to \overline{\mathbb{R}}$ be proper and $L \in \mathcal{L}(\mathbb{E}, \mathbb{E}')$ and $T \in \mathcal{L}(\mathbb{E}', \mathbb{E})$. Then the following hold:*

a) $(Lg)^* = g^* \circ L^*$.

b) $(g \circ T)^* = \mathrm{cl}\,(T^* g^*)$ *if* $g \in \Gamma$.

c) The closure in b) can be dropped and the infimum is attained when finite if $g \in \Gamma_0$ and

$$\mathrm{ri}\,(\mathrm{rge}\, T) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset. \tag{3.32}$$

Proof. a) For $y \in \mathbb{E}'$ we have

$$\begin{aligned}
(Lg)^*(y) &= \sup_{z\in\mathbb{E}'}\left\{\langle z,\, y\rangle - \inf_{x:\, L(x)=z} g(x)\right\} \\
&= \sup_{z\in\mathbb{E}',\, x\in L^{-1}(\{z\})} \{\langle z,\, y\rangle - g(x)\} \\
&= \sup_{x\in\mathbb{E}} \{\langle x,\, L^*(y)\rangle - g(x)\} \\
&= g^*(L^*(y)).
\end{aligned}$$

b) Follows from a) and the Fenchel-Moreau Theorem.
c) With $\phi : (x, y) \in \mathbb{E} \times \mathbb{E}' \mapsto g(y)$ we have

$$\begin{aligned}
(g\circ T)^*(z) &= \sup_{x\in\mathbb{E}}\{\langle z,\, x\rangle - g(T(x))\} \\
&= \sup_{(x,y)\in\operatorname{gph} T}\{\langle z,\, x\rangle - g(y)\} \\
&= \sup_{(x,y)\in\mathbb{E}\times\mathbb{E}'}\{\langle (z,0),\, (x,y)\rangle - (\delta_{\operatorname{gph} T} + \phi)(x,y)\} \\
&= (\delta_{\operatorname{gph} T} + \phi)^*(z,0).
\end{aligned}$$

We now observe that

$$\operatorname{ri}(\delta_{\operatorname{gph} T}) \cap \operatorname{ri}(\operatorname{dom}\phi) = \operatorname{ri}(\operatorname{gph} T) \cap \mathbb{E} \times \operatorname{ri}(\operatorname{dom} g).$$

Therefore, (3.30) for $\delta_{\operatorname{gph} T}$ and ϕ is satisfied if (and only if) (3.32) holds. We can hence invoke Attouch-Brézis to find that

$$\begin{aligned}
(g\circ T)^*(z) &= (\sigma_{\operatorname{gph} T}\#\phi^*)(z,0) \\
&= \inf_{(u,v)}\{\sigma_{\operatorname{gph} T}(u,v) + \delta_{\{0\}}(z-u) + g^*(-v)\} \\
&= \inf_{v} \delta_{\{0\}}(T^*(v) + z) + g^*(-v) \\
&= (T^* g^*)(z),
\end{aligned}$$

where the infimum is a minimum when finite. □

3.9 Subdifferential calculus and Fenchel-Rockafellar duality

In this section we develop a calculus for the convex subdifferential. Our primary goal is to establish a subdifferential rule for convex functions of the form $f + g\circ L$ for $f \in \Gamma(\mathbb{E}_1), g \in \Gamma(\mathbb{E}_2)$ and $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. There are various different ways to the subdifferential of that kind of convex function. We choose the path via Fenchel-Rockafellar duality which we can prove using Attouch-Brézis.

The central qualification condition is

$$0 \in \operatorname{ri}(\operatorname{dom} g - L(\operatorname{dom} f)) \tag{3.33}$$

or equivalently

$$\operatorname{ri}(L^{-1}\operatorname{dom} g) \cap \operatorname{ri}(\operatorname{dom} f) \neq \emptyset$$

which is simply (3.30) applied to f and $g\circ L$.

Theorem 3.96 (Fenchel-Rockafellar duality). *Let $f : \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}$, $g : \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ and $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. Then the following hold:*

a) (Weak duality) We have

$$p := \inf_{x\in\mathbb{E}_1} \{f(x) + g(L(x))\} \geq \sup_{y\in\mathbb{E}_2}\{-f^*(L^*(y)) - g^*(-y)\} =: d. \tag{3.34}$$

b) (Strong duality) If $f \in \Gamma_0(\mathbb{E}_1)$ and $g \in \Gamma_0(\mathbb{E}_2)$ (3.33) *holds then equality holds in* (3.34) *and the supremum is attained if finite.*

c) (Primal-dual recovery) If $f \in \Gamma_0(\mathbb{E}_1)$ and $g \in \Gamma_0(\mathbb{E}_2)$ the following are equivalent for $\bar{x} \in \mathbb{E}_1$ and $\bar{y} \in \mathbb{E}_2$:

i) $p = d$, $\bar{x} \in \mathrm{argmin} f(x) + g(L(x))$, $\bar{y} \in \mathrm{argmax} - f^*(L^*(y)) - g^*(-y)$;

ii) $L^*(\bar{y}) \in \partial f(\bar{x}), \quad -\bar{y} \in \partial g(L(\bar{x}))$;

iii) $\bar{x} \in \partial f^*(L^*(\bar{y})), \quad L(\bar{x}) \in \partial g^*(-\bar{y})$.

Proof. a) Follows immediately from the Fenchel-Young inequality.
b) By Attouch-Brézis (recall the remark about the qualification conditions above) and Proposition 3.95 c) we find that

$$\begin{aligned}
\inf_{x\in\mathbb{E}_1} \{f(x) + g(L(x))\} &= (f + g \circ L)^*(0) \\
&= \inf_{z\in\mathbb{E}_1} \{f^*(z) + (L^*g^*)(-z)\} \\
&= \inf_{y\in\mathbb{E}_2} \{f^*(L(y)) + g^*(-y)\}
\end{aligned}$$

where the second (and hence third) infimum is attained if finite. That shows the desired statement.
c) 'i)$\Leftrightarrow$ii)': Observe that

$$\begin{aligned}
ii) \iff\ & f(\bar{x}) + f^*(L^*(\bar{y})) = \langle \bar{x},\, L^*(\bar{y})\rangle\,,\ g(L(\bar{x})) + g^*(-\bar{y}) = \langle L(\bar{x}),\, -\bar{y}\rangle \\
\iff\ & f(\bar{x}) + f^*(L^*(\bar{y})) + g(L(\bar{x})) + g^*(-\bar{y}) = 0 \\
\iff\ & f(\bar{x}) + g(L(\bar{x})) = -f^*(L^*(\bar{y})) - g^*(-\bar{y}) \\
\iff\ & i).
\end{aligned}$$

Here the first equivalence is due to Theorem 3.86. In the second equivalence the '$\Leftarrow$'-direction uses the Fenchel-Young inequality. The third one is obvious, while the last equivalence uses a).

'ii)$\Leftrightarrow$iii)': Follows immediately from Theorem 3.86.

□

There is an abundance of applications of Fenchel-Rockafellar duality, but at this point in our study, it is primarily a tool for proving a generalized sum rule for the convex subdifferential.

Theorem 3.97 (Generalized sum rule). *Let $f:\mathbb{E}_1\to\overline{\mathbb{R}}$, $g:\mathbb{E}_2\to\overline{\mathbb{R}}$ and $L\in\mathcal{L}(\mathbb{E}_1,\mathbb{E}_2)$. Then the following hold:*

a) $\partial(f+g\circ L)(\bar{x})\supset\partial f(\bar{x})+L^*\left[\partial g(L(\bar{x}))\right]\quad(\bar{x}\in\mathbb{E}_1)$.

b) If $f\in\Gamma_0(\mathbb{E}_1)$ and $g\in\Gamma_0(\mathbb{E}_2)$ and condition (3.33) *is satisfied, equality holds in a).*

Proof. a) Let $v\in\partial f(\bar{x})+L^*\left[\partial g(L(\bar{x}))\right]$, i.e. $v=u+L^*(w)$ where $u\in\partial f(\bar{x})$ and $w\in\partial g(L(\bar{x}))$. By the respective subgradient inequalities we have

$$\langle u,\,x-\bar{x}\rangle+f(\bar{x})\le f(x)\quad(x\in\mathbb{E}_1)\tag{3.35}$$

and

$$\langle w,\,L(x)-L(\bar{x})\rangle+g(L(\bar{x}))\le g(L(x))\quad(x\in\mathbb{E}_1).$$

The latter is equivalent to

$$\langle L^*(w),\,x-\bar{x}\rangle+(g\circ L)(\bar{x})\le(g\circ L)(x)\quad(x\in\mathbb{E}_1).$$

Adding this to (3.35) yields

$$\langle v,\,x-\bar{x}\rangle+f(\bar{x})+(g\circ L)(\bar{x})\le f(x)+(g\circ L)(x)\quad(x\in\mathbb{E}_1),$$

which shows that $v\in\partial(f+g\circ L)(\bar{x})$.
b) Let $v\in\partial(f+g\circ L)(\bar{x})$. Observe (cf. Theorem 3.86) that $\bar{x}$ solves

$$\inf_{x\in\mathbb{E}_1}\{(f(x)-\langle v,\,x\rangle)+g(L(x))\}.$$

By Fenchel-Rockafellar duality (applied to $f-\langle v,\,\cdot\rangle$ and g noticing that $\operatorname{dom} f=\operatorname{dom}(f-\langle v,\,\cdot\rangle)$) we therefore infer that there exists $\bar{y}\in\mathbb{E}_2$ such that

$$f(\bar{x})-\langle v,\,\bar{x}\rangle+g(L(\bar{x}))=-f^*(L^*(\bar{y})+v)-g^*(-\bar{y}),$$

or equivalently

$$\langle v,\,\bar{x}\rangle=f(\bar{x})+g(L(\bar{x}))+f^*(L^*(\bar{y})+v)+g^*(-\bar{y}).$$

Hence,

$$\langle v+L^*(\bar{y}),\,\bar{x}\rangle+\langle-\bar{y},\,L(\bar{x})\rangle=f(\bar{x})+f^*(L^*(\bar{y})+v)+g(L(\bar{x}))+g^*(-\bar{y}).$$

By the Fenchel-Young inequality we hence must have

$$L^*(\bar{y})+v\in\partial f(\bar{x})\quad\text{and}\quad-\bar{y}\in\partial g(L(\bar{x})).$$

Therefore

$$v=v+L^*(\bar{y})+L^*(-\bar{y})\in\partial f(\bar{x})+L^*\partial g(L(\bar{x})),$$

which concludes the proof. □

The generalized subdifferential sum rule has many important consequences, two of which we present now: The first one is a sum rule, which in the literature is sometimes referred to as *Moreau-Rockafellar Theorem*, but this is moniker is not used uniformly.

Corollary 3.98 (Subdifferential sum rule)**.** *Let $f, g \in \Gamma$ then*

$$\partial(f+g)(x) \supset \partial f(x) + \partial g(x) \quad (x \in \mathbb{E}). \tag{3.36}$$

Under the qualification condition

$$0 \in \mathrm{ri}\,(\mathrm{dom}\, g - \mathrm{dom}\, f)$$

equality holds in (3.36)*.*

Likewise we obtain a chain rule.

Corollary 3.99 (Subdifferential chain rule)**.** *Let $g \in \Gamma(\mathbb{E}_2)$ and $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. Then*

$$\partial(g \circ L) \supset L^*(\partial g) \circ L. \tag{3.37}$$

Under the qualification condition

$$0 \in \mathrm{ri}\,(\mathrm{dom}\, g - \mathrm{rge}\, L)$$

equality holds in (3.37)*.*

We continue with a subdifferential rule for infimal convolution.

Theorem 3.100 (Subdifferential of infimal convolution)**.** *Let $f, g \in \Gamma_0$ and $x \in \mathrm{dom}\,(f\#g)$. Then*

$$\partial(f\#g)(\bar{x}) = \partial f(\bar{u}) \cap \partial g(\bar{x}-\bar{u}) \quad (\bar{u} \in \mathrm{argmin}_u f(u) + g(\bar{x}-u)).$$

Proof. Let $\bar{u} \in \mathrm{argmin}_u f(u) + g(\bar{x}-u)$. Then, by Theorem 3.86, we have

$$\begin{aligned} \bar{y} \in \partial(f\#g)(\bar{x}) \quad &\iff \quad \bar{x} \in \mathrm{argmin}_x\{(f\#g)(x) - \langle x,\, \bar{y}\rangle\} \\ &\iff \quad (f\#g)(\bar{x}) - \langle \bar{x},\, \bar{y}\rangle = \inf_x\{(f\#g)(x) - \langle x,\, \bar{y}\rangle\} \\ &\iff \quad f(\bar{u}) + g(\bar{x}-\bar{u}) - \langle \bar{x},\, \bar{y}\rangle = \inf_{(x,u)}\{f(u) + g(x-u) - \langle x,\, \bar{y}\rangle\} \\ &\iff \quad (\bar{x}, \bar{u}) \in \mathrm{argmin}_{x,u}\{f(u) + g(x-u) - \langle (x,u),\, (\bar{y}, 0)\rangle\}. \end{aligned}$$

Now consider the self-adjoint linear map

$$L : (x,u) \in \mathbb{E} \times \mathbb{E} \mapsto (u, x-u)$$

and the convex function

$$c : (v,w) \in \mathbb{E} \times \mathbb{E} \mapsto g(v) + h(w).$$

Using again Theorem 3.86 and Corollary 3.99 and Exercise 3.9.1, we find that $\bar{y} \in \partial(f\#g)(\bar{x})$ is equivalent to

$$(\bar{y}, 0) \in \partial(c \circ L)(\bar{x}, \bar{u}) = L^*\partial c(\bar{u}, \bar{x}-\bar{u}) = L\,[\partial f(\bar{u}) \times \partial g(\bar{x}-\bar{u}))].$$

This yields the desired result. □

Exercises for Section 3.9

3.9.1 **(Subdifferential of separable sum)** For $f_i : \mathbb{E}_i \to \overline{\mathbb{R}}$ $(i = 1, 2)$ the *separable sum of f_1 and f_2* is defined by

$$f_1 \oplus f_2 : (x_1, x_2) \in \mathbb{E}_1 \times \mathbb{E}_2 \mapsto f_1(x_1) + f_2(x_2).$$

Show that for $f_i \in \Gamma(\mathbb{E}_i)$ $(i = 1, 2)$ we have

$$\partial(f_1 \oplus f_2) = \partial f_1 \times \partial f_2.$$

3.10 Infimal projection

We would like to revisit the infimal projection setting from 3.53.

Theorem 3.101 (Infimal projection II)**.** *Let $\psi \in \Gamma_0(\mathbb{E}_1 \times \mathbb{E}_2)$ and define $p : \mathbb{E}_1 \to \overline{\mathbb{R}}$ by*

$$p(x) := \inf_v \psi(x, v). \tag{3.38}$$

Then the following hold:

a) p is convex.

b) $p^* = \psi^*(\cdot, 0)$ which is closed and convex.

c) The condition

$$\operatorname{dom} \psi^*(\cdot, 0) \neq 0 \tag{3.39}$$

is equivalent to having $p^* \in \Gamma_0$.

d) If (3.39) holds then $p \in \Gamma_0$ and the infimum in its definition is attained when finite.

Proof. a) Theorem 3.53.
b) We have

$$\begin{aligned}
p^*(y) &= \sup_x \{\langle y,\, x\rangle - \inf_v \psi(x, v)\} \\
&= \sup_{(x,v)} \{\langle (y, 0),\, (x, v)\rangle - \psi(x, v)\} \\
&= \psi^*(y, 0).
\end{aligned}$$

A conjugate function is always closed and convex as a pointwise supremum of affine, hence convex functions, cf. Proposition 3.50.
c) The is obvious from b).
d) Observe that by b) we have

$$\begin{aligned}
p^{**}(x) &= \sup_y \{\langle x,\, y\rangle - \psi^*(y, 0)\} \\
&= \sup_{(y,w)} \{\langle (x, 0),\, (y, w)\rangle - (\psi^* + \delta_{\mathbb{E}_1 \times \{0\}})(y, w)\} \\
&= (\psi^* + \delta_{\mathbb{E}_1 \times \{0\}})^*(x, 0).
\end{aligned}$$

Now we notice that

$$\mathrm{ri}\,(\mathrm{dom}\,\psi^*) \cap \mathrm{ri}\,(\mathrm{dom}\,\delta_{\mathbb{E}_1\times\{0\}}) = \mathrm{ri}\,(\mathrm{dom}\,\psi^*) \cap \mathbb{E}_1 \times \{0\}$$

see Exercise 3.1.8. Hence

$$\mathrm{ri}\,(\mathrm{dom}\,\psi^*) \cap \mathrm{ri}\,(\mathrm{dom}\,\delta_{\mathbb{E}_1\times\{0\}}) \neq \emptyset \quad \Longleftrightarrow \quad \mathrm{dom}\,\psi^*(\cdot, 0) \neq \emptyset$$

as the relative interior of a convex set is nonempty if and only if the set itself is not, see Theorem 3.22. Since we assume that the latter holds, we can hence apply the Attouch-Brézis Theorem to continue the above derivations and obtain

$$\begin{aligned} p^{**}(x) &= (\psi^* + \delta_{\mathbb{E}_1\times\{0\}})^*(x, 0) \\ &= (\psi \# \delta_{\{0\}\times\mathbb{E}_2})(x, 0) \\ &= \inf_v \psi(x, v) \\ &= p(x), \end{aligned}$$

and the infimum is attained when finite. □

4 Conjugacy of Composite Functions via K-Convexity and Infimal Convolution

In this section we are concerned with functions $g \circ F$, where g is closed, proper, convex and F is a (generally) nonlinear map. These kinds of functions are known under the moniker *convex-composite function*, and are, in general, nonconvex. However, if the monotonicity properties of g and the generalized convexity properties of F align in a certain way, this convex-composite may be convex after all. We want to study this case with a focus on the Fenchel conjugate of these kinds of (convex) convex-composite functions. Our main tools are infimal convolution, especially the Attouch-Brézis Theorem (see Theorem 3.94) and a notion of generalized convexity for vector-valued functions with respect to some cone-induced ordering.

4.1 K-convexity

Given a cone $K \subset \mathbb{E}$, the relation

$$x \leq_K y \quad :\Longleftrightarrow \quad y - x \in K \quad (x, y \in \mathbb{E})$$

induces a partial ordering on $\mathbb{E}$. We may then attach to $\mathbb{E}$ a *largest element* $+\infty_\bullet$ with respect to that ordering which satisfies

$$x \leq_K +\infty_\bullet \quad (x \in \mathbb{E}).$$

We will set $\mathbb{E}^\bullet := \mathbb{E} \cup \{+\infty_\bullet\}$. For a function $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ we define its *domain*, *graph* and *range*, respectively, as

$$\begin{aligned} \mathrm{dom}\, F &:= \{x \in \mathbb{E}_1 \mid F(x) \in \mathbb{E}_2\}, \\ \mathrm{gph}\, F &:= \{(x, F(x)) \in \mathbb{E}_1 \times \mathbb{E}_2 \mid x \in \mathrm{dom}\, F\}, \\ \mathrm{rge}\, F &:= \{F(x) \in \mathbb{E}_2 \mid x \in \mathrm{dom}\, F\}. \end{aligned}$$

We call F *proper* if $\operatorname{dom} F \neq \emptyset$. The following concept is central to our study.

Definition 4.1 (K-convexity)**.** *Let $K \subset \mathbb{E}_2$ be a cone and $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$. Then we call F* convex with respect to K *or* K-convex *if its K-epigraph*

$$K\text{-epi}\, F := \{(x, v) \in \mathbb{E}_1 \times \mathbb{E}_2 \mid F(x) \leq_K v\}$$

is convex (in $\mathbb{E}_1 \times \mathbb{E}_2$).

We point out that, in the setting of Definition 4.1, a K-convex function F has a convex domain as $\operatorname{dom} F = L(K\text{-epi}\, F)$ where $L \colon \mathbb{E}_1 \times \mathbb{E}_2 \to \mathbb{E}_1 \colon (x, v) \mapsto x$. Thus, F is K-convex if and only if

$$F(\lambda x + (1-\lambda)y) \leq_K \lambda F(x) + (1-\lambda)F(y) \quad (x, y \in \operatorname{dom} F,\ \lambda \in [0, 1]).$$

We will make use of the relative interior of the K-epigraph of a K-convex function which is described in the following lemma, the proof on which is based on results from Section 3.1.4.

Lemma 4.2 (Relative interior of K-epigraph)**.** *Let $K \subset \mathbb{E}_2$ be a convex cone, and let $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper and K-convex. Then*

$$\begin{aligned} \operatorname{ri}(K\text{-epi}\, F) &= \big\{(x, v) \bigm| x \in \operatorname{ri}(\operatorname{dom} F),\ F(x) \preceq_{\operatorname{ri}(K)} v\big\} \\ &= \big(\operatorname{ri} K\text{-epi}\, F\big) \bigcap \big(\operatorname{dom} F \times \mathbb{E}_2\big). \end{aligned}$$

In particular, if $\operatorname{dom} F = \mathbb{E}_1$, *then* $\operatorname{ri}(K\text{-epi}\, F) = \operatorname{ri} K\text{-epi}\, F$.

Proof. For each $x \in \mathbb{E}_1$, set $C_x = \{v \in \mathbb{E}_2 \mid (x, v) \in K\text{-epi}\, F\}$. It follows from Proposition 3.28 that

$$\operatorname{ri} C_x = \operatorname{ri}\big(L_x^{-1}(K)\big) = L_x^{-1}(\operatorname{ri} K) = \{v \in \mathbb{E}_2 \mid v \in F(x) + \operatorname{ri} K\},$$

where $L_x \colon v \mapsto v - F(x)$. Hence, by Theorem 3.29, we find that

$$\begin{aligned} (x, v) \in \operatorname{ri}(K\text{-epi}\, F) &\iff x \in \operatorname{ri}\ \{x \in \mathbb{E}_1 \mid C_x \neq \emptyset\} \text{ and } v \in \operatorname{ri} C_x \\ &\iff x \in \operatorname{ri}(\operatorname{dom} F) \text{ and } v \in \operatorname{ri} C_x \\ &\iff x \in \operatorname{ri}(\operatorname{dom} F) \text{ and } v \in F(x) + \operatorname{ri} K. \end{aligned}$$

□

It is clear from the definition that if F is K-convex and $L \supset K$ is a cone, then F is also L-convex. The following result due to Pennanen [23, Lemma 6.1] gives a nice description of the polar of smallest closed, convex cone with respect to which a given function is convex.

Lemma 4.3. *Let $f \colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ with a convex domain and let $K \subset \mathbb{E}_2$ be the smallest closed convex cone with respect to which F is convex. Then*

$$(-K)^\circ = \{v \in \mathbb{E}_2 \mid \langle v,\ F\rangle \ \textit{is convex}\} =: P_F.$$

Proof. First, we observe the following: As F is K-convex, for $\lambda \in (0,1)$ and $x, y \in \operatorname{dom} F$, we have

$$\begin{aligned} & \lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y) \in K \\ \iff\; & F(\lambda x + (1-\lambda)y) - \lambda F(x) - (1-\lambda)F(y) \in -K \\ \iff\; & \langle v,\, F(\lambda x + (1-\lambda)y) - \lambda F(x) - (1-\lambda)F(y)\rangle \leq 0 \ (v \in (-K)^\circ) \\ \iff\; & \langle v,\, F\rangle \ \text{ is convex } (v \in (-K)^\circ) \end{aligned}$$

where the third equivalence uses that $K = K^{\circ\circ}$ as K is a closed, convex cone, see Theorem 3.73. It hence, implies that

$$(-K)^\circ \subset P_F. \tag{4.1}$$

Next, it is clear that P_F is a convex cone, and, due to the continuity of the inner product, P_F is closed. By the above observation F is $-P_F^\circ$-convex and therefore $K \subset -P_F^\circ$, or equivalently,

$$-K \subset P_F^\circ. \tag{4.2}$$

Finally, by taking the polars in (4.2), using Theorem 3.73 and (4.1), we get

$$P_F = (P_F^\circ)^\circ \subset (-K)^\circ \subset P_F.$$

□

4.2 The convex-composite setting

For $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ and $g : \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ we define the composition of g and F as

$$(g \circ F)(x) := \begin{cases} g(F(x)) & \text{if} \quad x \in \operatorname{dom} F, \\ +\infty & \text{else} \end{cases}$$

which extends the usual convention $g(+\infty) = +\infty$ when $\mathbb{E}_2 = \mathbb{R}$. In particular, given $v \in \mathbb{E}_2$ and the linear form $\langle v,\, \cdot\rangle : \mathbb{E}_2 \to \mathbb{R}$ that goes with it, we set $\langle v,\, F\rangle := \langle v,\, \cdot\rangle \circ F$, i.e.

$$\langle v,\, F\rangle\,(x) = \begin{cases} \langle v,\, F(x)\rangle & \text{if} \quad x \in \operatorname{dom} F, \\ +\infty & \text{else.} \end{cases}$$

This scalarization of F is quite central to our study which is already foreshadowed in the next two auxiliary results.

Lemma 4.4. *Let $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ and $v \in \mathbb{E}_2$. Then the following hold:*

a) If F is K-convex then $\langle v,\, F\rangle$ is convex for all $v \in -K^\circ$.

b) The inverse of a) holds true if K is closed and convex.

Proof. a) Suppose that F is K-convex and let $v \in -K^\circ$, $x, y \in \mathbb{E}_1$, and $\lambda \in [0,1]$. Since

$$\lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y) \in K,$$

it follows that

$$\langle v,\, \lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y)\rangle \geq 0,$$

and hence,

$$\langle v,\, \lambda F(x) + (1-\lambda)F(y)\rangle \geq \langle v,\, F(\lambda x + (1-\lambda)y)\rangle .$$

b) Let $x, y \in \mathbb{E}_1$, $\lambda \in [0,1]$, and $v \in -K^\circ$. Set $z = \lambda x + (1-\lambda)y$. Since $\langle v,\, F\rangle$ is convex we have

$$\begin{aligned}\langle v,\, \lambda F(x) + (1-\lambda)F(y) - F(z)\rangle &= \lambda \langle v,\, F(x)\rangle + (1-\lambda)\langle v,\, F(y)\rangle - \langle v,\, F(z)\rangle \\ &= \lambda \langle v,\, F\rangle (x) + (1-\lambda)\langle v,\, F\rangle (y) - \langle v,\, F\rangle (\lambda x + (1-\lambda)y) \\ &\geq 0,\end{aligned}$$

and therefore,

$$\lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y) \in -(-K^\circ)^\circ = K^{\circ\circ} = K,$$

where $K^{\circ\circ} = K$ because K is closed and convex (and contains 0), see Theorem 3.73.

□

Lemma 4.5. *Let $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ and let $K \subset \mathbb{E}_2$ be a closed, convex cone. Then the following hold for all $(u, v) \in \mathbb{E}_1 \times \mathbb{E}_2$:*

a) $\sigma_{K\text{-epi}\, F}(u, v) = \sigma_{\mathrm{gph}\, F}(u, v) + \delta_{K^\circ}(v)$.

b) $\sigma_{\mathrm{gph}\, F}(u, -v) = \langle v,\, F\rangle^* (u)$.

c) If F is linear then $\langle v,\, F\rangle^* = \delta_{\{F^*(v)\}}$.

Proof. a) Observe that

$$\begin{aligned}\sigma_{K\text{-epi}\, F}(u, v) &= \sup_{(x,y)\in K\text{-epi}\, F} \langle (u, v),\, (x, y)\rangle \\ &= \sup_{(x,z)\in \mathbb{E}_1\times K} \langle (u, v),\, (x, F(x) + z)\rangle \\ &= \sup_{x\in\mathbb{E}_1} \langle (u, v),\, (x, F(x))\rangle + \sup_{z\in K} \langle z,\, v\rangle \\ &= \sigma_{\mathrm{gph}\, F}(u, v) + \delta_{K^\circ}(v),\end{aligned}$$

where the last identity uses Exercise 3.5.13.

b) We have

$$\begin{aligned}\sigma_{\mathrm{gph}\, F}(u, -v) &= \sup_{x\in \mathrm{dom}\, F} \langle (u, -v),\, (x, F(x))\rangle \\ &= \sup_{x\in\mathbb{E}_1} \{\langle u,\, x\rangle - \langle v,\, F\rangle (x)\} \\ &= \langle v,\, F\rangle^* (u).\end{aligned}$$

□

Proposition 4.6. *Let $K \subset \mathbb{E}_2$ be a convex cone, $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ K-convex and $g \in \Gamma(\mathbb{E}_2)$ such that* $\operatorname{rge} F \cap \operatorname{dom} g \neq \emptyset$. *If*

$$g(F(x)) \leq g(y) \quad ((x, y) \in K\text{-epi}\, F) \tag{4.3}$$

then the following hold:

a) $g \circ F$ is convex and proper.

b) If g is lsc and F is continuous then $g \circ F$ is lower semicontinuous.

Proof. a) As g is proper and $\operatorname{rge} F \cap \operatorname{dom} g \neq \emptyset$, the composite $g \circ F$ is proper. Now, let $v, w \in \operatorname{dom} F$ and $\lambda \in [0, 1]$. Then

$$\begin{array}{rcl} g(F(\lambda v + (1-\lambda)w)) & \leq & g(\lambda F(v) + (1-\lambda)F(w)) \\ & \leq & \lambda g(F(v)) + (1-\lambda) g(F(w)). \end{array}$$

Here the first inequality is due to the fact that $(\lambda v + (1-\lambda)w, \lambda F(v) + (1-\lambda)F(w)) \in K\text{-epi}\, F$ (as F is K-convex) and assumption (4.3).

b) Let $\{x_n\}$ be a sequence in $\mathbb{E}_1$ converging to x and set $t = \liminf_{n\to+\infty} g(F(x_n))$. Then there exists a subsequence (x_{k_n}) such that $g(F(x_{k_n})) \to t$. Due to the continuity of F, $F(x_{k_n}) \to F(x)$, and hence, by the lower semicontinuity of g, we have $t \geq g(F(x))$. □

4.3 The Fenchel conjugate of the convex convex-composite

Without further ado, we present the main result of this chapter.

Theorem 4.7 (Conjugacy for convex-composite)**.** *Let $K \subset \mathbb{E}_2$ be a closed convex cone, $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ K-convex such that K-*epi F *is closed and $g_0 \in \Gamma(\mathbb{E}_2)$ such that* (4.3) *is satisfied. Then the following hold:*

a) $(g \circ F)^* \leq \operatorname{cl} \eta$, where

$$\eta(p) = \inf_{v \in -K^\circ} g^*(v) + \langle v,\, F\rangle^*(p).$$

b) If

$$F(\operatorname{ri}(\operatorname{dom} F)) \cap \operatorname{ri}(\operatorname{dom} g - K) \neq \emptyset. \tag{4.4}$$

the function η in a) is closed, proper and convex and the infimum is attained if finite.

c) Under the assumptions of b) we have

$$(g \circ F)^*(p) = \min_{v \in -K^\circ} g^*(v) + \langle v,\, F\rangle^*(p)$$

with $\operatorname{dom}(g \circ F)^* = \{p \in \mathbb{E}_1 \mid \exists v \in \operatorname{dom} g^* \cap (-K^\circ) : \langle v,\, F\rangle^*(p) < +\infty\}$.

Proof. a) Define $\phi\colon \mathbb{E}_1 \times \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\} \colon (x,y) \mapsto g(y)$ and observe that

$$\phi^*(u,v) = \delta_{\{0\}}(u) + g^*(v). \tag{4.5}$$

Hence we find that

$$\begin{aligned}
(g \circ F)^*(p) &= \sup_{x \in \mathbb{E}_1} \{\langle x,\, p\rangle - g(F(x))\} \\
&= \sup_{(x,y) \in K\text{-epi}\, F} \{\langle x,\, p\rangle - g(y)\} \\
&= \sup_{(x,y) \in \mathbb{E}_1 \times \mathbb{E}_2} \{\langle (p,0),\, (x,y)\rangle - (g(y) + \delta_{K\text{-epi}\, F}(x,y)\} \\
&= (\phi + \delta_{K\text{-epi}\, F})^*(p,0) \\
&= \mathrm{cl}\, (\phi^* \# \sigma_{K\text{-epi}\, F})\, (p,0).
\end{aligned}$$

Here the second equality uses assumption (4.3), and last identity is then due to Proposition 3.93 as the functions in play are closed, proper and convex by assumption. Moreover, we have

$$\begin{aligned}
(\phi^* \# \sigma_{K\text{-epi}\, F})\, (p,0) &= \inf_{(u,v) \in \mathbb{E}_1 \times \mathbb{E}_2} \{\phi^*(u,v) + \sigma_{K\text{-epi}\, F}(p-u,-v)\} \\
&= \inf_{v \in \mathbb{E}_2} \{g^*(v) + \sigma_{K\text{-epi}\, F}(p,-v)\} \\
&= \inf_{v \in -K^\circ} \{g^*(v) + \sigma_{\mathrm{gph}\, F}(p,-v)\} \\
&= \inf_{v \in -K^\circ} \{g^*(v) + \langle v,\, F\rangle^*(p)\},
\end{aligned}$$

where the second identity uses (4.5) and the third and fourth one rely on Lemma 4.5. This shows the desired statement.

b) This follows from Theorem 3.94 while observing that

$$\begin{aligned}
\mathrm{ri}\,(\mathrm{dom}\, \phi) \cap \mathrm{ri}\,(\mathrm{dom}\, \delta_{K\text{-epi}\, F}) \neq \emptyset &\iff \mathbb{E}_1 \times \mathrm{ri}\,(\mathrm{dom}\, g) \cap \mathrm{ri}\,(K\text{-epi}\, F) \neq \emptyset \\
&\iff \exists x \in \mathrm{ri}\,(\mathrm{dom}\, F) : \; F(x) \in \mathrm{ri}\,(\mathrm{dom}\, g) - \mathrm{ri}\, K \\
&\iff F(\mathrm{ri}\,(\mathrm{dom}\, F)) \cap \mathrm{ri}\,(\mathrm{dom}\, g - K) \neq \emptyset.
\end{aligned}$$

Here the second equivalence relies on Lemma 4.2.

c) The first statement follows from a) and b) and Theorem 3.94. The expression of $\mathrm{dom}\,(g \circ F)^*$ is an immediate consequence of that.

□

We now continue with a whole sequence of rather immediate consequences of Theorem 4.7. The first one shows that our setting $g \circ F$ in fact covers the seemingly more general setting with an additional additive term.

Corollary 4.8 (Conjugate of additive composite functions)**.** *Under the assumptions of Theorem 4.7 let $f \in \Gamma_0$ such that*

$$F(\mathrm{ri}\,(\mathrm{dom}\, f \cap \mathrm{dom}\, F)) \cap \mathrm{ri}\,(\mathrm{dom}\, g - K) \neq \emptyset. \tag{4.6}$$

Then

$$(f + g \circ F)^*(p) = \min_{\substack{v \in -K^\circ, \\ y \in \mathbb{E}_1}} g^*(v) + f^*(y) + \langle v,\, F\rangle^*(p-y).$$

Proof. Define $\tilde{g} : \mathbb{R} \times \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ by $g(s, y) := s + g(y)$. Then $g \in \Gamma_0(\mathbb{R} \times \mathbb{E}_2)$ with

$$\tilde{g}^*(t, v) = \delta_{\{1\}}(t) + g^*(v). \tag{4.7}$$

Moreover, define $\tilde{F} : \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\} \times \mathbb{E}_2^\bullet$ by $\tilde{F}(x) = (f(x), F(x))$. Then $\operatorname{dom} \tilde{F} = \operatorname{dom} f \cap \operatorname{dom} F$. Setting $\tilde{K} := \mathbb{R}_+ \times K$, we find that $\tilde{F}$ is $\tilde{K}$-convex, $\tilde{g} \circ \tilde{F} = f + g \circ F \in \Gamma_0(\mathbb{E}_1)$ and $\tilde{g} \circ \tilde{F}$ satisfies (4.3) with $\tilde{K}$ (since $g \circ F$ satisfies it with K). Moreover, as $\operatorname{dom} \tilde{g} = \mathbb{R} \times \operatorname{dom} g$, we realize that condition (4.4) for $\tilde{g}, \tilde{F}$ and $\tilde{K}$ amounts to (4.6). Therefore, we obtain

$$\begin{aligned}
(f + g \circ F)^*(p) &= (\tilde{g} \circ \tilde{F})(p) \\
&= \min_{(t,v) \in -\tilde{K}^\circ} \tilde{g}^*(t, v) + \left\langle (t, v),\ \tilde{F} \right\rangle^* (p) \\
&= \min_{v \in -K^\circ} g^*(v) + \sup_{x \in \mathbb{E}_1} \{\langle p,\ v\rangle - (f(x) + \langle v,\ F\rangle (x))\} \\
&= \min_{v \in -K^\circ} g^*(v) + (f + \langle v,\ F\rangle)^*(p) \\
&= \min_{v \in -K^\circ} g^*(v) + (f^* \# \langle v,\ F\rangle^*)(p) \\
&= \min_{\substack{v \in -K^\circ, \\ y \in \mathbb{E}_1}} g^*(v) + f^*(y) + \langle v,\ F\rangle^* (p).
\end{aligned}$$

Here the second identity is due to Theorem 4.7, the third one uses (4.7), while the fifth relies once more on Theorem 3.94, realizing that $\operatorname{dom} \langle v,\ F\rangle = \operatorname{dom} F$ and (4.6) implies that $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} F) \supset \operatorname{ri}(\operatorname{dom} f \cap \operatorname{dom} F) \neq \emptyset$.

□

The next corollary follows simply from the fact that condition (4.3) is trivially satisfied if g has the following property: For a cone $K \subset \mathbb{E}$ we call $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ *K-increasing* if

$$f(u) \leq f(v) \quad (u \preceq_K v).$$

Corollary 4.9. *Let $K \subset \mathbb{E}_2$ be a closed, convex cone, $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ K-convex such that K-epi F is closed and let $g \in \Gamma_0$ and K-increasing such that (4.4) holds (which is true in particular when g is finite-valued or F is surjective). Then*

$$(g \circ F)^*(p) = \min_{v \in -K^\circ} g^*(v) + \langle v,\ F\rangle^* (p).$$

Proof. Since the fact that g is K-increasing implies (4.3) the assertion follows from Theorem 4.7. □

The next result shows that any closed, proper, convex function g is monotone with respect to its horizon cone, see Definition 3.80.

Lemma 4.10. *Let $g \in \Gamma_0$. Then g is $(-\operatorname{hzn} g)$-increasing.*

Proof. Put $K := -\operatorname{hzn} g$. Then Theorem 3.81 yields $K = -(\overline{\operatorname{cone}}\,(\operatorname{dom} g^*))^\circ$. Now let $x, y \in \mathbb{E}$ such that $x \preceq_K y$, i.e. $y = x + b$ for some $b \in K$. Since $g = g^{**}$ we hence find

$$\begin{aligned} g(x) &= \sup_{z \in \operatorname{dom} g^*} \{\langle x,\, z\rangle - g^*(z)\} \\ &= \sup_{z \in \operatorname{dom} g^*} \{\langle y,\, z\rangle - \langle b,\, z\rangle - g^*(z)\} \\ &\leq \sup_{z \in \operatorname{dom} g^*} \{\langle y,\, z\rangle - g^*(z)\} \\ &= g(y), \end{aligned}$$

where the inequality relies on the fact that $\langle b,\, z\rangle \geq 0$. $\square$

Corollary 4.11. *Let $g \in \Gamma_0(\mathbb{E}_2)$ and let $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be $(-\operatorname{hzn} g)$-convex with $-\operatorname{hzn} g$-$\operatorname{epi} F$ closed such that*

$$F(\operatorname{ri}(\operatorname{dom} F)) \cap \operatorname{ri}(\operatorname{dom} g + \operatorname{hzn} g) \neq \emptyset.$$

Then

$$(g \circ F)^*(p) = \min_{v \in \mathbb{E}_2} g^*(v) + \langle v,\, F\rangle^*(p).$$

Proof. This follows from combining Corollary 4.9 (with $K = -\operatorname{hzn} g$) and Lemma 4.10 while observing that $\operatorname{dom} g^* \subset \overline{\operatorname{cone}}\,(\operatorname{dom} g^*) = (\operatorname{hzn} g)^\circ$, cf. Theorem 3.81. $\square$

Finally, as another immediate consequence of our study we get the well known result for the case when F is linear, cf. Proposition 3.95.

Corollary 4.12 (The linear case)**.** *Let $g \in \Gamma(\mathbb{E}_2)$ and $F : \mathbb{E}_1 \to \mathbb{E}_2$ linear such that $\operatorname{rge} F \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset$. Then*

$$(g \circ F)^*(p) = \min_{v \in \mathbb{E}_2} \{g^*(v) \mid F^*(v) = p\}$$

with $\operatorname{dom}(g \circ F) = (F^*)^{-1}(\operatorname{dom} g^*)$.

Proof. We notice that F is $\{0\}$-convex. Hence we can apply Theorem 4.7 with $K = \{0\}$. Condition (4.4) then reads $\operatorname{rge} F \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset$, which is our assumption. Hence we obtain

$$(g \circ F)^*(p) = \min_{v \in -K^\circ} g^*(v) + \langle v,\, F\rangle^* p = \min_{v \in \mathbb{E}_2} g^*(v) + \delta_{F^*(v)}(p).$$

where the second identity uses Lemma 4.5. $\square$

4.4 Applications

In this section we present some eclectic applications of our study in Section 4.3 to illustrate the versatility of our findings.

4.4.1 Conic programming

We consider the general conic program

$$\min f(x) \quad \text{s.t.} \quad F(x) \in -K \tag{4.8}$$

where $f : \mathbb{E}_1 \to \mathbb{R}$ is convex, $F : \mathbb{E}_1 \to \mathbb{E}_2$ is K-convex and $K \subset \mathbb{E}_2$ is a closed, convex cone. Clearly, (4.8) can be written in the additive composite form

$$\min_{x\in\mathbb{E}_1} f(x) + (\delta_{-K} \circ F)(x). \tag{4.9}$$

This fits the additive composite setting of Corollary 4.8 with $g = \delta_{-K}$ which is K-increasing. Moreover, the qualification condition (4.4) for the conjugate calculus reads

$$\operatorname{rge} F \cap \operatorname{ri}(-K) = \emptyset, \tag{4.10}$$

which is simply a generalized version of *Slater's condition.*

The Fenchel (or Fenchel-Rockafellar) dual problem associated with (4.8) via (4.9) is

$$\max_{y\in\mathbb{E}_1} -f^*(y) - (\delta_{-K} \circ F)^*(-y),$$

while the Lagrangian dual is

$$\max_{v\in -K^\circ} \inf_{x\in\mathbb{E}_1} f(x) + \langle v,\, F(x)\rangle\,.$$

We obtain the following duality result.

Theorem 4.13. *(Strong duality and dual attainment for conic programming) Let $f : \mathbb{E}_1 \to \mathbb{R}$ is convex, $K \subset \mathbb{E}_2$ a closed, convex cone, and let $F : \mathbb{E}_1 \to \mathbb{E}_2$ be K-convex with closed K-epigraph. If* (4.10) *holds then*

$$\inf_{x\in\mathbb{E}_1} f(x) + (\delta_{-K} \circ F)(x) = \max_{v\in -K^\circ} -f^*(y) - (\delta_{-K} \circ F)^*(-y) = \max_{v\in -K^\circ} \inf_{x\in\mathbb{E}_1} f(x) + \langle v,\, F(x)\rangle\,.$$

Proof. Observe that, by Corollary 4.8, we have

$$\begin{aligned}
(f + \delta_{-K} \circ F)^*(0) &= \min_{\substack{v\in -K^\circ,\\ y\in\mathbb{E}_1}} f^*(y) + \langle v,\, F\rangle^*(-y) \\
&= \min_{v\in -K^\circ} (f + \langle v,\, F\rangle)^*(0) \\
&= \min_{v\in -K^\circ} \sup_{x\in\mathbb{E}_1} \{-f(x) - \langle v,\, F(x)\rangle\}.
\end{aligned}$$

Moreover, by Theorem 4.7 we have

$$(\delta_{-K} \circ F)^*(-y) = \min_{v\in -K^\circ} \langle v,\, F\rangle^*(-y).$$

Finally, since $\inf_{\mathbb{E}_1} f + \delta_{-K} \circ F = -(f + \delta_{-K} \circ F)^*(0)$, this shows everything. □

Theorem 4.13 furnishes various facts: It shows strong duality and dual attainment for both the Fenchel duality (first identity) and Lagrangian duality (second identity) scheme under a generalized Slater condition. Moreover, it shows the equivalence of both duality concepts. Of course, these are well-known results, but the proof based on Corollary 4.8 and Theorem 4.7, respectively, unifies this in a very elegant and convenient way.

4.4.2 Conjugate of pointwise maximum of convex functions

In what follows we denote the unit simplex in $\mathbb{R}^m$ by Δ_m, i.e.

$$\Delta_m = \left\{ v \in \mathbb{R}^m \;\middle|\; v_i \geq 0,\ \sum_{i=1}^m v_i = 1 \right\}.$$

The following result provides the conjugate for the pointwise maximum of finitely many convex functions. It therefore, slightly generalizes (at least in the finite dimensional case) the results established for the case of two functions in [15] and alternatively proven in [5].

Proposition 4.14. *For* $f_1, \dots, f_m \in \Gamma_0(\mathbb{E})$ *define* $f := \max_{i=1,\dots,m} f_i$*. Then* $f \in \Gamma_0(\mathbb{E})$ *with*

$$f^*(x) = \min_{v \in \Delta_m} \left(\sum_{i=1}^m v_i f_i \right)^*(x).$$

Proof. Define $F : \mathbb{E} \to (\mathbb{R}^m)^\bullet$ by

$$F(x) = \begin{cases} (f_1(x), \dots, f_m(x)) & \text{if } x \in \bigcap_{i=1}^m \operatorname{dom} f_i, \\ +\infty_\bullet & \text{otherwise,} \end{cases}$$

and $g\colon \mathbb{R}^m \to \mathbb{R}$ by $g(v) = \max_{1 \leq i \leq m} v_i$. Then $f = g \circ F$ (with the conventions made in Section 4.2) and we observe that F is $\mathbb{R}^m_+$-convex and g is $\mathbb{R}^m_+$-increasing with $\operatorname{dom} g = \mathbb{R}^m$. Hence, Theorem 4.7 c) is applicable with the qualification condition (4.4) trivially satisfied. Thus, for all $x \in \mathbb{E}$, we obtain

$$\begin{aligned} (g \circ F)^*(x) &= \min_{v \in \mathbb{R}^m_+} g^*(v) + \langle v,\, F \rangle^*(x) \\ &= \min_{v \in \mathbb{R}^m_+} \delta_{\Delta_m}(v) + \langle v,\, F \rangle^*(x) \\ &= \min_{v \in \Delta_m} \left(\sum_{i=1}^m v_i f_i \right)^*(x) \end{aligned}$$

where the second equality follows from Exercise 3.5.7. □

5 A New Class of Matrix Support Functionals

From this point on we set

$$\mathbb{E} := \mathbb{R}^{n \times m} \times \mathbb{S}^n \quad \text{and} \quad \kappa := \dim \mathbb{E}.$$

Then $\mathbb{E}$ is a Euclidean space equipped with the inner product

$$\langle (X, V),\, (Y, W) \rangle = \operatorname{tr}(X^T Y) + \operatorname{tr}(VW).$$

5.1 The matrix-fractional function

We consider the function $\phi : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ given by

$$\phi(X, V) = \begin{cases} \frac{1}{2}\mathrm{tr}\,(X^T V^{-1} X) & \text{if} \quad V \succ 0, \\ +\infty & \text{else} \end{cases} \tag{5.1}$$

which is referred to as the matrix-fractional function, see [6, Example 3.4] and [11, Example 3.6.0.0.2]. However, we reserve this moniker for its closure, see below. The function ϕ from (5.1) occurs in a myriad of different situations as the following two examples indicate; the first of which concerns a variational characterization of the nuclear norm.

Example 5.1 (Nuclear norm smoothing)**.** *The nuclear norm $\|\cdot\|_*$ is a popular regularizer in matrix optimization to promote low-rank solutions, see e.g. [13]. Hsieh and Olsen [20, Lemma 1] found an interesting variational representation*

$$\|X\|_* = \min_{V \in \mathbb{S}^n_{++}} \frac{1}{2}\mathrm{tr}\,(V) + \phi(X, V) \tag{5.2}$$

using ϕ from (5.1)*.*

The second example puts the function ϕ in the context of maximum-likelihood estimation.

Example 5.2 (Covariance estimation)**.** *Suppose $y_i \in \mathbb{R}^n$ $(i = 1, \dots, N)$ are measurements of a random vector y that is normally distributed with mean value $\mu \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{S}^n_{++}$, which are both unknown. In order to estimate them, one considers the optimization problem*

$$\max_{\mu \in \mathbb{R}^n, \Sigma \in \mathbb{S}^n_{++}} \frac{1}{(2\pi)^{n/2}} \prod_{i=1}^{N} \frac{1}{(\det \Sigma)^{1/2}} \exp\left(-\frac{1}{2}(y_i - \mu)^T \Sigma^{-1}(y_i - \mu)\right)$$

of maximizing the density with respect to mean value and covariance. This is, by the monotonicity of the logarithm, equivalent to the optimization problem

$$\min_{\mu \in \mathbb{R}^n, \Sigma \in \mathbb{S}^n_{++}} \sum_{i=1}^{N} \frac{1}{2}(y_i - \mu)^T \Sigma^{-1}(y_i - \mu) + \frac{N}{2} \log\det \Sigma.$$

Using the function ϕ from (5.1) *this reads*

$$\min_{(Y, \Sigma) \in \mathbb{R}^{n \times m} \times \mathbb{S}^n} \phi(Y - [\mu, \dots, \mu], \Sigma) - \left(-\frac{N}{2} \log\det \Sigma\right).$$

From a convex-analytical perspective, the function ϕ defined in (5.1) has one major drawback, namely it is not closed. As our study will show its closure is $\gamma : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ given by

$$\gamma(X, V) := \begin{cases} \frac{1}{2}\mathrm{tr}\,\left(X^T V^\dagger X\right) & \text{if} \quad V \succeq 0,\ \mathrm{rge}\, X \subset \mathrm{rge}\, V, \\ +\infty & \text{else.} \end{cases} \tag{5.3}$$

We will refer to γ as the *matrix-fractional function* while [11] refers to γ as the *pseudo matrix-fractional function.*

Proposition 5.3. *For the functions* γ *from* (5.3) *and* ϕ *from* (5.1) *we have:*

a) $\operatorname{epi}\gamma = \left\{(X,V,\alpha) \mid \exists\, Y\in\mathbb{S}^m:\ \begin{pmatrix} V & X\\ X^T & Y\end{pmatrix}\succeq 0,\ \frac{1}{2}\operatorname{tr}(Y)\le\alpha\right\}.$

b) $\operatorname{epi}\phi = \left\{(X,V,\alpha) \mid \exists\, Y\in\mathbb{S}^m:\ \begin{pmatrix} V & X\\ X^T & Y\end{pmatrix}\succeq 0,\ V\succ 0,\ \frac{1}{2}\operatorname{tr}(Y)\le\alpha\right\}.$

Moreover, $\operatorname{epi}\gamma = \operatorname{cl}(\operatorname{epi}\phi)$*, or equivalently,* $\gamma = \operatorname{cl}\phi$*.*

Proof. a) First note that

$$Y\succeq X^TV^\dagger X \ \Rightarrow\ \operatorname{tr}(Y)\ge\operatorname{tr}(X^TV^\dagger X). \tag{5.4}$$

We have

$$\begin{aligned}(X,V,\alpha)\in\operatorname{epi}\gamma \ &\Leftrightarrow\ V\succeq 0,\ \operatorname{rge}X\subset\operatorname{rge}V,\ \frac{1}{2}\operatorname{tr}(X^TV^\dagger X)\le\alpha\\ &\Leftrightarrow\ \exists\, Y\in\mathbb{S}^m:\ V\succeq 0,\ \operatorname{rge}X\subset\operatorname{rge}V,\ Y\succeq X^TV^\dagger X,\ \frac{1}{2}\operatorname{tr}(Y)\le\alpha\\ &\Leftrightarrow\ \exists\, Y\in\mathbb{S}^m:\ \begin{pmatrix} V & X\\ X^T & Y\end{pmatrix}\succeq 0,\ \frac{1}{2}\operatorname{tr}(Y)\le\alpha,\end{aligned}$$

where the second equivalence follows from (5.4) and the fact that we may take $Y = X^TV^\dagger X$, and the final equivalence follows from Lemma A.5.
b) Analogous to a) with the additional condition that $V\succ 0$ in each step.
In order to show that $\operatorname{cl}(\operatorname{epi}\phi) = \operatorname{epi}\gamma$, we first note that, as γ is lsc, $\operatorname{epi}\gamma$ is closed. Moreover, since obviously $\operatorname{epi}\phi\subset\operatorname{epi}\gamma$, the inclusion $\operatorname{cl}(\operatorname{epi}\phi)\subset\operatorname{epi}\gamma$ follows immediately.

In order to see the converse inclusion, let $(X,V,\alpha)\in\operatorname{epi}\gamma$. Now, set $V_k := V+\frac{1}{k}I_n\succ 0$ and put $\alpha_k := \alpha+\gamma(X,V_k)-\gamma(X,V)$ for all $k\in\mathbb{N}$. Then, by definition, we have

$$\alpha_k = \alpha+\gamma(X,V_k)-\gamma(X,V)\ge\gamma(X,V_k)=\phi(X,V_k),$$

i.e., $(X,V_k,\alpha_k)\in\operatorname{epi}\phi$ for all $k\in\mathbb{N}$. Clearly, $V_k\to V$ with V_k nonsingular, and so $V_k^{-1}\to V^+$ (e.g., see [22, p. 153]). Consequently, $\phi(X,V_k)=\gamma(X,V_k)\to\gamma(X,V)$ so that $\alpha_k\to\alpha$. Therefore, $(X,V_k,\alpha_k)\to(X,V,\alpha)$, and so $\operatorname{epi}\gamma=\operatorname{cl}(\operatorname{epi}\phi)$, i.e., $\gamma=\operatorname{cl}\phi$.

□

It is fairly easy to see now that the matrix-fractional function γ is closed, proper, convex as its epigraph is a closed, convex cone. By Hörmander's theorem (Theorem 3.68) it hence must be a support function - the question is which set it supports! We will answer this question as a by-product of our subequent analysis, in which we will consider a significantly generalized version of the matrix-fractional function. We would also like to point out that the fact that γ is a support function was already observed in the special case $m=n=1$ in [24, p. 83].

5.2 The generalized matrix-fractional function

We now want to study the GMF in more detail. To this end the following definition is crucial.

$$\mathcal{K}_{\mathcal{S}} := \left\{ V \in \mathbb{S}^n \;\middle|\; u^T V u \geq 0,\ (u \in \mathcal{S}) \right\}, \tag{5.5}$$

where $\mathcal{S}$ is a subspace of $\mathbb{R}^n$, that is, $\mathcal{K}_{\mathcal{S}}$ is the set of all symmetric matrices that are positive definite with respect to the given subspace $\mathcal{S}$. Observe that if $P \in \mathbb{S}^n$ is the orthogonal projection onto $\mathcal{S}$, then

$$\mathcal{K}_{\mathcal{S}} = \{ V \in \mathbb{S}^n \mid PVP \succeq 0 \}. \tag{5.6}$$

Clearly, $\mathcal{K}_{\mathcal{S}}$ is a convex cone, and, for $\mathcal{S} = \mathbb{R}^n$, it reduces to $\mathbb{S}^n_+$. Given a matrix $A \in \mathbb{R}^{p \times n}$, the cones $\mathcal{K}_{\ker A}$ play a special role in our analysis. For this reason, we simply write $\mathcal{K}_A$ to denote $\mathcal{K}_{\ker A}$, i.e. $\mathcal{K}_A := \mathcal{K}_{\ker A}$.

Proposition 5.4 ($\mathcal{K}_{\mathcal{S}}$ and its polar)**.** *Let $\mathcal{S}$ be a nonempty subspace of $\mathbb{R}^n$ and let P be the orthogonal projection onto $\mathcal{S}$. Then the following hold:*

a) $\mathcal{K}_{\mathcal{S}}^\circ = \mathrm{cone}\ \left\{ -vv^T \mid v \in \mathcal{S} \right\} = \{ W \in \mathbb{S}^n \mid W = PWP \preceq 0 \}$.

b) $\mathrm{int}\,\mathcal{K}_{\mathcal{S}} = \left\{ V \in \mathbb{S}^n \;\middle|\; u^T V u > 0\ (u \in \mathcal{S} \setminus \{0\}) \right\}$.

c) $\mathrm{aff}\,(\mathcal{K}_{\mathcal{S}}^\circ) = \mathrm{span}\ \left\{ vv^T \mid v \in \mathcal{S} \right\} = \{ W \in \mathbb{S}^n \mid \mathrm{rge}\, W \subset \mathcal{S} \}$.

d) $\mathrm{ri}\,(\mathcal{K}_{\mathcal{S}}^\circ) = \left\{ W \in \mathcal{K}_{\mathcal{S}}^\circ \;\middle|\; u^T W u < 0\ \ (u \in \mathcal{S} \setminus \{0\}) \right\}$ *when* $\mathcal{S} \neq \{0\}$ *and* $\mathrm{ri}\,(\mathcal{K}_{\{0\}}^\circ) = \{0\}$ *(since* $\mathcal{K}_{\{0\}} = \mathbb{S}^n$*).*

Proof. a) Put $B := \left\{ -ss^T \mid s \in \mathcal{S} \right\} \subset \mathbb{S}^n_-$ and observe that

$$\mathrm{cone}\ B = \left\{ -\sum_{i=1}^r \lambda_i s_i s_i^T \mid r \in \mathbb{N}, s_i \in \mathcal{S}, \lambda_i \geq 0\ (i = 1, \dots, r) \right\}.$$

We have $\mathrm{cone}\ B = \left\{ W \in \mathbb{S}^n_- \mid W = PWP \right\}$: To see this, first note that $\mathrm{cone}\ B \subset \left\{ W \in \mathbb{S}^n_- \mid W = PWP \right\}$. The reverse inclusion invokes the spectral decomposition of $W = \sum_{i=1}^n \lambda_i q_i q_i^T$ for $\lambda_1, \dots, \lambda_n \leq 0$. In particular, this representation of cone B shows that it is closed. We now prove the first equality in a): To this end, observe that

$$\begin{aligned} \mathcal{K}_{\mathcal{S}} &= \left\{ V \in \mathbb{S}^n \;\middle|\; s^T V s \geq 0\ (s \in \mathcal{S}) \right\} \\ &= \left\{ V \in \mathbb{S}^n \;\middle|\; \left\langle V, -ss^T \right\rangle \leq 0\ (s \in \mathcal{S}) \right\} \\ &= (\mathrm{cone}\ B)^\circ, \end{aligned}$$

where the third equality uses simply the linearity of the inner product in the second argument. Polarization then gives

$$\mathcal{K}_{\mathcal{S}}^\circ = (\mathrm{cone}\ B)^{\circ\circ} = \overline{\mathrm{cone}}\, B = \mathrm{cone}\ B.$$

b)The proof is straightforward and follows the pattern of proof for $\mathrm{int}\,\mathbb{S}^n_+ = \mathbb{S}^n_{++}$.

c) With B as defined above, observe that

$$\operatorname{aff} \mathcal{K}_{\mathcal{S}}^{\circ} = \operatorname{span} \mathcal{K}_{\mathcal{S}}^{\circ} = \operatorname{span} B,$$

since $0 \in \mathcal{K}_{\mathcal{S}}^{\circ}$, which shows the first equality. It is hence obvious that $\operatorname{aff} \mathcal{K}_{\mathcal{S}} \subset \{W \in \mathbb{S}^n \mid \operatorname{rge} W \subset \mathcal{S}\}$. On the other hand, every $W \in \mathbb{S}^n$ such that $\operatorname{rge} W \subset \mathcal{S}$ has a decomposition $W = \sum_{i=1}^{\operatorname{rank} W} \lambda_i q_i q_i^T$ where $\lambda_i \neq 0$ and $q_i \in \operatorname{rge} W \subset \mathcal{S}$ for all $i = 1, \ldots, \operatorname{rank} W$, i.e. $W \in \operatorname{span} B = \operatorname{aff} \mathcal{K}_{\mathcal{S}}^{\circ}$.

d) Set $R := \left\{W \in \mathcal{K}_{\mathcal{S}}^{\circ} \;\middle|\; u^T W u < 0 \;\; (u \in \mathcal{S} \setminus \{0\})\right\}$ and let $W \in \operatorname{ri}(\mathcal{K}_{\mathcal{S}}^{\circ}) \setminus R \subset \mathcal{K}_{\mathcal{S}}^{\circ}$. Then there exists $u \in \mathcal{S}$ with $\|u\| = 1$ such that $u^T W u = 0$. Then for every $\varepsilon > 0$ we have $u^T(W + \varepsilon u u^T)u = \varepsilon > 0$. Therefore $W + \varepsilon u u^T \in (B_\varepsilon(W) \cap \operatorname{aff}(\mathcal{K}_{\mathcal{S}}^{\circ})) \setminus \mathcal{K}_{\mathcal{S}}^{\circ}$ for all $\varepsilon > 0$, and hence $W \notin \operatorname{ri}(\mathcal{K}_{\mathcal{S}}^{\circ})$, which contradicts our assumption. Hence, $\operatorname{ri}(\mathcal{K}_{\mathcal{S}}^{\circ}) \subset R$.

To see the reverse implication assume there were $W \in R \setminus \operatorname{ri}(\mathcal{K}_{\mathcal{S}}^{\circ})$, i.e. for all $k \in \mathbb{N}$ there exists $W_k \in B_{\frac{1}{k}}(W) \cap \operatorname{aff}(\mathcal{K}_{\mathcal{S}}^{\circ}) \setminus \mathcal{K}_{\mathcal{S}}^{\circ}$. In particular, there exists $\{u_k \in \mathcal{S} \mid \|u_k\| = 1\}$ such that $u_k^T W_k u_k \geq 0$ for all $k \in \mathbb{N}$. W.l.o.g. we can assume that $u_k \to u \in \mathcal{S} \setminus \{0\}$. Letting $k \to \infty$, we find that $u^T W u \geq 0$ since $W_k \to W$. This contradicts the fact that $W \in R$.

□

5.2.1 Quadratic optimization with affine equality constraints

Given $(A, b) \in \mathbb{R}^{p \times n} \times \mathbb{R}^p$ with $b \in \operatorname{rge} A$ we consider the quadratic optimization problem

$$\min_{u \in \mathbb{R}^n} \frac{1}{2} u^T V u - x^T u \quad \text{s.t.} \quad Au = b, \tag{5.7}$$

and its optimal value function

$$v(x, V) := \inf_{u \in \mathbb{R}^n} \left\{ \frac{1}{2} u^T V u - x^T u \mid Au = b \right\}. \tag{5.8}$$

The following result is well known.

Lemma 5.5 (Solvabilty of equality constrained QPs). *For $(x, V, A, b) \in \mathbb{R}^n \times \mathbb{S}^n \times \mathbb{R}^{p \times n} \times \mathbb{R}^p$ consider the quadratic optimization problem (5.7). Then (5.7) has a solution if and only if the following three conditions hold:*

i) $b \in \operatorname{rge} A$ (i.e. (5.7) is feasible);

ii) $x \in \operatorname{rge} \begin{bmatrix} V & A^T \end{bmatrix}$;

iii) $V \in \mathcal{K}_A$.

If, however, $b \in \operatorname{rge} A$, but ii) or iii) are violated, we have $v(x, V) = -\infty$.

Proof. This is a standard result. It is an immediate consequence of [19, Exercise 5, page 17]. We leave the details to the reader as an exercise. □

We now provide an explicit formula for the optimal value function (5.8). Here, given $A \in \mathbb{R}^{p\times n}$, the matrix

$$M(V) := \begin{pmatrix} V & A^T \\ A & 0 \end{pmatrix}, \tag{5.9}$$

which is often referred to as a *bordered matrix* in the literature [10, 12, 22], will be useful. It also plays a central role in our subsequent analysis. We record an interesting result on invertibility of the bordered matrix before we continue our study.

Proposition 5.6 (Invertibility of bordered matrix). *The bordered matrix $M(V)$ is invertible if and only if* $\operatorname{rank} A = p$ *and $V \in \operatorname{int} \mathcal{K}_A$ in which case*

$$M(V)^{-1} = \begin{pmatrix} P(P^TVP)^{-1}P^T & \left(I - P(P^TVP)^{-1}P^TV\right)A^\dagger \\ (A^\dagger)^T\left(I - VP(P^TVP)^{-1}P^T\right) & (A^\dagger)^T\left(VP(P^TVP)^{-1}P^TV - V\right)A^\dagger \end{pmatrix},$$

where $P \in \mathbb{R}^{n\times(n-p)}$ is any matrix whose columns form an orthonormal basis of $\ker A$.

Proof. For the characterization of the invertibility of $M(V)$ see e.g. [10, Th. 7]. The inversion formula can be found in [12, Th. 1] and is easily verified by direct matrix multiplication. □

We now provide an explicit formula for the optimal value function in (5.8).

Theorem 5.7. *For $b \in \operatorname{rge} A$ and v given by* (5.8) *we have*

$$v(x,V) = \begin{cases} -\frac{1}{2}\begin{pmatrix} x \\ b \end{pmatrix}^T M(V)^\dagger \begin{pmatrix} x \\ b \end{pmatrix} & \text{if} \quad x \in \operatorname{rge}[V\,A^T],\ V \in \mathcal{K}_A, \\ -\infty & \text{else.} \end{cases}$$

Proof. If $b \in \operatorname{rge} A$, Lemma 5.5 tells us that if $x \notin \operatorname{rge}\left[V\,A^T\right]$ or V is not positive semidefinite on $\ker A$, we have $v(V,x) = -\infty$. Hence, we need only show the expression for v when $x \in \operatorname{rge}\left[V\,A^T\right]$ and V is positive semidefinite on $\ker A$. Again, Lemma 5.5 tells us that, in this case, a solution to (5.7) exists. The first-order necessary optimality conditions at

$$\bar{u} \in \operatorname{argmin}_{u\in\mathbb{R}^n}\left\{\frac{1}{2}u^TVu - x^Tu \quad \text{s.t.} \quad Au = b\right\} \neq \emptyset$$

are that there exists $\bar{y} \in \mathbb{R}^p$ for which

$$\begin{pmatrix} V & A^T \\ A & 0 \end{pmatrix}\begin{pmatrix} \bar{u} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x \\ b \end{pmatrix},$$

or equivalently,

$$\begin{pmatrix} \bar{u} \\ \bar{y} \end{pmatrix} \in \begin{pmatrix} V & A^T \\ A & 0 \end{pmatrix}^\dagger \begin{pmatrix} x \\ b \end{pmatrix} + \ker\begin{pmatrix} V & A^T \\ A & 0 \end{pmatrix}.$$

Plugging such a pair $\binom{\bar u}{\bar y}$ into the objective function yields

$$\begin{aligned}
\frac{1}{2}\bar u^T V\bar u - x^T\bar u &= \frac{1}{2}\bar u^T(\underbrace{V\bar u - x}_{=-A^T\bar y}) - \frac{1}{2}x^T\bar u\\
&= -\frac{1}{2}\underbrace{\bar u^T A^T}_{=b^T}\bar y - \frac{1}{2}x^T\bar u\\
&= -\frac{1}{2}\begin{pmatrix} x\\ b\end{pmatrix}^T\begin{pmatrix}\bar u\\ \bar y\end{pmatrix}\\
&= -\frac{1}{2}\begin{pmatrix} x\\ b\end{pmatrix}^T\begin{pmatrix} V & A^T\\ A & 0\end{pmatrix}^\dagger\begin{pmatrix} x\\ b\end{pmatrix}^T,
\end{aligned}$$

where the last equation is due to the fact that

$$\begin{pmatrix} x\\ b\end{pmatrix}\in \operatorname{rge}\begin{pmatrix} V & A^T\\ A & 0\end{pmatrix} = \left(\ker\begin{pmatrix} V & A^T\\ A & 0\end{pmatrix}\right)^\perp.$$

Since all such points yield the same optimal value, this concludes the proof. □

5.2.2 The generalized matrix fractional function

Given $A\in\mathbb{R}^{p\times n}$ and $B\in\mathbb{R}^{p\times m}$ such that $\operatorname{rge}B\subset\operatorname{rge}A$, Theorem 5.7 motivates an inspection of a generalization of the matrix fractional function defined by

$$\varphi_{A,B} : (X,V)\in\mathbb{E}\mapsto\begin{cases}\frac{1}{2}\operatorname{tr}\left(\binom{X}{B}^T M(V)^\dagger\binom{X}{B}\right) & \text{if } \operatorname{rge}\binom{X}{B}\subset\operatorname{rge}M(V),\ V\in\mathcal{K}_A,\\ +\infty, & \text{else.}\end{cases} \tag{5.10}$$

Theorem 5.7 then compactly reads

$$-v(x,V) = \varphi_{A,b}(x,V)$$

for some $b\in\operatorname{rge}A$. Moreover, for $A=0$ and $B=0$ we recover the matrix-fractional function from (5.3), i.e.

$$\varphi_{0,0}=\gamma.$$

this motivates us to call $\varphi_{A,B}$ the *generalized matrix-fractional function (GMF)*. We already argued in Section 5.1 that the matrix-fractional function γ is a support function. Naturally, this begs the question whether this is also true for the GMF. The following result gives a positive answer to this by showing that the GMF defined by A,B is the support function of the set

$$\mathcal{D}(A,B) := \left\{\left(Y, -\frac{1}{2}YY^T\right)\in\mathbb{E}\;\middle|\; Y\in\mathbb{R}^{n\times m} :\ AY=B\right\}. \tag{5.11}$$

Theorem 5.8. *Let $A\in\mathbb{R}^{p\times n}$ and $B\in\mathbb{R}^{p\times m}$ such that* $\operatorname{rge}B\subset\operatorname{rge}A$ *and let $\mathcal{D}(A,B)$ be given by* (5.11)*. Then*

$$\sigma_{\mathcal{D}(A,B)}(X,V)=\varphi_{A,B}(X,V).$$

In particular,

$$\operatorname{dom}\sigma_{D(A,B)} = \operatorname{dom}\partial\sigma_{D(A,B)} = \left\{(X,V)\in\mathbb{E}\;\middle|\;\operatorname{rge}\begin{pmatrix}X\\B\end{pmatrix}\subset\operatorname{rge}M(V),\;V\in\mathcal{K}_A\right\}.$$

Moreover, we have

$$\operatorname{int}(\operatorname{dom}\sigma_{D(A,B)}) = \{(X,V)\in\mathbb{E}\mid V\in\operatorname{int}\mathcal{K}_A\}.$$

Proof. By direct computation,

$$\begin{aligned}
\sigma_{\mathcal{D}(A,B)}(X,V) &= \sup_{(U,W)\in\mathcal{D}(A,B)}\langle (X,V),(U,W)\rangle\\
&= \sup_{U:\,AU=B}\left\{\operatorname{tr}(X^TU)-\frac{1}{2}\operatorname{tr}(UU^TV)\right\}\\
&= \sup_{U:\,AU=B}\left\{\operatorname{tr}(X^TU)-\frac{1}{2}\operatorname{tr}\left(\sum_{i=1}^m u^i(u^i)^TV\right)\right\}\\
&= \sup_{U:\,AU=B}\left\{\sum_{i=1}^m (x^i)^Tu^i-\frac{1}{2}(u^i)^TVu^i\right\}\\
&= -\sum_{i=1}^m\left\{\inf_{u:\,Au=b^i}\frac{1}{2}u^TVu-(x^i)^Tu\right\} \qquad (5.12)\\
&= \sum_{i=1}^m\begin{cases}\frac{1}{2}\binom{x^i}{b^i}^TM(V)^\dagger\binom{x^i}{b^i} & \text{if}\quad \binom{x^i}{b^i}\in\operatorname{rge}M(V),\;V\in\mathcal{K}_A,\\ +\infty & \text{else}\end{cases}\\
&= \begin{cases}\dfrac{1}{2}\operatorname{tr}\left(\begin{pmatrix}X\\B\end{pmatrix}^TM(V)^\dagger\begin{pmatrix}X\\B\end{pmatrix}\right) & \text{if }\operatorname{rge}\begin{pmatrix}X\\B\end{pmatrix}\subset\operatorname{rge}M(V),\,V\in\mathcal{K}_A,\\ +\infty & \text{else.}\end{cases}
\end{aligned}$$

Here, the sixth equation exploits Theorem 5.7. This establishes the representation for $\sigma_{\mathcal{D}(A,B)}$ as well as for its domain. In addition, since

$$\partial\sigma_{\mathcal{D}(A,B)}(X,V) = \operatorname{argmax}\{\langle (X,V),\,(U,W)\rangle\mid (U,W)\in\mathcal{D}(A,B)\},$$

Theorem 5.7 also yields the equivalence $\operatorname{dom}\sigma_{\mathcal{D}(A,B)} = \operatorname{dom}\partial\sigma_{\mathcal{D}(A,B)}$.

In order to see that $\operatorname{dom}\sigma_{\mathcal{D}(A,B)}$ is closed, let $\{(X_k,V_k)\in\operatorname{dom}\sigma_{\mathcal{D}(A,B)}\}\to(V,X)$. In particular, $\operatorname{rge}X_k\subset\operatorname{rge}[V_k\;\;A^T]$ and $V_k\in\mathcal{K}_A$. These properties are preserved by passing to the limit, hence $(X,V)\in\operatorname{dom}\sigma_{\mathcal{D}(A,B)}$, i.e., $\operatorname{dom}\sigma_{\mathcal{D}(A,B)}$ is closed.

It remains to prove the expression for $\operatorname{int}(\operatorname{dom}\sigma_{\mathcal{D}(A,B)})$. First we show that $O := \{(X,V)\in\mathbb{E}\mid V\succ_{\ker A}0\}$ is open. For this purpose, let $(X,V)\in O$. Suppose $q:=\operatorname{rank}A$ and let $P\in\mathbb{R}^{n\times(n-q)}$ be such that its columns form a basis of $\ker A$ so that P^TVP is positive definite. Using the fact that $\mathbb{S}^n_{++}=\operatorname{int}\mathbb{S}^n_+$ (e.g., see [3, Ch. 1, Ex. 1]), there exists an $\varepsilon>0$ such that

$$\|W-V\|\le\varepsilon\quad\Rightarrow\quad P^TWP\succ 0\quad\forall W\in\mathbb{S}^n.$$

Hence, $B_\varepsilon(X,V) \subset O$ so that O is open.

Next, we show that $O \subset \mathrm{dom}\, \sigma_{\mathcal{D}(A,B)}$. Again let $(X,V) \in O$, so that $V \succ_{\ker A} 0$. By Finsler's lemma (see Lemma A.6) we readily infer that $\mathrm{rge}\,[V\ \ A^T] = \mathbb{R}^n$, and so $\mathrm{rge}\, X \subset \mathrm{rge}\,[V\ \ A^T]$. Hence $(X,V) \in \mathrm{dom}\, \sigma_{\mathcal{D}(A,B)}$ yielding $O \subset \mathrm{dom}\, \sigma_{\mathcal{D}(A,B)}$. Since O is open and $O \subset \mathrm{dom}\, \sigma_{\mathcal{D}(A,B)}$, we have $O \subset \mathrm{int}\, \left(\mathrm{dom}\, \sigma_{\mathcal{D}(A,B)}\right)$.

We now show the reverse inclusion. Let $(X,V) \in \mathrm{dom}\, \sigma_{\mathcal{D}(A,B)} \setminus O$. Hence V is positive semidefinite, but not positive definite on $\ker A$. Therefore, there exists $x \in \ker A \setminus \{0\}$ such that $x^T V x = 0$. Define

$$V_k := V - \frac{1}{k} I \in \mathbb{S}^n \quad (k \in \mathbb{N}).$$

Then, for all $k \in \mathbb{N}$, $x^T V_k x = -\frac{\|x\|^2}{k} < 0$, so that $(X, V_k) \notin \mathrm{dom}\, \sigma_{\mathcal{D}(A,B)}$ for all $k \in \mathbb{N}$ while $(X, V_k) \to (X,V)$. Hence $(X,V) \in \mathrm{bd}\, \left(\mathrm{dom}\, \sigma_{\mathcal{D}(A,B)}\right)$, that is, $(X,V) \notin \mathrm{int}\, (\mathrm{dom}\, \sigma_{\mathcal{D}(A,B)})$. Consequently, $O \supset \mathrm{int}\, \left(\mathrm{dom}\, \sigma_{\mathcal{D}(A,B)}\right)$. $\square$

Justified by Theorem 5.8, from now on we will $\sigma_{\mathcal{D}(A,B)} = \varphi_{A,B}$ interchangeably and also refer to $\sigma_{\mathcal{D}(A,B)}$ as the generalized matrix-fractional function or simply GMF.

We state the case $A = 0$ and $B = 0$ explicitly.

Corollary 5.9. *We have* $\gamma = \sigma_{\mathcal{D}(0,0)}$.

5.2.3 Convex analysis of the generalized matrix-fractional function

Theorem 5.8 allows us to invoke the whole machinery for support functions and its duality correspondence with indicator functions, see Section 3.5.3, and its rich subdifferential calculus, see Corollary 3.87. These results already foreshadow that we need a good description of the closed convex hull of $\mathcal{D}(A,B)$. This will be achieved by the following set

$$\Omega(A,B) := \left\{ (Y,W) \in \mathbb{E} \;\middle|\; AY = B \ \text{ and } \ \frac{1}{2} Y Y^T + W \in \mathcal{K}_A^\circ \right\} \tag{5.13}$$

as the next result shows. We also set $\Omega := \Omega(0,0)$.

Theorem 5.10. *Let $\mathcal{D}(A,B)$ and $\Omega(A,B)$ be s given by* (5.11) *and* (5.13)*, respectively. Then*

$$\overline{\mathrm{conv}}\, \mathcal{D}(A,B) = \Omega(A,B).$$

Proof. We first show that $\Omega(A,B)$ is itself a closed convex set. Obviously, $\Omega(A,B)$ is closed since $\mathcal{K}_A^\circ$ is closed and the mappings $Y \mapsto AY$ and $(Y,W) \mapsto \frac{1}{2} Y Y^T + W$ are continuous.

So we need only show that $\Omega(A,B)$ is convex: To this end, let $(Y_i, W_i) \in \Omega(A,B),\ i = 1,2$ and $0 \le \lambda \le 1$. Then there exist $M_i \in \mathcal{K}_A^\circ,\ i = 1,2$ such that $W_i = -\frac{1}{2} Y_i Y_i^T + M_i$.

Observe that $A((1-\lambda)Y_1+\lambda Y_2)=B$. Moreover, we compute that

$$
\begin{aligned}
&\frac{1}{2}((1-\lambda)Y_1+\lambda Y_2)((1-\lambda)Y_1+\lambda Y_2)^T+((1-\lambda)W_1+\lambda W_2)\\
=&\frac{1}{2}((1-\lambda)Y_1+\lambda Y_2)((1-\lambda)Y_1+\lambda Y_2)^T+\left((1-\lambda)(-\frac{1}{2}Y_1Y_1^T+M_1)+\lambda(-\frac{1}{2}Y_2Y_2^T+M_2)\right)\\
=&\frac{1}{2}\lambda(1-\lambda)(-Y_1Y_1^T+Y_1Y_2^T+Y_2Y_1^T-Y_2Y_2^T)+(1-\lambda)M_1+\lambda M_2\\
=&\lambda(1-\lambda)\left(-\frac{1}{2}(Y_1-Y_2)(Y_1-Y_2)^T\right)+(1-\lambda)M_1+\lambda M_2.
\end{aligned}
$$

Since $\operatorname{rge}(Y_1-Y_2)\subset\ker A$, this shows $\lambda(1-\lambda)\left(-\frac{1}{2}(Y_1-Y_2)(Y_1-Y_2)^T\right)+(1-\lambda)M_1+\lambda M_2\in\mathcal{K}_A^\circ$. Consequently, $\Omega(A,B)$ is a closed convex set.

Next note that if $(Y,-\frac{1}{2}YY^T)\in\mathcal{D}(A,B)$, then $(Y,-\frac{1}{2}YY^T)\in\Omega(A,B)$ since $0\in\mathcal{K}_A^\circ$. Hence, $\overline{\operatorname{conv}}\,\mathcal{D}(A,B)\subset\Omega(A,B)$.

It therefore remains to establish the reverse inclusion: For these purposes, let $(Y,W)\in\Omega(A,B)$. By Carathéodory's theorem, there exist $\mu_i\geq 0, v_i\in\ker A$ $(i=1,\ldots,N)$ such that

$$
W=-\frac{1}{2}YY^T-\sum_{i=1}^N\mu_iv_iv_i^T,
$$

where $N=\frac{n(n+1)}{2}+1$. Let $0<\varepsilon<1$. Set $\lambda_1:=1-\varepsilon$ and $\lambda_2=\ldots=\lambda_{N+1}=\lambda:=\varepsilon/N$. Denote $Y_1:=Y/\sqrt{1-\varepsilon}$. Take $Z_i\in\mathbb{R}^{n\times m},\ i=1,\ldots,N$ such that $AZ_i=B$. Finally, set

$$
V_i=\left[\sqrt{\frac{2\mu_i}{\lambda}}v_i,0,\ldots,0\right]\in\mathbb{R}^{n\times m}\quad\text{and}\quad Y_{i+1}=Z_i+V_i,\ (i=1,\ldots,N).
$$

Observe that

$$
\sum_{i=1}^{N+1}\lambda_iY_i=\sqrt{1-\varepsilon}Y+\frac{\varepsilon}{N}\sum_{i=2}^{N+1}Y_i=\sqrt{1-\varepsilon}Y+\frac{\varepsilon}{N}\sum_{i=1}^{N}Z_i+\sqrt{\frac{\varepsilon}{N}}\sum_{i=1}^N\bar{V}_i,
$$

where $\bar{V}_i=[\sqrt{2\mu_i}v_i,0,\ldots,0],\ i=1,\ldots,N$, and

$$
\begin{aligned}
-\frac{1}{2}\sum_{i=1}^{N+1}\lambda_iY_iY_i^T&=-\frac{1}{2}YY^T-\frac{1}{2}\sum_{i=1}^N\frac{\varepsilon}{N}\left(Z_iZ_i^T+Z_iV_i^T+V_iZ_i^T\right)-\sum_{i=1}^N\mu_iv_iv_i^T\\
&=W-\sum_{i=1}^N\frac{1}{2}\left(\frac{\varepsilon}{N}Z_iZ_i^T+\sqrt{\frac{\varepsilon}{N}}Z_i\bar{V}_i^T+\sqrt{\frac{\varepsilon}{N}}\bar{V}_iZ_i^T\right),
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left(\sqrt{1-\varepsilon}Y+\frac{\varepsilon}{N}\sum_{i=1}^NZ_i+\sqrt{\frac{\varepsilon}{N}}\sum_{i=1}^N\bar{V}_i,\quad W-\sum_{i=1}^N\frac{1}{2}\left(\frac{\varepsilon}{N}Z_iZ_i^T+\sqrt{\frac{\varepsilon}{N}}Z_i\bar{V}_i^T+\sqrt{\frac{\varepsilon}{N}}\bar{V}_iZ_i^T\right)\right)\\
&\qquad=\left(\sum_{i=1}^{N+1}\lambda_iY_i,\quad -\frac{1}{2}\sum_{i=1}^{N+1}\lambda_iY_iY_i^T\right).
\end{aligned}\tag{5.14}
$$

By Carathéodory's theorem (see Theorem 3.5) we have

$$\operatorname{conv}\mathcal{D}(A,B)=\left\{\left(\sum_{i=1}^{\kappa+1}\lambda_i Y_i,\,-\frac{1}{2}\sum_{i=1}^{\kappa+1}\lambda_i Y_i Y_i^T\right)\,\middle|\, \begin{array}{c}\lambda\in\mathbb{R}_+^{\kappa+1},\sum_{i=1}^{\kappa+1}\lambda_i=1,\,Y_i\in\mathbb{R}^{n\times m}\\ AY_i=B\ (i=1,\dots,\kappa+1)\end{array}\right\}.$$

By letting $\varepsilon\downarrow 0$ in (5.14), we find $(Y,W)\in\overline{\operatorname{conv}}\,\mathcal{D}(A,B)$ thereby concluding the proof. □

Let us state the immediate consequence of the above theorem.

Corollary 5.11. *Let $\Omega(A,B)$ be given by*(5.13)*. Then*

$$\varphi_{A,B}=\sigma_{\mathcal{D}(A,B)}=\sigma_{\Omega(A,B)}\quad\textit{and}\quad\gamma=\sigma_\Omega.$$

The conjugacy relation between indicator and support functions, see Proposition 3.66, now gives the following immediate consequence of Theorem 5.10.

Corollary 5.12 (Conjugate of GMF)**.** *We have*

$$\sigma^*_{\mathcal{D}(A,B)}=\delta_{\Omega(A,B)}.$$

In order to derive the subdifferential for $\varphi_{A,B}=\sigma_{\mathcal{D}(A,B)}$, we use the relation

$$\partial\sigma_C(x)=\{z\in\overline{\operatorname{conv}}\,C\mid x\in N_{\overline{\operatorname{conv}}\,C}(z)\}\tag{5.15}$$

from Corollary 3.87. Therefore, we first need the normal cone to $\Omega(A,B)$.

Proposition 5.13 (The normal cone to $\Omega(A,B)$)**.** *Let $\Omega(A,B)$ be as given by* (5.13) *and let $(Y,W)\in\Omega(A,B)$. Then*

$$N_{\Omega(A,B)}(Y,W)=\left\{(X,V)\in\mathbb{E}\;\middle|\;\begin{array}{c}V\in\mathcal{K}_A,\ \left\langle V,\,\frac{1}{2}YY^T+W\right\rangle=0\\ \textit{and}\quad\operatorname{rge}(X-VY)\subset(\ker A)^\perp\end{array}\right\}$$

Proof. Observe that $\Omega(A,B)=C_1\cap C_2\subset\mathbb{E}$ where

$$C_1:=\left\{Y\in\mathbb{R}^{n\times m}\mid AY=B\right\}\times\mathbb{S}^n\quad\text{and}\quad C_2:=\{(Y,W)\mid F(Y,W)\in\mathcal{K}_A^\circ\},$$

with $F(Y,W):=\frac{1}{2}YY^T+W$. Clearly, C_1 is affine, hence convex, and C_2 is also convex, which can be seen by an analogous reasoning as for the convexity of $\Omega(A,B)$ (cf. the proof of Theorem 5.10). Therefore, [24, Corollary 23.8.1] tells us that

$$N_{\Omega(A,B)}(Y,W)=N_{C_1}(Y,W)+N_{C_2}(Y,W),\tag{5.16}$$

where

$$N_{C_1}(Y,W)=\left\{R\in\mathbb{R}^{n\times m}\;\middle|\;\operatorname{rge}R\subset(\ker A)^\perp\right\}\times\{0\}.$$

We now compute $N_{C_2}((Y,W))$. First recall that for any nonempty closed convex cone $C\subset\mathcal{E}$, we have $N_C(x)=\{z\in C^\circ\mid\langle z,\,x\rangle=0\}$ for all $x\in C$. Next, note that

$$\nabla F(Y,W)^*U=(UY,\;U)\quad(U\in\mathbb{S}^n),$$

so that $\nabla F(Y,W)^*U = 0$ if and only if $U = 0$. Hence, by [25, Exercise 10.26 Part (d)],

$$N_{C_2}(Y,W) = \left\{ (VY, V) \;\middle|\; V \in \mathcal{K}_A,\ \left\langle V, \frac{1}{2}YY^T + W \right\rangle = 0 \right\}.$$

Therefore, by (5.16), $N_{\Omega(A,B)}(Y,W)$ is given by

$$\left\{ (X,V) \;\middle|\; \operatorname{rge}(X - VY) \subset (\ker A)^\perp,\ V \in \mathcal{K}_A,\ \left\langle V, \frac{1}{2}YY^T + W \right\rangle = 0 \right\},$$

which proves the result. $\square$

By combining (5.15) and Proposition 5.13 we obtain a simplified representation of the subdifferential of the support function $\sigma_{\mathcal{D}}(A,B)$.

Corollary 5.14 (The subdifferential of $\sigma_{\mathcal{D}(A,B)}$). *Let* $\mathcal{D}(A,B)$ *be as given in* (5.11). *Then, for all* $(X,V) \in \operatorname{dom} \sigma_{\mathcal{D}(A,B)}$, *we have*

$$\partial\sigma_{\mathcal{D}(A,B)}(X,V) = \left\{ (Y,W) \in \Omega(A,B) \;\middle|\; \begin{array}{l} \exists Z \in \mathbb{R}^{p\times m}:\ X = VY + A^TZ, \\ \left\langle V, \frac{1}{2}YY^T + W \right\rangle = 0 \end{array} \right\}.$$

Proof. This follows directly from the normal cone description in Proposition 5.13 and the relation (5.15). $\square$

We infer from Corollary 5.14 that the GMF is continuously differentiable on the interior of its domain, and we give an explicit formula for the gradient in this case.

Corollary 5.15. *Let* $\mathcal{D}(A,B)$ *be as given in* (5.11). *Then* $\sigma_{\mathcal{D}(A,B)}$ *is (continuously) differentiable on the interior of its domain with*

$$\nabla\sigma_{\mathcal{D}(A,B)}(X,V) = \left(Y, -\frac{1}{2}YY^T\right) \quad ((X,V) \in \operatorname{int}(\operatorname{dom}\sigma_{\mathcal{D}(A,B)}))$$

where $Y := A^\dagger B + (P(P^TVP)^{-1}P^T)(X - A^\dagger X)$, $P \in \mathbb{R}^{n\times(n-p)}$ *is any matrix whose columns form an orthonormal basis of* $\ker A$ *and* $p := \operatorname{rank} A$.

Proof. We first recall from Theorem 5.8 that $(X,V) \in \operatorname{int}\operatorname{dom}\sigma_{\mathcal{D}(A,B)}$ if and only if $V \in \operatorname{int}\mathcal{K}_A$. By Corollary 5.14, $(Y,W) \in \partial\sigma_{\mathcal{D}(A,B)}(X,V)$ if and only if there exists $Z \in \mathbb{R}^{p\times m}$ such that

$$AY = B \tag{5.17}$$
$$VY + A^TZ = X \tag{5.18}$$
$$\left\langle V, \frac{1}{2}YY^T + W \right\rangle = 0 \tag{5.19}$$
$$\frac{1}{2}YY^T + W \in \mathcal{K}_A^\circ. \tag{5.20}$$

We now observe that (5.17)-(5.18) are equivalent to

$$M(V)\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} X \\ B \end{pmatrix}.$$

As $V \in \operatorname{int} \mathcal{K}_A$, by Proposition 5.6 this implies

$$Y = A^\dagger B + P(P^T V P)^{-1} P^T (X - A^\dagger X).$$

Moreover (5.19), (5.20) and $V \in \operatorname{int} \mathcal{K}_A$ readily imply that $\frac{1}{2} Y Y^T + W = 0$, which concludes the proof. □

We state the special case $A = 0$ and $B = 0$ (i.e. $\sigma_{\mathcal{D}(A,B)} = \gamma$) as a separate result, which by the way, was well known all along, see e.g. [6], and can be derived by standard calculus methods.

Corollary 5.16. *The matrix-fractional function* γ *from* (5.3) *is (continuously) differentiable on the interior of its domain with*

$$\nabla \gamma(X, V) = \left(V^{-1} X, -\frac{1}{2} V^{-1} X X^T V^{-1} \right) \quad ((X, V) \in \mathbb{R}^{n \times m} \times \mathbb{S}^n_{++}).$$

Remark 5.17. *We would like to point out that in [7, Proposition 4.3] the following description of the closed convex hull of* $\mathcal{D}(A, B)$ *was established:*

$$\overline{\operatorname{conv}}\, \mathcal{D}(A, B) = \left\{ (Z(d \otimes I_m), -\frac{1}{2} Z Z^T) \mid (d, Z) \in \mathcal{F}(A, B) \right\}, \tag{5.21}$$

where $d \otimes I_m := (d_1 I_m, \ldots, d_{\kappa+1} I_m)^T \in \mathbb{R}^{m(\kappa+1) \times m}$ *and*

$$\mathcal{F}(A, B) := \left\{ (d, Z) \in \mathbb{R}^{\kappa+1} \times \mathbb{R}^{n \times m(\kappa+1)} \;\middle|\; \begin{array}{l} d \geq 0, \ \|d\| = 1, \\ A Z_i = d_i B \ (i = 1, \ldots, \kappa + 1) \end{array} \right\}. \tag{5.22}$$

The description in (5.21) *was obtained by computing the convex hull of* $\mathcal{D}(A, B)$ *first, see [7, Lemma 4.2], using "brute force" in the form of Carathéodory's Theorem (cf. Theorem 3.5), and then determining the closure of said convex hull. Although the representation of* $\overline{\operatorname{conv}}\, \mathcal{D}(A, B)$ *from* (5.21) *was successfully used in [7] to study some convex-analytical properties of the GMF and it yielded some interesting applications, the description from Theorem 5.10 is much more powerful due to its simplicity.*

5.3 The geometry of $\Omega(A, B)$

We now compute the relative interior and the affine hull of $\Omega(A, B)$. We will rely heavily and expand on the results established in Section 3.1.4. In particular, Theorem 3.29 will be very useful as we will use this result to get a representation for the relative interior of $\Omega(A, B)$ directly, and then mimic its technique of proof to tackle its affine hull.

Lemma 5.18. *Let* $A, B \subset \mathbb{E}$ *be convex with* $\operatorname{ri} A \cap \operatorname{ri} B \neq \emptyset$. *Then* $\operatorname{aff}(A \cap B) = \operatorname{aff} A \cap \operatorname{aff} B$.

Proof. The inclusion $\operatorname{aff}(A \cap B) \subset \operatorname{aff} A \cap \operatorname{aff} B$ is clear since the latter set is affine and contains $A \cap B$.

For proving the reverse inclusion, we can assume w.l.o.g. that $0 \in \operatorname{ri} A \cap \operatorname{ri} B = \operatorname{ri}(A \cap B)$, where for the latter equality we refer to Proposition 3.26. In particular we have

$$\operatorname{aff} A = \mathbb{R}_+ A, \ \operatorname{aff} B = \mathbb{R}_+ B \quad \text{and} \quad \operatorname{aff}(A \cap B) = \mathbb{R}_+(A \cap B), \tag{5.23}$$

see Exercise 3.1.7 and the discussion afterwards. Now, let $x \in \operatorname{aff} A \cap \operatorname{aff} B$. If $x = 0$ there is nothing to prove. If $x \neq 0$, by (5.23), we have $x = \lambda a = \mu b$ for some $\lambda, \mu > 0$ and $a \in A, b \in B$. W.l.o.g we have $\lambda > \mu$, and hence, by convexity of B, we have

$$a = \left(1 - \frac{\mu}{\lambda}\right) 0 + \frac{\mu}{\lambda} b \in B.$$

Therefore $x = \lambda a \in \mathbb{R}_+(A \cap B) = \operatorname{aff}(A \cap B)$, see (5.23). $\square$

We now prove a result analogous to Theorem 3.29.

Proposition 5.19. *In addition to the assumptions of Theorem 3.29 assume that D is affine. Then $(y, z) \in \operatorname{aff} C$ if and only if $y \in D$ and $z \in \operatorname{aff} C_y$.*

Proof. We imitate the proof of Theorem 3.29: Let $L : (y, z) \mapsto z$. Since D is assumed to be affine (hence $D = \operatorname{aff} D = \operatorname{ri} D$), we have

$$D = L(C) = L(\operatorname{ri} C) = L(\operatorname{aff} C), \tag{5.24}$$

where we invoke the fact that linear mappings commute with the relative interior (Proposition 3.28) and the affine hull (Exercise 3.1.6). Now fix $y \in D = \operatorname{ri} D$ and define the affine set $M_y := \{(y, z) \mid z \in \mathbb{E}_2\} = \{y\} \times \mathbb{E}_2$. Then, by (5.24), there exists $z \in \mathbb{E}_2$ such that $y = L(y, z)$ and $(y, z) \in \operatorname{ri} C$. Hence, $\operatorname{ri} M_y \cap \operatorname{ri} C \neq \emptyset$ and we can invoke Lemma 5.18 to obtain

$$\operatorname{aff} M_y \cap \operatorname{aff} C = \operatorname{aff}(M_y \cap C) = \operatorname{aff}(\{y\} \times C_y) = \{y\} \times \operatorname{aff} C_y.$$

Hence, if $y \in D, z \in \operatorname{aff} C_y$, we have $(y, z) \in \{y\} \times \operatorname{aff} C_y = M_y \cap \operatorname{aff} C \subset \operatorname{aff} C$.

In turn, for $(y, z) \in C$, we have $(y, z) \in M_y \cap \operatorname{aff} C = \{y\} \times C_y$, hence $z \in C_y \neq \emptyset$, so $y \in D$. $\square$

We are now in a position to prove the desired result on the relative interior and the affine hull of $\Omega(A, B)$.

Proposition 5.20. *For $\Omega(A, B)$ given by* (5.13) *the following hold:*

a) $\operatorname{ri} \Omega(A, B) = \left\{(Y, W) \in \mathbb{E} \;\middle|\; AY = B \ \text{ and } \ \frac{1}{2} Y Y^T + W \in \operatorname{ri}(\mathcal{K}_A^\circ)\right\}.$

b) $\operatorname{aff} \Omega(A, B) = \left\{(Y, W) \in \mathbb{E} \;\middle|\; AY = B \ \text{ and } \ \frac{1}{2} Y Y^T + W \in \operatorname{span} \mathcal{K}_A^\circ\right\},$
where $\operatorname{span} \mathcal{K}_A^\circ = \operatorname{span}\left\{v v^T \mid v \in \ker A\right\}$.

Proof. We apply the format of Theorem 3.29 and Proposition 5.19, respectively, for $C := \Omega(A, B)$. Then

$$D = \{Y \mid AY = B\} \quad \text{and} \quad C_Y = \begin{cases} \mathcal{K}_A^\circ - \frac{1}{2} Y Y^T, & \text{if} \quad AY = B, \\ \emptyset, & \text{else} \end{cases} \qquad (Y \in \mathbb{R}^{n \times m}).$$

a) Apply Theorem 3.29 and observe that $\mathrm{ri}\,(\mathcal{K}_A^\circ - \frac{1}{2}YY^T) = \mathrm{ri}\,(\mathcal{K}_A^\circ) - \frac{1}{2}YY^T$.

b) Apply Proposition 5.19 and observe that D is affine, and that $\mathrm{aff}\,(\mathcal{K}_A^\circ - \frac{1}{2}YY^T) = \mathrm{aff}\,(\mathcal{K}_A^\circ) - \frac{1}{2}YY^T$.

$\square$

As a direct consequence of Propositions 5.4 and 5.20, we obtain the following result for the special case $(A, B) = (0, 0)$.

Corollary 5.21. *It holds that*

$$\overline{\mathrm{conv}}\left\{(Y, -\frac{1}{2}YY^T) \mid Y \in \mathbb{R}^{n\times m}\right\} = \left\{(Y, W) \in \mathbb{E} \;\middle|\; W + \frac{1}{2}YY^T \preceq 0\right\},$$

and

$$\mathrm{int}\left(\overline{\mathrm{conv}}\left\{(Y, -\frac{1}{2}YY^T) \mid Y \in \mathbb{R}^{n\times m}\right\}\right) = \left\{(Y, W) \in \mathbb{E} \;\middle|\; W + \frac{1}{2}YY^T \prec 0\right\}.$$

We conclude this section by giving representations for the horizon cone and polar of $\Omega(A, B)$.

Proposition 5.22 (The polar of $\Omega(A, B)$)**.** *Let $\Omega(A, B)$ be as given in* (5.13)*. Then*

$$\Omega(A, B)^\circ = \left\{(X, V) \;\middle|\; \begin{array}{c} \mathrm{rge}\,\binom{X}{B} \subset \mathrm{rge}\, M(V),\ V \in \mathcal{K}_A, \\ \frac{1}{2}\mathrm{tr}\left(\binom{X}{B}^T M(V)^\dagger \binom{X}{B}\right) \le 1 \end{array}\right\}.$$

Moreover,

$$\Omega(A, B)^\infty = \{0_{n\times m}\} \times \mathcal{K}_A^\circ \tag{5.25}$$

and

$$(\Omega(A, B)^\circ)^\infty = \left\{(X, V) \;\middle|\; \begin{array}{c} \mathrm{rge}\,\binom{X}{B} \subset \mathrm{rge}\, M(V),\ V \in \mathcal{K}_A, \\ \frac{1}{2}\mathrm{tr}\left(\binom{X}{B}^T M(V)^\dagger \binom{X}{B}\right) \le 0 \end{array}\right\}. \tag{5.26}$$

Proof. For any nonempty convex set $C \subset \mathbb{E}$, observe that

$$\{z \mid \sigma_C(z) \le 1\} = \{z \mid \langle z, x\rangle \le 1, (x \in C)\} = C^\circ,$$

see Definition 3.72. Consequently, our expression for $\Omega(A, B)^\circ$ follows from Theorem 5.8.

To see (5.25), let $(Y, W) \in \Omega(A, B)$ and recall that $(S, T) \in \Omega(A, B)^\infty$ if and only if $(Y + tS, W + tT) \in \Omega(A, B)$ for all $t \ge 0$. In particular, for $(S, T) \in \Omega(A, B)^\infty$, we have $A(Y + tS) = B$ and

$$\frac{1}{2}\left[YY^T + t(SY^T + YS^T) + \frac{t^2}{2}SS^T\right] + (W + tT) \in \mathcal{K}_A^\circ \quad (t > 0). \tag{5.27}$$

Consequently, $AS = 0$ and, if we divide (5.27) by t^2 and let $t \uparrow \infty$, we see that $SS^T \in \mathcal{K}_A^\circ$. But $SS^T \in \mathcal{K}_A$ since $\mathrm{rge}\, S \subset \ker A$, so we must have $S = 0$. If we now divide (5.27) by

t and let $t \uparrow \infty$, we find that $T \in \mathcal{K}_A^\circ$. Hence the set on the left-hand side of (5.25) is contained in the one on the right. To see the reverse inclusion, simply recall that $\mathcal{K}_A^\circ$ is a closed convex cone so that $\mathcal{K}_A^\circ + \mathcal{K}_A^\circ \subset \mathcal{K}_A^\circ$.

Finally, we show (5.26). Since $(0,0) \in \Omega(A,B)^\circ$, we have $(S,T) \in (\Omega(A,B)^\circ)^\infty$ if and only if $(tS, tT) \in \Omega(A,B)^\circ$ for all $t > 0$, or equivalently, for all $t > 0$,

$$tT \in \mathcal{K}_A \quad \text{and} \quad \exists\ (Y_t, Z_t) \in \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times m} \ \text{ s.t. } \begin{pmatrix} tS \\ B \end{pmatrix} = M(tT) \begin{pmatrix} Y_t \\ Z_t \end{pmatrix}$$
$$\text{with } \ \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} Y_t \\ Z_t \end{pmatrix}^T M(tT) \begin{pmatrix} Y_t \\ Z_t \end{pmatrix} \right) \le 1,$$

or equivalently, by taking $\widehat{Z}_t := t^{-1} Z_t$,

$$T \in \mathcal{K}_A \quad \text{and} \quad \exists\ (Y_t, \widehat{Z}_t) \in \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times m} \ \text{ s.t. } \begin{pmatrix} S \\ B \end{pmatrix} = M(T) \begin{pmatrix} Y_t \\ \widehat{Z}_t \end{pmatrix}$$
$$\text{with } \ \frac{t}{2}\mathrm{tr}\left(\begin{pmatrix} Y_t \\ \widehat{Z}_t \end{pmatrix}^T M(T) \begin{pmatrix} Y_t \\ \widehat{Z}_t \end{pmatrix} \right) \le 1.$$

If we take $\binom{Y_t}{\widehat{Z}_t} := M(T)^\dagger \binom{S}{B}$, we find that $(S,T) \in (\Omega(A,B)^\circ)^\infty$ if and only if

$$T \in \mathcal{K}_A \quad \text{and} \quad \frac{t}{2}\mathrm{tr}\left(\begin{pmatrix} S \\ B \end{pmatrix}^T M(T)^\dagger \begin{pmatrix} S \\ B \end{pmatrix} \right) \le 1 \quad (t > 0),$$

which proves the result. □

5.4 $\sigma_{\Omega(A,0)}$ as a gauge

Note that the origin is an element of $\Omega(A,B)$ if and only if $B = 0$. In this case the support function of $\Omega(A,0)$ equals the gauge of $\Omega(A,0)^\circ$ This fact and an explicit representation for both $\gamma_{\Omega(A,0)^\circ}$ and $\gamma_{\Omega(A,0)}$ will be given in the following theorem.

Theorem 5.23 ($\sigma_{\mathcal{D}(A,0)}$ is a gauge)**.** *Let $\Omega(A,B)$ be as given in* (5.13)*. Then*

$$\sigma_{\Omega(A,0)}(X,V) = \gamma_{\Omega(A,0)^\circ}(X,V) = \gamma^\circ_{\Omega(A,0)}(X,V), \tag{5.28}$$

and

$$\gamma_{\Omega(A,0)}(Y,W) = \sigma_{\Omega(A,0)^\circ}(Y,W) = \begin{cases} \frac{1}{2}\sigma_{\min}^{-1}(-Y^\dagger W (Y^\dagger)^T) & \text{if } \operatorname{rge} Y \subset \ker A \cap \operatorname{rge} W,\ W \in \mathcal{K}_A^\circ, \\ +\infty & \text{else,} \end{cases} \tag{5.29}$$

where $\sigma_{\min}(-Y^\dagger W (Y^\dagger)^T)$ is the smallest singular value of $-Y^\dagger W (Y^\dagger)^T$ when such a singular value exists and $+\infty$ otherwise, e.g. when $Y = 0$. Here we interpret $\frac{1}{\infty}$ as 0 $(0 = \frac{1}{\infty})$, and so, in particular, $\gamma_{\Omega(A,0)}(0,W) = \delta_{\mathcal{K}_A^\circ}(W)$.

Proof. The statement in (5.28) follows from Proposition 3.74. To show (5.29), first observe that

$$t\,\Omega(A,0)=\left\{(Y,W)\;\middle|\; AY=0 \quad\text{and}\quad \frac{1}{2}YY^T+tW\in\mathcal{K}_A^\circ\right\}, \tag{5.30}$$

whose straightforward proof is left to the reader.

Given $\bar{t}\ge 0$, by (5.30), $(Y,W)\in t\Omega(A,0)$ for all $t>\bar{t}$ if and only if $AY=0$ and $\frac{1}{2}YY^T+tW\in\mathcal{K}_A^\circ$ for all $t>\bar{t}$. By Proposition 5.4 a), this is equivalent to $AY=0$ and

$$\frac{1}{2}YY^T+tW=P\left(\frac{1}{2}YY^T+tW\right)P\preceq 0 \quad (t>\bar{t}), \tag{5.31}$$

where, again, P is the orthogonal projection onto $\ker A$. Dividing this inequality by t and taking the limit as $t\uparrow\infty$ tells us that $W=PWP\preceq 0$. Since YY^T is positive semidefinite, inequality (5.31) also tells us that $\ker W\subset\ker Y^T$, i.e. $\operatorname{rge} Y\subset\operatorname{rge} W$. Consequently,

$$\operatorname{dom}\gamma_{\Omega(A,0)}\subset\left\{(Y,W)\mid \operatorname{rge}Y\subset\ker A\cap\operatorname{rge}W,\, W\in\mathcal{K}_A^\circ\right\}.$$

Now suppose $(Y,W)\in\operatorname{dom}\gamma_{\Omega(A,0)}$. Let $Y=U\Sigma V^T$ be the reduced singular-value decomposition of Y where Σ is an invertible diagonal matrix and U,V have orthonormal columns. Since $\operatorname{rge}Y\subset\operatorname{rge}W=(\ker W)^\perp$, we know that U^TWU is negative definite, and so $\Sigma^{-1}U^TWU\Sigma^{-1}$ is also negative definite. Multiplying (5.31) on the left by $\Sigma^{-1}U^T$ and on the right by $U\Sigma^{-1}$ gives

$$\mu I\preceq -2\Sigma^{-1}U^TWU\Sigma^{-1} \quad (0<\mu\le\bar{\mu}),$$

where $\bar{\mu}=\bar{t}^{-1}$. The largest $\bar{\mu}$ satisfying this inequality is

$$\sigma_{\min}(-2Y^\dagger W(Y^\dagger)^T)=\sigma_{\min}(-2\Sigma^{-1}U^TWU\Sigma^{-1})>0,$$

or equivalently, the smallest possible $\bar{t}$ in (5.31) is $1/\sigma_{\min}(-2Y^\dagger W(Y^\dagger)^T)$, which proves the result. $\square$

5.5 Applications

5.5.1 Conjugate of variational Gram functions

Given a set $M\subset\mathbb{S}^n_+$, the associated *variational Gram function (VGF)* [9, 21] is given by

$$\Omega_M:\mathbb{R}^{n\times m}\to\mathbb{R}\cup\{+\infty\},\quad \Omega_M(X)=\frac{1}{2}\sigma_M(XX^T).$$

Since $\sigma_M=\sigma_{\overline{\operatorname{conv}}\,M}$ there is no loss in generality to assume that M is closed and convex. For the remainder we let

$$F:\mathbb{R}^{n\times m}\to\mathbb{S}^n,\quad F(X)=\frac{1}{2}XX^T. \tag{5.32}$$

Then $\Omega_M=\sigma_M\circ F$ fits the composite scheme studied in Section 4.3. It is obvious that $\operatorname{rge}F=\mathbb{S}^n_+$. The following lemma clarifies the K-convexity properties of F.

Lemma 5.24. *Let F be given by* (5.32)*. Then $\mathbb{S}^n_+$ is the smallest closed convex cone in $\mathbb{S}^n$ with respect to which F is convex.*

Proof. Let K be the smallest closed convex cone in $\mathbb{S}^n$ such that F is K-convex. On the one hand, since F is $\mathbb{S}^n_+$-convex, $K \subset \mathbb{S}^n_+$. On the other hand, by Lemma 4.3,

$$(-K)^\circ = K_F := \{V \in \mathbb{S}^n \mid \langle V, F\rangle \text{ is convex}\}.$$

Now fixing $V \in \mathbb{S}^n$ for all $X \in \mathbb{R}^{n\times m}$ the mapping $\nabla^2 \langle V, F\rangle (X) : \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \to \mathbb{R}$ is given by

$$\nabla^2 \langle V, F\rangle (X)[D, H] = \left\langle V, \frac{1}{2}(HD^T + DH^T)\right\rangle.$$

Clearly, this symmetric bilinear form is positive semidefinite if and only if $V \in \mathbb{S}^n_+$, which proves that

$$K_F = \mathbb{S}^n_+.$$

Finally, by taking the polarity and using the fact that $(\mathbb{S}^n_+)^\circ = -\mathbb{S}^n_+$ we get

$$K = -(-K)^\circ = -K_F^\circ = -(\mathbb{S}^n_+)^\circ = \mathbb{S}^n_+.$$

□

Corollary 5.25. *Let $M \subset \mathbb{S}^n_+$ be nonempty, closed and convex, and let F be given by* (5.32)*. Then the following hold:*

a) *We have $-\operatorname{hzn}\sigma_M \supset \mathbb{S}^n_+$. In particular, F is $(-\operatorname{hzn}\sigma_M)$-convex.*

b) $\operatorname{ri}(\operatorname{hzn}\sigma_M) \subset \operatorname{ri}(\operatorname{dom}\sigma_M)$.

c) *The following are equivalent:*

i) $\operatorname{rge} F \cap \operatorname{ri}(\operatorname{dom}\sigma_M - \mathbb{S}^n_+) \neq \emptyset$;

ii) $\operatorname{rge} F \cap \operatorname{ri}(\operatorname{dom}\sigma_M + \operatorname{hzn}\sigma_M) \neq \emptyset$;

iii) *M is bounded.*

Proof. a) We have $\overline{\operatorname{cone}}\, M \subset \mathbb{S}^n_+$ as $M \subset \mathbb{S}^n_+$, and hence,

$$-\operatorname{hzn}\sigma_M = -(\overline{\operatorname{cone}}\, M)^\circ \supset -(\mathbb{S}^n_+)^\circ = \mathbb{S}^n_+.$$

b) Since M is a subset of the closed convex cone $\overline{\operatorname{cone}}\, M$, $\overline{\operatorname{cone}}\, M = (\overline{\operatorname{cone}}\, M)^\infty \supset M^\infty$. Furthermore, because they lie entirely in $\mathbb{S}^n_+$, they are both pointed and hence, by Exercise 3.2.5, $\emptyset \neq \operatorname{int}[(\overline{\operatorname{cone}}\, M)^\circ] \subset \operatorname{int}[(M^\infty)^\circ]$. Therefore,

$$\begin{aligned}\operatorname{ri}(\operatorname{hzn}\sigma_M) &= \operatorname{ri}[(\overline{\operatorname{cone}}\, M)^\circ] = \operatorname{int}[(\overline{\operatorname{cone}}\, M)^\circ]\\ &\subset \operatorname{int}[(M^\infty)^\circ] = \operatorname{ri}[(M^\infty)^\circ]\\ &= \operatorname{ri}(\overline{\operatorname{dom}\sigma_M}) = \operatorname{ri}(\operatorname{dom}\sigma_M),\end{aligned}$$

where the equality between the second line and the third line follows from Theorem 3.81 and the equality in the last line follows from Proposition 3.23 b).

c) i) $\Rightarrow$ ii): This follows immediately from a) and the fact that both $\mathrm{dom}\,\sigma_M$ and $\mathrm{hzn}\,\sigma_M$ have nonempty interiors and Corollary 3.27.

ii)$\Leftrightarrow$iii): Observe that

$$\emptyset \neq \mathrm{int}\,[(M^\infty)^\circ] = \{W \in \mathbb{S}^n \mid \mathrm{tr}\,(WV) < 0 \; (V \in M^\infty \setminus \{0\})\}.$$

In particular, also taking into account a), we have

$$\begin{aligned} \mathrm{ri}\,(\mathrm{dom}\,\sigma_M + \mathrm{hzn}\,\sigma_M) &= \mathrm{ri}\,(\mathrm{dom}\,\sigma_M) + \mathrm{ri}\,(\mathrm{hzn}\,\sigma_M) \\ &= \mathrm{ri}\,[(M^\infty)^\circ] + \mathrm{ri}\,[(\overline{\mathrm{cone}}\; M)^\circ] \\ &= \mathrm{ri}\,[(M^\infty)^\circ + (\overline{\mathrm{cone}}\; M)^\circ] \\ &= \mathrm{ri}\,[(M^\infty)^\circ] = \mathrm{int}\,[(M^\infty)]^\circ \end{aligned}$$

where $(M^\infty)^\circ + (\overline{\mathrm{cone}}\; M)^\circ = (M^\infty)^\circ$ as $0 \in (\overline{\mathrm{cone}}\; M)^\circ \subset (M^\infty)^\circ$ are convex cones. As $\mathrm{rge}\, F = \mathbb{S}^n_+$, condition ii) is equivalent to the condition

$$F := \left\{W \in \mathbb{S}^n_+ \mid \mathrm{tr}\,(VW) < 0 \; (V \in M^\infty \setminus \{0\})\right\} \neq \emptyset.$$

We claim that

$$F := \begin{cases} \mathbb{S}^n_+ & \text{if } \; M \text{ is bounded,} \\ \emptyset & \text{if } \; \text{else.} \end{cases}$$

The first case is clear, since $M^\infty = \{0\}$ if (and only if) M is bounded, see Proposition 3.39, in which case the condition restricting F is vacuous.

On the other hand, if M is unbounded, then there exists $V \in M^\infty \setminus \{0\} \subset \mathbb{S}^n_+ \setminus \{0\}$. But then $\mathrm{tr}\,(VW) \geq 0$ for all $W \in \mathbb{S}^n_+$, see Exercise 1.0.1, which proves the second case.

All in all, we have established the equivalence between ii) and iii).

iii) $\Rightarrow$ i): Follows readily from the fact that $(M^\infty)^\circ = \{0\}^\circ = \mathbb{S}^n$ here.

□

Our analysis in Section 4.3 combined with our findings in Section 5.2.3 allows for a very short proof of the conjugate function Ω^*_M in case M is bounded (hence compact). This covers what was proven in [21, Proposition 3.4] entirely and one case of [9, Proposition 5.10].

Corollary 5.25 c) shows that our framework does not apply when M is unbounded as the crucial condition (4.4) is violated then.

Theorem 5.26. *Let $M \subset \mathbb{S}^n_+$ be nonempty, convex and compact. Then Ω^*_M is finite-valued and given by*

$$\Omega^*(X) \;\; = \;\; \frac{1}{2} \min_{V \in M} \left\{\mathrm{tr}\,(X^T V^\dagger X) \mid \mathrm{rge}\, X \subset \mathrm{rge}\, V\right\}.$$

Proof. Let $K := \mathbb{S}^n_+$. Recall that $(\overline{\mathrm{cone}}\; M)^\circ = \mathrm{hzn}\,\sigma_M$. Then F given by (5.32) is K-convex by Lemma 5.24 and Corollary 5.25 a), and g is K-increasing by Lemma 4.10 and Corollary 5.25 a). By Corollary 5.25 c) we find that condition (4.4) for $\sigma_M \circ M$ and K is satisfied if (and only if) M is bounded. Hence, we can apply Corollary 4.9 and Lemma 4.5 a) and Corollary 5.11 to infer that

$$\begin{aligned}
\Omega^*(X) &= \min_{V\in -K^\circ} \delta_M(V) + \langle V,\, F\rangle^*(X)\\
&= \min_{V\in M} \sigma_{K\text{-epi}\,F}(X,-V)\\
&= \min_{V\in M} \sigma_\Omega(X,V)\\
&= \min_{V\in M} \gamma(X,V)\\
&= \min_{V\in M}\left\{\frac{1}{2}\mathrm{tr}\,(X^TV^\dagger X)\mid \mathrm{rge}\,X\subset \mathrm{rge}\,V\right\}
\end{aligned}$$

As M is compact this proves also the finite-valuedness. □

5.5.2 Relation of the GMF and the nuclear norm

We now want to exapand on the intriguing relation between the GMF and the nuclear that was already foreshadowed in Example 5.1. Here we use the following notation for the matrix $A\in\mathbb{R}^{p\times n}$.

$$\mathrm{Ker}A := \left\{V\in\mathbb{R}^{n\times n}\mid AV=0\right\}\quad\text{and}\quad \mathrm{Rge}A := \left\{W\in\mathbb{R}^{n\times n}\mid \mathrm{rge}\,W\subset\mathrm{rge}\,A\right\}.$$

Theorem 5.27. *Let $p:\mathbb{R}^{n\times m}\to\overline{\mathbb{R}}$ be defined by*

$$p(X)=\inf_{V\in\mathbb{S}^n}\sigma_{\Omega(A,0)}(X,V)+\left\langle \bar U,\, V\right\rangle$$

for some $\bar U\in\mathbb{S}^n_+\cap\mathrm{Ker}A$ and set $C(\bar U):=\left\{Y\;\middle|\;\frac{1}{2}YY^T\preceq\bar U\right\}$. Then we have:

a) $p^*=\delta_{C(\bar U)\cap\mathrm{Ker}A}$ is closed, proper, convex.

b) $p=\sigma_{C(\bar U)\cap\mathrm{Ker}A}=\gamma_{C(\bar U)^\circ+\mathrm{Rge}A^T}$ is sublinear, finite-valued, nonnegative and symmetric (i.e. a seminorm).

c) If $\bar U\succ 0$ with $2\bar U=LL^T$ ($L\in\mathbb{R}^{n\times n}$) and $A=0$ then $p=\sigma_{C(\bar U)}=\|L^T(\cdot)\|_*$, i.e. p is a norm with $C(\bar U)^\circ$ as its unit ball and $\gamma_{C(\bar U)}$ as its dual norm.

Proof. a) Observe that $\sigma_{\Omega(A,0)}(X,V)+\left\langle\bar U,\, V\right\rangle=\sigma_{\Omega(A,0)+\{0\}\times\{\bar U\}}(X,V)$. As $\Omega(A,0)+\{0\}\times\{\bar U\}$ is nonempty, closed and convex, Theorem 3.101 yields $p^*=\delta_{\Omega(A,0)+\{0\}\times\{\bar U\}}(\cdot,0)$ which is closed and convex. Now observe that

$$\begin{aligned}
\mathrm{dom}\,\delta_{\Omega(A,0)+\{0\}\times\{\bar U\}}(\cdot,0) &= \left\{Y\;\middle|\; Y\in\mathrm{Ker}A,\,\frac{1}{2}YY^T-\bar U\in\mathcal{K}_A^\circ\right\}\\
&= \mathrm{Ker}A\cap C(\bar U)\\
&\neq \emptyset.
\end{aligned}$$

Here we use the first identity uses the definition of $\Omega(A,0)$. The second one is due to the fact that with $\mathrm{Rge}YY^T=\mathrm{Rge}Y\subset\mathrm{Ker}A$ and $\bar U\in\mathrm{Ker}A$. The nonemptiness is clear as

$0 \in \mathrm{Ker}A \cap C(\bar{U})$. All in all, we have $p^* = \delta_{\mathrm{Ker}A\cap C(\bar{U})}$ which is closed proper, convex, see Theorem 3.101.

b) We have

$$\begin{aligned} p &= p^{**} \\ &= \sigma_{C(\bar{U})\cap \mathrm{Ker}A} \\ &= \gamma_{(C(\bar{U})\cap \mathrm{Ker}A)^\circ} \\ &= \gamma_{\mathrm{cl}\,(C(\bar{U})^\circ + \mathrm{Rge}A^T)} \\ &= \gamma_{C(\bar{U})^\circ + \mathrm{Rge}A^T}. \end{aligned}$$

The first identity is due to Theorem 3.101 c). The second uses a), the third follows from Proposition 3.74. The sublinearity of p is clear. The finite-valuedness is also clear since the domain of p is the whole space (clear?) and p is proper. Since $0 \in C(\bar{U})$ the nonnegativity follows as well, and the symmetry is due to the symmetry of $C(\bar{U})$.

c) Consider the case $\bar{U} = \frac{1}{2}I$: By part a), we have $p^* = \delta_{\{Y \,|\, YY^T \preceq I\}}$. Observe that

$$\left\{Y \;\middle|\; YY^T \preceq I\right\} = \{Y \mid \|Y\|_2 \leq 1\} =: \mathbb{B}_\Lambda$$

is the closed unit ball of the spectral norm. Therefore, $p = \sigma_{\mathbb{B}_\Lambda} = \|\cdot\|_{\mathbb{B}_\Lambda^\circ} = \|\cdot\|_*$, cf. Corollary 3.76.

To prove the general case suppose that $2\bar{U} = LL^T$. Then it is clear that $C(\bar{U}) = \left\{Y \;\middle|\; L^{-1}Y \in C(\frac{1}{2}I)\right\}$, and therefore

$$\begin{aligned} p(X) &= \sigma_{C(\bar{U})}(X) \\ &= \sup_{Y:L^{-1}Y\in C(\frac{1}{2}I)} \langle Y, X\rangle \\ &= \sup_{Y:L^{-1}Y\in C(\frac{1}{2}I)} \left\langle L^{-1}Y, L^T X\right\rangle \\ &= \sigma_{C(\frac{1}{2}I)}(L^T X) \\ &= \|L^T X\|_*. \end{aligned}$$

Here the first identity is due to part b) (with $A = 0$) and the last one follows from the special case considered above.

□

We point out that Theorem 5.27 significantly generalizes the result by Hsieh and Olsen eluded to in Example 5.1.

For another result along these lines, which also generalized the Hsieh and Olsen result, we refer the interested reader to [7, Theorem 5.7].

A Background from Linear Algebra

Here we gather some background material and notation from linear algebra that we use in our study. Most of the results (unless stated otherwise) can be found in e.g. [17, 18].

Symmetric and orthogonal matrices

Recall that the set

$$\mathrm{O}(n) := \left\{ A \in \mathbb{R}^{n\times n} \;\middle|\; A^T A = I \right\}$$

is a group (in fact, a subgroup of the the invertible matrices in $\mathbb{R}^{n\times n}$) called the *orthogonal group* with its members being called *orthogonal matrices*.

An important subspace of $\mathbb{R}^{n\times n}$ is

$$\mathbb{S}^n = \left\{ A \in \mathbb{R}^{n\times n} \;\middle|\; A^T = A \right\},$$

the space of all $n \times n$ *symmetric matrices*.

In Linear Algebra it is shown that any symmetric matrix is orthogonally similar to a diagonal matrix with real entries (the eigenvalues), which is usually subsumed in the following theorem.

Theorem A.1 (Spectral Theorem)**.** *Let $A \in \mathbb{S}^n$. Then there exists $U \in \mathrm{O}(n)$ orthogonal such that*

$$A = U^T \mathrm{diag}\,(\lambda_1, \dots, \lambda_n) U$$

and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Some important subsets of $\mathbb{S}^n$ are:

- $\mathbb{S}^n_+ := \left\{ A \in \mathbb{S}^n \;\middle|\; x^T A x \geq 0 \; (x \in \mathbb{R}^n) \right\}$ *(positive semidefinite matrices)*,
- $\mathbb{S}^n_{++} := \left\{ A \in \mathbb{S}^n \;\middle|\; x^T A x > 0 \; (x \in \mathbb{R}^n \setminus \{0\}) \right\}$ *(positive definite matrices)*,
- $\mathbb{S}^n_- := \left\{ A \in \mathbb{S}^n \;\middle|\; x^T A x \leq 0 \; (x \in \mathbb{R}^n) \right\}$ *(negative semidefinite matrices)*,
- $\mathbb{S}^n_{--} := \left\{ A \in \mathbb{S}^n \;\middle|\; x^T A x < 0 \; (x \in \mathbb{R}^n \setminus \{0\}) \right\}$ *(negative definite matrices)*.

Note that

$$\mathbb{S}^n_- = -\mathbb{S}^n_+ \quad \text{and} \quad \mathbb{S}^n_{--} = -\mathbb{S}^n_{++}.$$

For $A, B \in \mathbb{S}^n$, we also make use of the convention

$$A \succeq B \quad :\Longleftrightarrow \quad A - B \in \mathbb{S}^n_+$$

and

$$A \succ B \quad :\Longleftrightarrow \quad A - B \in \mathbb{S}^n_{++}.$$

An immediate consequence of the spectral theorem is the existence of a square root of a positive semidefinite matrix.

Corollary A.2 (Square root of a positive semidefinite matrix)**.** *For $A \in \mathbb{S}^n_+$ there exists a unique matrix $B \in \mathbb{S}^n_+$ such that $B^2 = A$.*

In the scenario of Corollary A.2, we put $\sqrt{A} := A^{1/2} := B$ and call it the square root of A. Moreover, we define $A^{-1/2} := \sqrt{A^{-1}} = (\sqrt{A})^{-1}$.

Singular value decomposition and the Moore-Penrose Pseudoinverse

An important theoretical and computational tool for matrix analysis is the following well-known theorem which is also a consequence of the spectral theorem applied to the positive semidefinite matrix $\sqrt{A^T A}$.

Theorem A.3 (Singular-value decomposition)**.** *Let $A \in \mathbb{R}^{m\times n}$ and put $r := \operatorname{rank} A$. Then there exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ as well as a matrix $\Sigma \in \mathbb{R}^{m\times n}$ with*

$$\Sigma = \begin{pmatrix} \operatorname{diag}(\sigma_1, \dots \sigma_r) & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_1, \dots, \sigma_r > 0$$

such that $A = U\Sigma V^T$.

The scalars $\sigma_1, \dots, \sigma_r$ are called the singular-values of A, and they coincide with the positive eigenvalues of $\sqrt{A^T A}$.

The singular-value decomposition of a matrix gives rise to a whole class of matrix norms, namely the *Schatten norms*. For $A \in \mathbb{R}^{m\times n}$ the *Schatten p-norm* is the p-norm of the vector of singular values, i.e.

$$\|A\|_p := \|\sigma\|_p := \left(\sum_{i=1}^{r} \sigma_i^p\right)^{\frac{1}{p}},$$

where $\sigma = (\sigma_1, \dots, \sigma_r)$ is the vector of singular-values of A. The *nuclear norm* is the Schatten 1-norm, and often denoted by $\|\cdot\|_*$.

Theorem A.4 (Moore-Penrose pseudoinverse)**.** *Let $A \in \mathbb{S}^n_+$ with $\operatorname{rank} A = r$ and the spectral decomposition*

$$A = Q\Lambda Q^T \quad \text{with} \quad \Lambda = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}, \quad Q \in \mathrm{O}(n).$$

Then the matrix

$$A^\dagger := Q\Lambda^\dagger Q^T \quad \text{with} \quad \Lambda^\dagger := \begin{pmatrix} \lambda_1^{-1} & & & & \\ & \ddots & & & \\ & & \lambda_r^{-1} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

called the (Moore-Penrose) pseudoinverse *of A, has the following properties:*

a) $AA^\dagger A = A$ *and* $A^\dagger A A^\dagger = A^\dagger$*;*

b) $(AA^\dagger)^T = AA^\dagger$ *and* $(A^\dagger A)^T = A^\dagger A$*;*

c) $(A^\dagger)^T = (A^T)^\dagger$*;*

d) If $A \succ 0$, then $A^\dagger = A^{-1}$;

e) $AA^\dagger = P_{\operatorname{rge} A}$, i.e. $AA^\dagger$ is the projection onto the image of A. In particular, if $b \in \operatorname{rge} A$, we have
$$\{x \in \mathbb{R}^n \mid Ax = b\} = A^\dagger b + \ker A.$$

In fact, the Moore-Penrose pseudoinverse can be uniquely defined through properties a) and b) from above for any matrix $A \in \mathbb{C}^{m\times n}$, see, e.g. [18], but we confine ourselves with the positive semidefinite case.

Using the Moore-Penrose pseudoinverse, one has the following extension of the Schur complement, see [16, Th. 16.1] for a proof or [6, App. A.5.5].

Lemma A.5 (Schur complement)**.** *Let* $S \in \mathbb{S}^n, T \in \mathbb{S}^m, R \in \mathbb{R}^{n\times m}$. *Then*

$$\begin{pmatrix} S & R \\ R^T & T \end{pmatrix} \succeq 0 \quad \Longleftrightarrow \quad [S \succeq 0, \quad \operatorname{rge} R \subset \operatorname{rge} S, \quad T - R^T S^\dagger R \succeq 0].$$

The next result is referred to in the literature as *Finsler's Lemma* and originally goes back to [14], and can also be found in [10, Th. 2].

Lemma A.6 (Finsler's lemma)**.** *Let* $A \in \mathbb{R}^{p\times n}$ *and* $V \in \mathbb{S}^n$. *Then*

$$V \succ_{\ker A} 0 \quad \Longleftrightarrow \quad \exists \varepsilon > 0: \; V + \varepsilon A^T A \succ 0.$$

Exercises for Section A

1.0.1 **(Trace of product of semdefinite matrices)** Let $A, B \in \mathbb{S}^n_+$.

a) Is AB symmetric positive definite?

b) Show that $\operatorname{tr}(AB) \geq 0$.

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