The maximum entropy on the mean method for linear inverse problems

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Higher level approach to linear inverse problems

The canonical linear inverse problem $Cx \approx b$ is usually solved via an optimization problem

$$\min_{x \in \mathbb{R}^d} \left\{ \frac{1}{2} \|Cx - b\|^2 + R(x) \right\}$$

- $C$: linear (forward) operator
- $b$: measurement vector
- $R$: (convex) regularizer

Higher level approach: Interpret the ground truth as a random vector with unknown distribution. Solve for a distribution $Q$ that is close to a prior (guess) $P$ and such that its expectation $\mathbf{E}_Q$ satisfies $C \cdot \mathbf{E}_Q \approx b$.

What is the information theoretic foundation for this?

Principle of Maximum Entropy: "The probability distribution which is maximally non-committal with regard to missing information among all the distributions that agree with the present knowledge is the one with the maximum entropy." (E.T. Jaynes, 1957)

\[ i.e. \quad \mathbf{E}_Q = \int_{\Omega} y dQ(y) \]
Let $P$ be a (prior) distribution, i.e. a probability measure on $\Omega \subset \mathbb{R}^n$.

The measure of compliance of another distribution $Q$ with $P$ is measured by the \textbf{Kullback-Leibler divergence} $\text{KL}(\cdot \mid \cdot) : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^2 \to \mathbb{R} \cup \{+\infty\}$,

$$\text{KL}(Q \mid P) = \begin{cases} \int_{\Omega} \log \left(\frac{dQ}{dP}\right) dQ, & Q \ll P, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $\frac{dQ}{dP}$ is the \textit{Radon-Nikodym derivative}.

- $\text{KL}(\cdot \mid \cdot)$ is convex, $\text{KL}(\cdot \mid P)$ strictly convex for all $P \in \mathcal{P}(\Omega)$.
- $\text{KL}(Q \mid P) \geq 0$; equality if and only if $Q = P$ a.e.
KL divergence concretely

Let $P \in \mathcal{P}(\Omega)$ be our prior/reference distribution. We are mainly interested in two cases.

1. $\Omega = \mathbb{R}^n$ and $P$ is absolutely continuous w.r.t. the Lebesgue measure $\mu$, i.e. has a density $p = \frac{dP}{d\mu}$. In this case, if $Q \ll P$, $Q$ has a density $q$, and

   $$\text{KL}(Q \mid P) = \int_{\mathbb{R}^n} \log \left( \frac{q(x)}{p(x)} \right) q(x)dx.$$ 

2. $P$ is a discrete probability distribution, i.e. $\Omega$ is countable, and the probability mass function $p(x) = P(\{x\})$ has $\sum_{x \in \Omega} p(x) = 1$. Then $Q \ll P$ implies that $Q$ has a probability mass function $q$ and it holds that

   $$\text{KL}(Q \mid P) = \sum_{x \in \Omega} q(x) \log \left( \frac{q(x)}{p(x)} \right).$$

**Example:** Let $P$ be the uniform distribution on $\Omega := \{1, \ldots, N\}$, i.e. $p(i) = 1/N$ for all $i = 1, \ldots, N$. Then for $Q \ll P$ with PMF $q$, we have

   $$\text{KL}(Q \mid P) = \log(N) + \sum_{i=1}^{N} \log(q(i))q(i).$$
Given a prior \( P \in \mathcal{P}(\Omega) \), the \textit{maximum entropy on the mean method (MEMM)} for the linear inverse problem \( Cx \approx b \) reads:

Determine \( \bar{Q} \) as the solution of

\[
\min_{Q \in \mathcal{P}(\Omega)} \left\{ \frac{\alpha}{2} \| C \cdot E_Q - b \|^2 + \text{KL}(Q \mid P) \right\},
\]

and set \( \bar{x} := E_{\bar{Q}} \) to be the estimate for the ground truth.

A dual approach for finding \( \bar{x} \): Let \( \psi_P : \mathbb{R}^d \to \mathbb{R} \) be given by the \textit{cumulant generating function} of \( P \), i.e.

\[
\psi_P(y) = \log \int_{\Omega} \exp \langle y, \cdot \rangle \, dP = \log(M_P(y)).
\]

Under suitable assumptions\(^4\), the (Fenchel) dual of (1) reads (Rioux et al. ’21):

\[
\max_{\lambda \in \mathbb{R}^d} \left\{ \langle b, \lambda \rangle - \frac{1}{2\alpha} \| \lambda \|^2 - \psi_P(C^T \lambda) \right\}.
\]

Given the maximizer \( \bar{\lambda} \) of (2) one can recover \( \bar{x} \) via \( \bar{x} = \nabla \psi_P(C^T \bar{\lambda}) \).

\(^4\)E.g. \( \Omega \) compact.
Applications

To solve the dual problem, one can use standard solvers like e.g. L-BFGS which was successfully done for (blind and non-blind) deblurring of

- **Barcodes/QR-codes.**
  
  **Prior** $P$: Bernoulli.
  

- **General images.**
  
  **Prior** $P$: Uniform on box.
  

![QR code](image-url)

*Fig. 11. Out of focus image of a QR code.*

![Processed image](image-url)

*Fig. 12. Result of applying our method to a processed version of Fig. 11.*

[ Rioux et al. (2021) ]
The reformulated problem and the MEM functional

We observe that the (primal) MEMM problem can be reformulated as follows:

\[
\inf_{Q \in \mathcal{P}(\Omega)} \left\{ \frac{\alpha}{2} \| C \cdot E_Q - b \|^2 + \text{KL}(Q \mid P) \right\} = \inf_{y \in \mathbb{R}^d} \left\{ \frac{\alpha}{2} \| C \cdot y - b \|^2 + \inf_{\substack{Q \in \mathcal{P}(\Omega) : \ \text{E}_Q = y}} \text{KL}(Q \mid P) \right\}
\]

We define the MEM functional \( \kappa_P : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \),

\[
\kappa_P(y) = \inf_{Q \in \mathcal{P}(\Omega)} \{ \text{KL}(Q \mid P) + \delta_{\{0\}}(E_Q - y) \}.
\]

Then we obtain the reformulated problem

\[
\min_{y \in \mathbb{R}^d} \frac{\alpha}{2} \| C \cdot y - b \|^2 + \kappa_P(y).
\]

Since \( \kappa_P \geq 0 \), and \( \kappa_P(y) = 0 \) iff \( y = E_P \), \( \kappa_P \) can be interpreted as a regularizer promoting proximity to the prior distribution.

Q: Is this reformulation useful at all?
Interlude: convex analysis basics - the epigraphical perspective

Let $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$.

- $\text{dom } f := \{ x \in \mathbb{R}^d \mid f(x) < +\infty \}$ (domain);
- $\text{epi } f := \{ (x, \alpha) \in \mathbb{R}^d \times \mathbb{R} \mid f(x) \leq \alpha \}$ (epigraph).

We call $f$

- **convex** if $\text{epi } f$ is convex;
- **closed** (or **lower semicontinuous**) if $\text{epi } f$ is closed;
- **proper** if $\text{dom } f \neq \emptyset$.

$\Gamma_0 := \{ f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\} \mid f \text{ closed, proper, convex } \}$.

**Affine minorization principle:** Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ convex and proper, and $\bar{x} \in \text{ri } (\text{dom } f)^5$. Then there exists $v \in \mathbb{R}^n$ such that

$$f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle \quad \forall x \in \mathbb{R}^n.$$

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5The relative interior of a convex set is its interior in the relative topology w.r.t. its affine hull.

6 ‘What’s dead may never die!’
Let $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the function whose epigraph encodes the affine minorants of $\text{epi} f$ in that

$$\text{epi} f^* = \{(v, \beta) \mid \langle v, x \rangle - \beta \leq f(x) \quad \forall x \in \mathbb{R}^n \}.$$ 

Thus

$$f^*(v) \leq \beta \iff \sup_{x \in \mathbb{R}^n} \{\langle v, x \rangle - f(x)\} \leq \beta \quad \forall (v, \beta) \in \mathbb{R}^n \times \mathbb{R}.$$ 

Therefore

$$f^*(v) = \sup_{x \in \mathbb{R}^n} \{\langle v, x \rangle - f(x)\} \quad \forall v \in \mathbb{R}^n,$$

which is called the (Fenchel) conjugate of $f$. We set $f^{**} := (f^*)^*$. 

- $f^*$ closed and convex - proper if $f$ has an affine minorant
- If $f$ is convex and proper, then $f^*$ is proper (closed, convex), and

$$f^{**}(x) = (\text{cl} f)(x)^7.$$

- $f = f^{**} \iff f \in \Gamma_0$ (Fenchel-Moreau)

$^7(\text{cl} f) : x \in \mathbb{R}^n \mapsto \lim \inf_{z \to x} f(z)$, the closure of $f$, is the largest lsc minorant of $f$. 


Recall the *cumulant generating function* $\psi_P : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ of $P \in \mathcal{P}(\Omega)$, given by

$$\psi_P(\theta) := \log \int_{\Omega} \exp(\langle \theta, \cdot \rangle) dP = \log(M_P(\theta)).$$

The conjugate $\psi_P^* : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$,

$$\psi_P^*(y) := \sup_{\theta \in \mathbb{R}^d} \{\langle y, \theta \rangle - \psi_P(\theta)\}$$

is called *Cramér’s function* (fundamental in *large deviations theory*).

The key to computational tractability of the reformulated MEMM problem is to establish conditions (on $P$) under which Cramér’s function equals the MEM functional, i.e.

$$\kappa_P = \psi_P^*.$$

**Key ingredient:** Exponential families and Legendre-type functions.

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8Named after Swedish mathematician and statistician Harald Cramér who is considered as ‘one of the giants of statistical theory’.
Legendre-type functions

A function $\psi \in \Gamma_0$ is \textit{essentially smooth} if it satisfies the following conditions:

1. $\text{int}(\text{dom } \psi) \neq \emptyset$
2. $\psi$ is differentiable on $\text{int}(\text{dom } \psi)$
3. $\|\nabla \psi(x^k)\| \to \infty$ for any $\{x^k \in \text{int}(\text{dom } \psi)\} \to \bar{x} \in \text{bd}(\text{dom } \psi)$

If, in addition, $\psi$ is strictly convex on $\text{int}(\text{dom } \psi)$ then $\psi$ is called of \textit{Legendre type}.

\textbf{Rockafellar (1970):} For $\psi \in \Gamma_0$ of Legendre type, we have:

- $\psi^*$ is of Legendre type.
- $\nabla \psi : \text{int}(\text{dom } \psi) \to \text{int}(\text{dom } \psi^*)$ is a bijection (with $(\nabla \psi)^{-1} = \nabla \psi^*$).

$\Gamma_0$ : set of all closed, proper, convex functions (on $\mathbb{R}^d$)
Exponential families

Let $(\Omega, \mathcal{A}, P)$ be a probability space$^{10}$ with $P \ll \nu^{11}$. The natural parameter space for $P$ is defined by

$$\Theta_P := \left\{ \theta \in \mathbb{R}^d \mid \int_{\Omega} \exp(\langle \theta, \cdot \rangle) dP < +\infty \right\} (= \text{dom } \psi_P).$$

The standard exponential family generated by $P$ is given by

$$\mathcal{F}_P := \left\{ f_{P_\theta} \mid f_{P_\theta}(y) := \exp(\langle y, \theta \rangle - \psi_P(\theta)), \quad \theta \in \Theta_P \right\}.$$

Properties and connections

- $\int_{\Omega} f_{P_\theta} dP = 1$, thus $P_\theta := P \circ f_{P_\theta}^{-1}$ is a probability measure with $\frac{dP_\theta}{dP} = f_\theta$ ($\theta \in \Theta_P$).
- Under suitable assumptions: $\arg\min_{Q: E_Q = y} \{\text{KL}(Q \mid P) \} \in \mathcal{F}_P$.
- $\theta_1 \in \Theta_P, \theta_2 \in \text{int } (\Theta_P)$: $\text{KL}(P_{\theta_2} \mid P_{\theta_1}) = D_{\psi_P}(\theta_1, \theta_2)$ (Bregman distance).

$^{10}(\Omega, \mathcal{A})$ measurable and $P$ $\sigma$-finite works, too.
$^{11}\nu$: Lebesgue measure ($\Omega = \mathbb{R}^d$) or counting measure ($\Omega$ countable).
The (standard) exponential family \( \mathcal{F}_P \) is called

- **minimal** \(^{12}\) if \( \text{int} \, \Theta_P \neq \emptyset \) and \( \text{int} \, (\text{conv} \, S_P) \neq \emptyset \); \(^{13}\)
- **steep** if \( \psi_P \) is essentially smooth (automatically satisfied if \( \Theta_P \) open).

**Theorem (Regularity of \( \psi_P \), Brown 1986)**

Let \( \mathcal{F}_P \) be a minimal exponential family. Then:

(a) The log-cumulant generating function \( \psi_P \) is strictly convex on (the convex set) \( \Theta_P \).
(b) \( \psi_P \in C^\infty(\text{int} \, \Theta_P) \), and then \( \nabla \psi_P(\theta) = \mathbb{E}_{P,\theta} \).

**Corollary**

Let the exponential family \( \mathcal{F}_P \) be minimal and steep. Then:

(a) \( \psi_P \) (and hence \( \psi^*_P \)) is of Legendre type.
(b) \( \theta = \nabla \psi^*_P(\mathbb{E}_{P,\theta}) \).

\(^{12}\)This can essentially be assumed w.l.o.g.

\(^{13}\)\( S_P \): support of \( P \), i.e. the smallest closed set \( A \subset \Omega \) s.t. \( P(\Omega \setminus A) = 0 \).
Domain correspondences and the key inequality

Given $\psi$ of Legendre type, its Bregman distance is:

$$D_\psi(y, x) := \psi(y) - \psi(x) - \langle \nabla \psi(x), y - x \rangle \quad \forall (x, y) \in \text{int} \, (\text{dom } \psi) \times \text{dom } \psi.$$  

- $D_\psi \geq 0$ and $D_\psi(x, y) = 0 \iff x = y$;
- $D_\psi$ not symmetric in general, but $D_{\frac{1}{2} \| \cdot \|_2} = \frac{1}{2} \| x - y \|_2^2$;
- 

Lemma (Vaisbourd et al.)

Suppose $P \in \mathcal{P}(\Omega)$ generates a minimal and steep exponential family. Then:

(a) (Domain relations)

(i) If $S_P$ is countable, then $\text{dom } \kappa_P = \text{conv } S_P \subset \text{dom } \psi^*_P$;
(ii) If $S_P$ is uncountable, then $\text{dom } \kappa_P = \text{int} \, (\text{conv } S_P) = \text{dom } \psi^*_P$.

(b) For all $y \in \text{dom } \kappa_P$, $Q \ll P$ s.t. $E_Q = y$ and for all $\theta \in \text{int } \Theta_P$ we have

$$\psi^*_P(y) \leq \kappa_P(y) \leq \psi^*_P(y) + KL(Q \mid P_\theta) - D_{\psi^*_P}(y, \nabla \psi_P(\theta)).$$  (3)
Equivalence of MEM functional and Cramér’s function

\[
\psi^*_P = \kappa_P? \]

- \( y \in \text{int} (\text{conv } S_P): \)

\[
\text{int} (\text{conv } S_P) \subset \text{int} (\text{dom } \psi^*), \ \psi^* \text{ Legendre-type} \\
\implies \exists \theta \in \text{int} (\text{dom } \psi) = \text{int} \Theta_P : y = \nabla \psi_P(\theta) = E_{P\theta} \\
\xrightarrow{(3)} \psi^*_P(y) \leq \kappa_P(y) \leq \psi^*(y) + \underbrace{\text{KL}(P_\theta | P_\theta)}_{=0} - \underbrace{D_{\psi^*_P}(\nabla \psi_P(\theta), \nabla \psi_P(\theta))}_{=0}
\]

- \( y \in \text{bd} (\text{conv } S_P): \text{Can only occur when } S_P \text{ is countable.} \)

**Theorem \( \psi^*_P = \kappa_P, \text{ Vaisbourd et al.} \)**

*Suppose \( P \in \mathcal{P}(\Omega) \) generates a minimal and steep exponential family. Moreover, suppose one of the following holds:

- \( S_P \text{ is uncountable} \)
- \( S_P \text{ is countable and } \text{conv } S_P \text{ is closed (which is always the case if } S_P \text{ is finite).} \)

Then \( \kappa_P = \psi^*_P. \) In this case \( 0 \leq \kappa_P \in \Gamma_0 \) is of Legendre type and coercive.
How is $\kappa_P = \psi^*_P$ useful?

If $P \in \mathcal{P}(\Omega)$ is separable (i.e. $P = P_1 \times P_2 \times \cdots \times P_d$), then $M_P(\theta) = \prod_{i=1}^d M_{P_i}(\theta_i)$. Hence

$$\psi^*_P(y) = \sup_{\theta \in \mathbb{R}^d} \{ \langle y, \theta \rangle - \log M_P(\theta) \}$$

$$= \sum_{i=1}^d \sup_{\theta_i \in \mathbb{R}} \{ y_i \theta_i - \log M_{P_i}(\theta_i) \}.$$ 

In many cases this yields analytic formulas for $\psi^*_P$, i.e. $\kappa_P$ (even without separability!).

**Example:** If $P$ is the multivariate normal distribution $N(\mu, \Sigma)$ for $\Sigma \succ 0$, i.e. $M_P(\theta) = \exp \left( \langle \mu, \theta \rangle + \frac{1}{2} \theta^T \Sigma \theta \right)$, then

$$\psi^*_P(y) = \sup_{\theta \in \mathbb{R}^n} \{ \langle y, \theta \rangle - \log M_P(\theta) \}$$

$$= \sup_{\theta \in \mathbb{R}^n} \left\{ \langle y - \mu, \theta \rangle - \frac{1}{2} \theta^T \Sigma \theta \right\}$$

$$= \frac{1}{2} (y - \mu)^T \Sigma^{-1} (y - \mu).$$
Examples of Cramér’s function

<table>
<thead>
<tr>
<th>Reference Distribution ((P))</th>
<th>Cramér Rate Function ((\psi^*_P(y)))</th>
<th>(\text{dom } \psi^*_P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multivariate Normal (\mu \in \mathbb{R}^d, \Sigma \in \mathbb{S}^d, \Sigma &gt; 0)</td>
<td>(\frac{1}{2} (y - \mu)^T \Sigma^{-1} (y - \mu))</td>
<td>(\mathbb{R}^d)</td>
</tr>
<tr>
<td>Poisson ((\lambda \in \mathbb{R}_{++}))</td>
<td>(y \log(y/\lambda) - y + \lambda)</td>
<td>(\mathbb{R}_+)</td>
</tr>
<tr>
<td>Gamma ((\alpha, \beta \in \mathbb{R}_{++}))</td>
<td>(\beta y - \alpha + \alpha \log \left( \frac{\alpha}{\beta y} \right))</td>
<td>(\mathbb{R}_{++})</td>
</tr>
</tbody>
</table>

Normal-inverse Gaussian

\(\alpha, \beta, \delta \in \mathbb{R} : \alpha \geq |\beta|, \delta > 0, \gamma := \sqrt{\alpha^2 - \beta^2}\)

\[ \alpha \sqrt{\delta^2 + (y - \mu)^2} - \beta (y - \mu) - \delta \gamma \]

\(\mathbb{R}\)

| Multinomial \((p \in \Delta_d, n \in \mathbb{N})\) | \(\sum_{i=1}^{d} y_i \log \left( \frac{y_i}{np_i} \right)\) | \(n \Delta_d \cap I(p)^{14}\) |

In addition: Laplace, (Negative) Multinomial, Continuous/Discrete Uniform, Logistic, Exponential/Chi-Squared/Erlang (via Gamma), Binomial/Bernoulli/Categorical (via Multinomial), Negative Binomial & Shifted Geometric (via Negative Multinomial).

\[ I(p) := \{ x \in \mathbb{R}^d \mid x_i = 0 \text{ if } p_i = 0 \} \]
Let the following be given:

- \( \hat{y} \in \mathbb{R}^d \): observed data;
- \( S^* \subset \mathbb{R}^d \): admissible parameters;
- \( F_\Lambda := \{ P_\lambda \mid \lambda \in \Lambda \subset \mathbb{R}^d \} \): parameterized family of distributions\(^{15}\);
- \( P_{\hat{\lambda}} \in F_\Lambda \): reference distribution such that \( \hat{y} = E_{P_{\hat{\lambda}}} \).

We define the **MEM estimator** \( y_{MEM} \in \mathbb{R}^d \) by

\[
y_{MEM}(\hat{y}, F_\Lambda, S^*) := \arg\min_{y \in S^*} \psi_{P_{\hat{\lambda}}}^*(y).
\]

Under suitable assumptions on \( P_{\hat{\lambda}} \), the function \( \psi_{P_{\hat{\lambda}}}^* \) is coercive and strictly convex, which guarantees well-definedness of the MEM estimator.

\(^{15}\)not necessarily exponential
MEM vs. ML estimation

Let the following be given:

- \( \hat{y} \in \mathbb{R}^d \): observation;
- \( S \subset \mathbb{R}^m \): set of admissible parameters;
- \( F_{\lambda} := \{ P_{\lambda} | \lambda \in \Lambda \subset \mathbb{R}^m \} \): parameterized family of distributions with densities \( f_{\lambda} \);

The ubiquitous maximum likelihood estimator is given by

\[
\lambda_{ML}(\hat{y}, F_{\lambda}, S) := \text{argmax}_{\lambda \in S \cap \Lambda} \log f_{\lambda}(\hat{y}).
\]

It induces a distribution that is most likely to produce the given observation.

When \( F_{\lambda} \) is an exponential family induced by \( P \), and \( \hat{\lambda} := \nabla \psi^*_P(\hat{y}) \) then (under some technical assumptions) we have

\[
y_{MEM} = \psi^*_P(\lambda_{MEM})
\]

for

\[
\lambda_{MEM} = \text{argmin}_{\lambda \in S} \text{KL}(P_{\lambda} | P_{\hat{\lambda}}),
\]

whereas

\[
\lambda_{ML} = \text{argmin}_{\lambda \in S} \text{KL}(P_{\hat{\lambda}} | P_{\lambda}).
\]
Linear model based on MEM

Consider the linear inverse problem $C x \approx \hat{y}$ for some

- $\hat{y} \in D \subset \mathbb{R}^m$: measurement vector;
- $C \in \mathcal{C} \subset \mathbb{R}^{m \times d}$: measurement matrix (dictated by the problem).

Now consider:

- $F_{\Lambda} = \{ P_{\lambda} \mid \lambda \in \Lambda \subset \mathbb{R}^m \} \subset \mathcal{P}(\Omega)$: reference family;
- $\hat{P} := P_{\hat{\lambda}}$: reference distribution with $E_{\hat{P}} = \hat{y}$;
- $S^* := \{ Cx \mid x \in X \}$: set of admissible parameters.

The linear model based on the MEM functional reads

$$\min_{x \in X} \psi^*_{\hat{P}}(C x).$$

<table>
<thead>
<tr>
<th>Reference Family</th>
<th>Objective Function ($\psi^*_{\hat{P}} \circ C$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>$\frac{1}{2} | C x - \hat{y} |^2$</td>
</tr>
<tr>
<td>Poisson</td>
<td>$\sum_{i=1}^{m} [\langle c_i, x \rangle \log(\langle c_i, x \rangle / \hat{y}_i) - \langle c_i, x \rangle + \hat{y}_i]$</td>
</tr>
<tr>
<td>Gamma ($\beta = 1$)</td>
<td>$\sum_{i=1}^{m} [\langle c_i, x \rangle - \hat{y}_i \log(\langle c_i, x \rangle) - (\hat{y}_i - \hat{y}_i \log \hat{y}_i)]$</td>
</tr>
</tbody>
</table>
In case of ill-posedness or to incorporate prior information we consider the
MEM regularized linear model:

\[
\min \left\{ \kappa_{P_{\hat{\theta}}}(Ax) + \varphi(x) : x \in X \right\},
\]

where

\[
\varphi(x) = \begin{cases} \\
\kappa_R(x) \\
\kappa_R(Lx) & (L \in \mathbb{R}^{r \times d}) \\
\sum_{i=1}^{d} \kappa_R(L_ix) & (L_i \in \mathbb{R}^{r \times d}, i = 1, 2, \ldots, d),
\end{cases}
\]

with \( R \in \mathcal{P}(\Omega) \) reference distribution.

Q: How can we efficiently solve this problem?
The regularized model falls into the additive composite framework

$$\min_{x \in \mathbb{R}^d} \{ f(x) + g(x) \} \quad (g \in \Gamma_0, f \in C^1(\cap \Gamma_0)).$$

The Bregman proximal gradient algorithm

**Initialization.** Pick $t \in (0, 1/L]$ and $x^0 \in \text{int} (\text{dom} h)$.

**Procedure.** For $k = 0, 1, 2, \ldots$:

$$x^{k+1} = \text{prox}^h_{tg} \left( \nabla h^* \left( \nabla h(x^k) - t \nabla f(x^k) \right) \right)$$

is specified by a *kernel* $h \in \Gamma_0 \cap C^1$ that [Bauschke et al., 2017]:

- is *smooth adaptable* w.r.t. $f$ i.e. $Lh - f$ is convex for some $L > 0$.
- has computationally tractable *Bregman proximal operator* with respect to $g$:

$$\text{prox}^h_g(\bar{x}) := \arg\min_{x \in \mathbb{R}^d} \{ g(x) + D_h(x, \bar{x}) \}.$$
The $h$-Bregman proximal operator of $\psi_R^*$ is always well defined under mild assumptions (on $R$ and $h$), and can be efficiently evaluated, often has closed form:

<table>
<thead>
<tr>
<th>Reference Distribution</th>
<th>Proximal Operator</th>
<th>Kernel ($h(x)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multivariate Normal $\mu \in \mathbb{R}^d, \Sigma \in \mathbb{S}^d, \Sigma \succ 0$</td>
<td>$x^+ = \left( tI + \Sigma \right)^{-1} \left( \Sigma \bar{x} + t\mu \right)$</td>
<td>$(1/2)|x|_2^2$</td>
</tr>
<tr>
<td>Gamma $(\alpha, \beta \in \mathbb{R}^{++})$</td>
<td>$x^+ = \frac{\bar{x} - t\beta + \sqrt{(\bar{x} - t\beta)^2 + 4t\alpha}}{2}$</td>
<td>$(1/2)|x|_2^2$</td>
</tr>
</tbody>
</table>
| Laplace $(\mu \in \mathbb{R}, b \in \mathbb{R}^{++})$ | $x^+ = \begin{cases} 
\mu, & \mu = \bar{x}, \\
\mu + b\rho, & \mu \neq \bar{x},
\end{cases}$ | $-\sum \log x_i$ |

where $\rho$ is the unique real root of a cubic$^{16}$

| Poisson $(\lambda \in \mathbb{R}^{++})$ | $x^+ = \left( \bar{x} \lambda^t \right)^{\frac{1}{t+1}}$ | $\sum x_i \log x_i$ |
| Multinomial $(p \in \Delta_d, n \in \mathbb{N})$ | $x^+ = \left( \frac{n(np_i)^{\frac{1}{t+1}} \bar{x}_i^{\frac{1}{t+1}}}{\sum_{i=1}^d (np_i)^{\frac{1}{t+1}} \bar{x}_i^{\frac{1}{t+1}}} \right)^d_{i=1}$ | $\sum x_i \log x_i$ |

In addition: Normal-inverse Gaussian, Negative Multinomial, Continuous/Discrete Uniform, Logistic, Exponential/Chi-Squared/Erlang (via Gamma), Binomial/Bernoulli/Categorical (via Multinomial), Negative Binomial & Shifted Geometric (via Negative Multinomial).

$^{16}$With closed-form coefficients dependent on $b, \mu, \bar{x}, t$
All models are wrong, but some are useful.

George E. P. Box

- MEM is a useful tool for incorporating prior information into models for inverse problems.
- The application of MEM to inverse problems is scarce in the literature.
- We unify and extend much of the theory that appears in the literature, while providing an algorithmic framework.
- arXiv preprint and computational toolbox of Cramér functions, prox operators, and algorithms, to appear online shortly.
- Ongoing work: Obtain the Cramér function (or log-MGF) via (deep) learning.


