A note on the K-epigraph

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Abstract

We study the question as to when the closed convex hull of the graph of a K-convex map equals its K-epigraph. In particular, we shed light onto the smallest cone K such that a given map has convex and closed K-epigraph, respectively. We apply our findings to several examples in matrix space as well as to convex composite functions.

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1 Introduction

Motivation In a recent paper, Burke et al. [8, Corollary 9] show that the closed convex hull of the set $\mathcal{D} := \left\{(X, \frac{1}{2}XX^T) \mid X \in \mathbb{R}^{n\times m}\right\}$ is given by

$$\overline{\operatorname{conv}}\,\mathcal{D} = \left\{(X,Y) \in \mathbb{R}^{n\times m} \times \mathbb{S}^n \;\middle|\; Y \succeq \frac{1}{2}XX^T\right\}.$$

Here, '$\succeq$' is the *Löwner partial ordering* [12] on the symmetric matrices $\mathbb{S}^n$ induced by the positive semidefinite cone $\mathbb{S}^n_+$ via '$A \succeq B$ if and only if $A - B \in \mathbb{S}^n_+$'. At second glance, the set $\mathcal{D} \subset \mathbb{R}^{n\times m} \times \mathbb{S}^n$ is simply the graph of the matrix-valued map $F : X \in \mathbb{R}^{n\times m} \mapsto \frac{1}{2}XX^T \in \mathbb{S}^n$; and $\overline{\operatorname{conv}}\,\mathcal{D}$ in (1) then appears to be a 'generalized epigraph' of F where the partial ordering on the image space $\mathbb{S}^n$ (induced by $\mathbb{S}^n_+$) plays the role of the ordinary ordering of $\mathbb{R}$ (induced by $\mathbb{R}_+$) for scalar-valued functions.

More generally, given a map $F : \mathbb{E}_1 \to \mathbb{E}_2$ between two (Euclidean) spaces $\mathbb{E}_1$ and $\mathbb{E}_2$ and a cone $K \subset \mathbb{E}_2$, we can order $\mathbb{E}_2$ via '$y \geq_K z$ if and only if $y - z \in K$'. In view of the above identity, the natural question that arises is the following: *When is*

$$\overline{\operatorname{conv}}\,(\operatorname{gph} F) = \{(x,y) \mid y \geq_K F(x)\} \tag{1}$$

valid? Clearly, this can only hold if the set on the right, which will later be called the *K-epigraph of F*, is itself closed and convex, in which case we say that F is *K-closed* and *K-convex*, respectively, or *closed* and *convex, with respect to (w.r.t.)* K, respectively.

Related work The study of K-convexity has a long tradition in convex analysis and is now part of many textbooks, e.g. [2,19]: Borwein [3] pursued an ambitious program of extending most of convex analysis to cone convex functions including conjugacy, subdifferential analysis, and duality, laying out much of the groundwork. Kusraev and Kutateladze [14] take this idea to an even more general setting by considering *convex operators* with values in arbitrary ordered vector spaces. Pennanen [17] develops a deep theory of generalized differentiation for *graph-convex mappings* (these are called *convex correspondences* in [14]) which contains some results on K-convexity, highly relevant to our study. One of the most important features of a K-convex map F is the fact that the composition $g \circ F$ with a convex function g, which is increasing with respect to the ordering induced by K, is convex; a fact that has been well observed and utilized widely in the literature [4–6, 10, 11, 17].

Road map and contributions We start our study in Section 2 with the necessary tools from convex and variational analysis. In Section 3, we formally introduce and expand on the central notions of K-convexity and K-closedness. In particular, in Sections 3.1-3.3 we characterize the functions which are convex w.r.t. a given subspace, half-space and polyhedral cone, respectively. In Section 3.4, we elaborate on Pennanen's characterization of the dual cone of the smallest closed, necessarily convex (Proposition 11) cone with respect to which a given F is convex. We extend this in Section 3.5 to study the smallest, necessarily closed and convex (Proposition 11) cone with respect to which F is convex *and* closed. Section 4 is fully devoted to the question as to when (1) holds. Theorem 31 in Section 4.1 provides a characterization for (1), which consitutes one of the main workhorses for the the rest of Section 4. Section 4.2 is mainly devoted to necessary conditions for (1). Partly mimicking the scalar case (Theorem 37), in Section 4.2.2 we present necessary conditions based on affine K-minorization and K-majorization. Section 4.3, in turn, provides sufficient conditions. Section 4.4 presents different examples of K-convex maps by which we illustrate the theory developed in Section 3 and, more importantly, Section 4. In particular, we apply our findings to the following maps:

- $F : X \in \mathbb{R}^{n\times m} \mapsto \frac{1}{2}XX^T \in \mathbb{S}^n$;
- $F : X \in \mathbb{S}^n_{++} \to X^{-1} \in \mathbb{S}^n$ (inverse matrix);
- $F : X \in \mathbb{S}^n \mapsto \lambda(X)^1 \in \mathbb{R}^n$ (spectral map);
- $F : \mathbb{E} \to \mathbb{R}^m$ where F_i is convex for all $i = 1, \dots, m$ (component-wise convex).

Most of the criteria worked out in the previous sections for the validity of (1) are brought to bear directly or indirectly here.

Section 5 taps into the composite framework alluded to above, where, primarily, we study the following question: given a vector-valued map F and a (closed, proper) convex function g such that $g \circ F$ is convex, does there exist a cone K such that F is K-convex and g is increasing in a K-related ordering?

Notation: In what follows, $\mathbb{E}$ denotes a Euclidean space, i.e. a finite-dimensional real inner product space with inner product denoted by $\langle \cdot, \cdot \rangle$. Given a set $S \subset \mathbb{E}$, we denote its closure, convex hull, closed convex hull, and convex conical hull by $\mathrm{cl}\, S$, $\mathrm{conv}\, S$, $\overline{\mathrm{conv}}\, S$ and $\mathrm{cone}\, S$, respectively. For a vector $u \in \mathbb{E}$, we denote its (convex) conical hull by $\mathbb{R}_+ u$, and $\mathbb{R}_{++}u = \{\lambda u \mid \lambda > 0\}$. The indicator function $\delta_S : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ of $S \subset \mathbb{E}$ is given by $\delta_S(x) = 0$ if $x \in S$ and $\delta_S(x) = +\infty$ otherwise.

2 Preliminaries

Throughout we make use of the *relative topology* for convex sets [18, §6]. The *relative interior* $\mathrm{ri}\, C$ of a convex set $C \subset \mathbb{E}$ is its interior in the subspace topology induced by its *affine hull* $\mathrm{aff}\, C := \{\lambda x + (1-\lambda)y \mid \lambda \in \mathbb{R},\ x, y \in C\}$. For a convex set $C \subset \mathbb{E}$ and (any) $x_0 \in C$, the *subspace parallel to* C is $\mathrm{par}\, C := \mathrm{aff}\, C - x_0$. For convex sets, we have a handy description of the affine hull.

Lemma 1. *Let $C \subset \mathbb{E}$ be a convex. Then* $\mathrm{aff}\, C = \{\alpha x - \beta y \mid \alpha, \beta \geqslant 0, \alpha - \beta = 1, x, y \in C\}$.

Proof. Set $A := \{\alpha x - \beta y \mid \alpha, \beta \geqslant 0, \alpha - \beta = 1, x, y \in C\}$. Thus $\mathrm{aff}\, C \supset A$. Conversely, for $z \in \mathrm{aff}\, C$, there exist $\lambda \in \mathbb{R}$ and $x, y \in C$ such that $z = \lambda x + (1-\lambda)y$. If $\lambda \in (0,1)$, then $z \in C$ by convexity, and hence $z = 1 \cdot z - 0 \cdot z \in A$. If $\lambda > 1$, set $\alpha := \lambda \geqslant 0$, $\beta := \lambda - 1 \geqslant 0$, and we get $\alpha - \beta = 1$ and $z = \alpha x - \beta y \in A$. Finally, if $\lambda < 0$, then $1 - \lambda \geqslant 0$, and thus ($\beta := \lambda, \alpha := 1 - \lambda$) $z \in A$. □

Let $f : \mathbb{E} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. We call f *proper* if its *domain* $\mathrm{dom}\, f := \{x \in \mathbb{E} \mid f(x) < +\infty\}$ is nonempty and f doesn't take the value $-\infty$. We say that f is *convex* if its *epigraph* $\mathrm{epi}\, f := \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid f(x) \leq \alpha\}$ is convex which coincides with the usual definition via a secant condition

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \quad \forall x, y \in \mathrm{dom}\, f,\ \alpha \in (0,1),$$

if f does not take the value $-\infty$. Although pointwise convergence is not a suitable for preservation of many variational properties, see e.g. [19, Chapter 7], it still preserves convexity in the limit.

Lemma 2. *Let $\{f_k : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}\}$ converge pointwise to $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$, i.e. $f_k(x) \to f(x)$ for all $x \in \mathbb{E}$. If f_k is convex for all $k \in \mathbb{N}$ (sufficiently large), then so is f.*

[1] Here $\lambda(X) = (\lambda_1, \dots, \lambda_n)^T$ is the vector of eigenvalues of $X \in \mathbb{S}^n$ in decreasing order.

Proof. For $\alpha \in (0,1)$ and $x, y \in \mathbb{E}$, the convexity of f_k yields $\alpha f_k(x) + (1-\alpha) f_k(y) \geqslant f_k(\alpha x + (1-\alpha)y)$. Passing to the limit $k \to \infty$ on both sides gives $\alpha f(x) + (1-\alpha) f(y) \geqslant f(\alpha x + (1-\alpha)y)$. $\square$

We call $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ *closed* or *lower semicontinuous (lsc)* if $\operatorname{epi} f$ is closed. We set

$$\begin{aligned} \Gamma(\mathbb{E}) &:= \{ f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\} \mid f \text{ proper, convex} \}, \\ \Gamma_0(\mathbb{E}) &:= \{ f \in \Gamma(\mathbb{E}) \mid f \text{ closed} \}. \end{aligned}$$

For $f \in \Gamma(\mathbb{E})$, its *closure* $\operatorname{cl} f \in \Gamma_0(\mathbb{E})$ is defined via $\operatorname{cl} (\operatorname{epi} f) = \operatorname{epi} (\operatorname{cl} f)$. More generally, given a convex subset $D \subset \mathbb{E}$, we call f *D-closed* if $\operatorname{epi} f$ is closed in the subspace topology induced by $D \times \mathbb{R}$, i.e. f is D-closed if and only if $(D \times \mathbb{R}) \cap \operatorname{epi} f$ is closed in $D \times \mathbb{R}$. On the other hand, we note that $D \times \mathbb{R}$ is a metric space, hence closedness is sequential closedness and, in particular, f is D-closed if and only if

$$\liminf_{k \to \infty} f(x_k) \geq f(\bar{x}) \quad \forall \{x_k \in D\} \to \bar{x} \in D.$$

We define $\Gamma_0(D) := \{ f \in \Gamma(\mathbb{E}) \mid f \ D\text{-closed} \}$.

Lemma 3. *Let $f \in \Gamma(\mathbb{E})$. Then the following are equivalent:*

i) $f \in \Gamma_0(\operatorname{dom} f)$*;*

ii) $f(x) = (\operatorname{cl} f)(x)$ *for all* $x \in \operatorname{dom} f$.

Proof. Since $f \in \Gamma_0(\mathbb{E})$, we have

$$\begin{aligned} f \in \Gamma_0(\operatorname{dom} f) &\iff \operatorname{epi} f \text{ is closed in } \operatorname{dom} f \times \mathbb{R} \\ &\iff \operatorname{epi} f = \operatorname{cl} (\operatorname{epi} f) \cap \operatorname{dom} f \times \mathbb{R} \\ &\iff \operatorname{epi} f = \operatorname{epi} (\operatorname{cl} f) \cap \operatorname{dom} f \times \mathbb{R} \\ &\iff f(x) = \operatorname{cl} f(x) \quad \forall x \in \operatorname{dom} f. \end{aligned}$$

Here the first equivalence is simply the definition of $\Gamma_0(\operatorname{dom} f)$. The second is due to the fact that the closed sets in the $\operatorname{dom} f \times \mathbb{R}$ subspace topology are exactly the intersections of closed sets (in $\mathbb{E} \times \mathbb{R}$) with $\operatorname{dom} f \times \mathbb{R}$. The third one is clear as $\operatorname{epi} (\operatorname{cl} f) = \operatorname{cl} (\operatorname{epi} f)$, and the fourth one follows from elementary considerations. $\square$

Remark 4. *We point out that $f \in \Gamma_0(\mathbb{E})$ implies that $f \in \Gamma_0(\operatorname{dom} f)$, since the closed set $\operatorname{epi} f \subset \mathbb{E} \times \mathbb{R}$ intersected with $\operatorname{dom} f \times \mathbb{R}$ is (trivially) closed in the subspace topology induced by $\operatorname{dom} f \times \mathbb{R}$. However, the converse statement is not true. Consider for instance $\delta_{(0,1)} \in \Gamma_0((0,1)) \setminus \Gamma_0(\mathbb{R})$, as $(0,1) \times \mathbb{R}_+$ is a closed set in the topology induced by $(0,1) \times \mathbb{R}$, but is not a closed set in $\mathbb{R} \times \mathbb{R}$.*

A nonempty subset $K \subset \mathbb{E}$ is called a *cone* if $\lambda x \in K$ for all $\lambda \geq 0$ and $x \in K$. If the latter only holds for all $\lambda > 0$, we call K a *pre-cone*. For instance if K is a convex cone, then $\operatorname{ri} K$ is a (convex) pre-cone, use e.g. [18, Corollary 6.6.1]. Combining this with the *line segment principle* [18, Theorem 6.1] and [18, Theorem 6.3], we find the following result.

Lemma 5. *Let $K \subset \mathbb{E}$ be a convex cone. Then $\operatorname{cl} K + \operatorname{ri} K \subset \operatorname{ri} K$.*

The *dual cone* of a (pre-)cone K is given by $K^+ := \{ v \in \mathbb{E}_2 \mid \forall u \in K \colon \langle u, v \rangle \geq 0 \}$, and K^+ is referred to as the *dual cone*. The dual cone is the negative *polar cone*, i.e. $K^+ = -K^\circ$. Recall that $\overline{\operatorname{conv}} K = (K^+)^+ =: K^{++}$ by the bipolar theorem [19, Corollary 6.21], and that the duality operation is order reversing. The *horizon cone* of $C \subset \mathbb{E}$ is given by $C^\infty := \{ u \in \mathbb{E} \mid \exists \{t_k\} \downarrow 0, \{x_k \in C\} : \lim_{k \to \infty} t_k x_k = u \}$. If C is a nonempty closed, convex set, then $C + C^\infty = C$ and for a cone K, we have $K^\infty = \operatorname{cl} K$.

A cone $K \subset \mathbb{E}$ induces an ordering on $\mathbb{E}$ via

$$y \geq_K x :\iff y - x \in K \quad \forall x, y \in \mathbb{E}.$$

Lemma 6. *If K is a closed and convex cone of $\mathbb{E}_2$, then*

$$y \geq_K x \quad \iff \quad \langle u, x \rangle \geqslant \langle u, y \rangle \quad \forall u \in K^+.$$

Proof. By the bipolar theorem [19, Corollary 6.21], we have $K^{++} = K$ and hence

$$x \geqslant_K y \iff x - y \in K = K^{++} \iff \langle u, x - y \rangle \geqslant 0 \quad \forall u \in K^+.$$

$\square$

A cone $K \subset \mathbb{E}$ is said to be *pointed* if $K \cap (-K) = \{0\}$. Such a cone induces a partial ordering when convex.

Lemma 7 (Ordering induced by a pointed cone)**.** *Let $K \subset \mathbb{E}$ be pointed. Then*

$$x = y \iff x \geq_K y \quad \text{and} \quad y \geq_K x.$$

In particular, the relation'$\geq_K$' induces partial ordering *if K is (pointed and) convex.*

Proof. The equivalence is straightforward from pointedness. For a partial ordering, we also need to see that $x \geq_K x$ for all $x \in \mathbb{E}$, which is true since $0 \in K$, and that $x \geq_K y$ and $y \geq_K z$ implies $x \geq_K z$, which is true since $x - z = x - y + y - z \in K$, by convexity of the cone. □

3 K-convexity and K-closedness

We commence this section with the central definitions of this paper.

Definition 8 (K-epigraphs, K-convexity and K-closedness)**.** *Let $K \subset \mathbb{E}_2$ be a cone and let $F \colon D \subset \mathbb{E}_1 \to \mathbb{E}_2$. Then the K-*epigraph *of F is given by*

$$K\text{-epi}\, F = \{(x, y) \in D \times \mathbb{E}_2 \mid F(x) \leq_K y\} \subset \mathbb{E}_1 \times \mathbb{E}_2. \tag{2}$$

We say that F is

- *i)* proper *if K-epi $F \neq \emptyset$ (i.e. $D \neq \emptyset$);*
- *ii)* K-convex *if K-epi F is convex;*
- *iii)* K-closed *if K-epi F is closed.*

For $D \subset \mathbb{E}_1$ convex and $K \subset \mathbb{E}_2$ a cone, we point out that $F : D \to \mathbb{E}_2$ is K-convex if and only if K is convex and

$$\alpha F(x) + (1 - \alpha)F(y) \geqslant_K F(\alpha x + (1 - \alpha)y) \quad \forall x, y \in D, \alpha \in (0, 1).$$

Moreover, we always have

$$K\text{-epi}\, F = \text{gph}\, F + \{0\} \times K. \tag{3}$$

This has, in particular, the following immediate consequence.

Lemma 9. *Let $F \colon D \subset \mathbb{E}_1 \to \mathbb{E}_2$ be proper, and $K_1 \subsetneq K_2 \subset \mathbb{E}_2$ be cones. Then K_1-epi $F \subsetneq K_2$-epi F. In particular, there is at most one cone $K \subset \mathbb{E}_2$ such that K-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$.*

In the convex case we can extract the following.

Lemma 10. *Let $F \colon D \subset \mathbb{E}_1 \to \mathbb{E}_2$ be proper, and let $K_1 \subset K_2 \subset \mathbb{E}_2$ be convex cones. Then K_2-epi $F = K_1$-epi $F + \{0\} \times K_2$. In particular, if F is K_1-convex, then F is K_2-convex.*

Proof. This is due to (3) combined with the fact that $K_1 + K_2 = K_2$ because K_1 and K_2 are convex cones. □

Given a cone $K \subset \mathbb{E}$ and its induced ordering, we attach to $\mathbb{E}$ a formal largest element $+\infty_\bullet$ with respect to that ordering, and set $\mathbb{E}^\bullet := \mathbb{E} \cup \{+\infty_\bullet\}$. For $G \colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ its *domain* is $\text{dom}\, G := \{x \in \mathbb{E}_1 : G(x) \in \mathbb{E}_2\}$. The *graph* of G is given by $\text{gph}\, G := \{(x, G(x)) \mid x \in \text{dom}\, G\}$. We adopt the notions in Definition 8 via the restriction $F := G_{|\text{dom}\, G}$. We record in the next result that K-closedness and K-convexity requires certain conditions about the underlying cone K.

Proposition 11. *Let $K \subset \mathbb{E}_2$ be a cone, and let $F \colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper. Then the following hold:*

- *a) If F is K-closed, then K is closed.*
- *b) If F is K-convex, then K is convex.*

Proof. a) Let $\{y_k \in K\} \to y$ and pick $x \in \text{dom}\, F$. Then $(x, F(x) + y_n) \in K$-epi F for all $k \in \mathbb{N}$ and $(x, F(x) + y_k) \to (x, F(x) + y) \in K$-epi F as K-epi F is closed. Thus, $y \in F(x) + y - F(x) = y \in K$.

b) Let $y_1, y_2 \in K, \alpha \in (0, 1)$. For $x \in \text{dom}\, F$ we hence find $(x, F(x) + y_1) \in K$-epi F, and $(x, F(x) + y_2) \in K$-epi F. As K-epi F is a convex, we have $(x, F(x) + \alpha y_1 + (1 - \alpha)y_2) \in K$-epi F, and consequently $\alpha y_1 + (1 - \alpha)y_2 \in K$. Thus, K is convex. □

The following proposition shows that a function $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ is fully determined by its K-epigraph when K is a pointed cone.

Proposition 12. *Let $K \subset \mathbb{E}_2$ be a pointed cone, and let $F, G\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper. Then*

$$K\text{-epi}\, F = K\text{-epi}\, G \iff F = G.$$

Proof. Suppose that K-epi $F = K$-epi G. In particular, for all $x \in \operatorname{dom} F$, $(x, F(x)) \in K$-epi F, so $(x, F(x)) \in K$-epi G, hence $x \in \operatorname{dom} G$ and $F(x) \geqslant_K G(x)$. Likewise, for all $x \in \operatorname{dom} G$, we have $x \in \operatorname{dom} F$ and $G(x) \geqslant_K F(x)$. Thus $\operatorname{dom} F = \operatorname{dom} G$ and for any $x \in \operatorname{dom} F = \operatorname{dom} G$ we have $F(x) = G(x)$ by Lemma 7. □

Given $F : D \subset \mathbb{E}_1 \to \mathbb{E}_2$ and $u \in \mathbb{E}_2$, we define the *scalarization* $\langle u, F\rangle\colon \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}$ by

$$\langle u, F\rangle(x) = \begin{cases} \langle u, F(x)\rangle, & x \in D, \\ +\infty & \text{else.} \end{cases} \tag{4}$$

We adapt this notion for $D = \operatorname{dom} F$ if $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ where $\mathbb{E}_2$ is ordered by some cone K. In this case, note that, for $u, v \in \mathbb{E}_1$ and $\alpha > 0$, we have $\langle u + v,\, F\rangle = \langle u,\, F\rangle + \langle v,\, F\rangle$, $\langle \alpha u,\, F\rangle = \alpha \langle u,\, F\rangle$, and $\operatorname{dom} \langle u + v,\, F\rangle = \operatorname{dom} \alpha \langle u,\, F\rangle = \operatorname{dom} F$. Equipped with this concept, the following proposition gives a characterization of K-epi F (and gph F) via the epigraphs (and graphs) of the scalarizations $\langle u,\, F\rangle$ for $u \in K^+$. For $u \in \mathbb{E}_2$, we use in the following proposition the map $(\mathrm{id}, \langle u,\, \cdot\rangle) : (x, y) \in \mathbb{E}_1 \times \mathbb{E}_2 \mapsto (x, \langle u,\, y\rangle)$.

Proposition 13. *Let $K \subset \mathbb{E}_2$ be a closed and convex cone, and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper. Then:*

a) K-epi $F = \bigcap_{u\in K^+} (\mathrm{id}, \langle u,\, \cdot\rangle)^{-1}\big(\mathrm{epi}\ \langle u,\, F\rangle\big)$;

b) If K is pointed, then gph $F = \bigcap_{u\in K^+} (\mathrm{id}, \langle u,\, \cdot\rangle)^{-1}\big(\mathrm{gph}\ \langle u,\, F\rangle\big)$.

Proof. We deduce from Lemma 6 that

$$\begin{aligned} K\text{-epi}\, F &= \{(x, v) \mid v \geq_K F(x)\} \\ &= \big\{(x, v) \,\big|\, \langle u,\, v\rangle \geq \langle u,\, F(x)\rangle\ \forall u \in K^+\big\} \\ &= \bigcap_{u\in K^+} (\mathrm{id}, \langle u,\, \cdot\rangle)^{-1}\big(\mathrm{epi}\ \langle u,\, F\rangle\big). \end{aligned}$$

Similarly, if K is pointed, we obtain

$$\begin{aligned} \mathrm{gph}\, F &= \{(x, v) \mid x \in \mathbb{E}_1,\ v = F(x)\} \\ &= \{(x, v) \mid x \in \mathbb{E}_1,\ v - F(x) \in K \cap (-K)\} \\ &= \{(x, v) \mid x \in \mathbb{E}_1,\ v \geq_K F(x)\} \bigcap \{(x, v) \mid x \in \mathbb{E}_1,\ F(x) \geq_K v\} \\ &= \bigcap_{u\in K^+} (\mathrm{id}, \langle u,\, \cdot\rangle)^{-1}\big(\mathrm{gph}\ \langle u,\, F\rangle\big). \end{aligned}$$

□

As an immediate consequence of the latter proposition, one obtains Pennanen's sufficient condition for K-closedness [17, Lemma 6.2], which unfortunately excludes functions with domains that are not closed. We therefore provide the following, stronger version in the next result's part b) whose proof is simply a refinement of Pennanen's proof. Part a) is a refinement of the scalarization characterization of K-convexity.

Proposition 14. *Let $K \subset \mathbb{E}_2$ be a closed, convex cone, let $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper. Then:*

a) The following are equivalent:

i) $\langle u,\, F\rangle$ is convex for all $u \in \mathrm{ri}\,(K^+)$;

ii) $\langle u,\, F\rangle$ is convex for all $u \in K^+$;

iii) F is K-convex.

b) F is K-closed if $\langle u,\, F\rangle$ is lower semicontinuous for all $u \in K^+ \setminus \{0\}$ and $K \neq \mathbb{E}_2$.

In particular, if $K \neq \mathbb{E}_2$ and $\langle u,\, F\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $u \in K^+ \setminus \{0\}$, then F is K-closed and K-convex.

Proof. a) 'i)⇒ ii)': Let $u \in K^+ \setminus \{0\}$. Then u is a limit $\{u_k \in \mathrm{ri}\,(K^+)\} \to u$, and hence $\langle u,\, F\rangle$ is a pointwise limit of convex functions $\langle u_k,\, F\rangle$, hence convex by Lemma 2.
'ii)⇒ iii)': Follows from Proposition 13 a).
'iii)⇒ i)': Follows from Lemma 6.

b) Assume that F is not K-closed i.e. there exists $\{(x_k, y_k) \in K\text{-epi}\,F\} \to (x, y) \notin K\text{-epi}\,F$. Then $x \notin \mathrm{dom}\,F$ or $y - F(x) \notin K = K^{++}$. In the latter case there exists $u^* \in K^+$ such that

$$\langle u^*,\, y\rangle < \langle u^*,\, F(x)\rangle \quad \text{and} \quad \langle u^*,\, y_k\rangle \geq \langle u^*,\, F(x_k)\rangle \quad \forall k \in \mathbb{N}. \tag{5}$$

If $x \in \mathrm{dom}\,F$, then necessarily $u^* \neq 0$. On other hand, if $x \notin \mathrm{dom}\,F$, since $K^+ \supsetneq \{0\}$ by assumption, we can choose $u^* \neq 0$. All in all, there exists $u^* \in K^+ \setminus \{0\}$ such that (5) holds. We hence obtain

$$\langle u^*,\, F\rangle\,(x) > \langle u^*,\, y\rangle = \liminf_{k\to\infty}\,\langle u^*,\, y_k\rangle \geq \liminf_{k\to\infty}\,\langle u^*,\, F\rangle\,(x_k),$$

and, consequently, $\langle u^*,\, F\rangle$ is not lsc, which concludes the proof of part b). □

We close out this preparatory paragraph with the following useful result.

Lemma 15. *Let $D \subset \mathbb{E}_1$ be nonempty, let $F\colon D \to \mathbb{E}_2$, and let $(K_i)_{i\in I}$ be a family of cones of $K_i \subset \mathbb{E}_2$. Then*

$$\left(\bigcap_{i\in I} K_i\right)\text{-epi}\,F = \bigcap_{i\in I}\left(K_i\text{-epi}\,F\right).$$

In particular, if F is K_i-closed for all $i \in I$, then F is $(\cap_{i\in I} K_i)$-closed. Moreover, if F is K_i-convex for all $i \in I$ then F is $(\cap_{i\in I} K_i)$-convex. The latter is an equivalence if K_i is convex for all $i \in I$.

Proof. For any $x \in \mathrm{dom}\,F$ and $y \in \mathbb{E}_2$, we have

$$\begin{aligned}
(x, y) \in \left(\bigcap_{i\in I} K_i\right)\text{-epi}\,F \quad &\iff \quad y - F(x) \in \bigcap_{i\in I} K_i \\
&\iff \quad \forall i \in I:\; y - F(x) \in K_i \\
&\iff \quad \forall i \in I:\; (x, y) \in K_i\text{-epi}\,F \\
&\iff \quad (x, y) \in \bigcap_{i\in I} K_i\text{-epi}\,F.
\end{aligned}$$

The addendum follows from the fact that intersection preserves closedness and convexity, and that $\bigcap_{i\in I} K_i \subset K_i$, so $\bigcap_{i\in I} K_i$-convexity implies K_i-convexity for all $i \in I$ if these are convex, see Lemma 10. □

3.1 Affine and {0}-convex functions

We extend the notion of affine functions to affine subsets of $\mathbb{E}$, see, e.g., Rockafellar [18, §1] for the standard case.

Definition 16. *Let $A \subset \mathbb{E}$ be an affine set and let $x_0 \in A$. Then a function $F\colon A \to \mathbb{E}_2$ is said to be affine if there exists a linear map $L : \mathrm{par}\,A \mapsto \mathbb{E}_2$ and a vector $b \in \mathbb{E}_2$ such that we have $F(x) = L(x - x_0) + b$ for all $x \in A$.*

Lemma 17. *Let $A \subset \mathbb{E}_1$ be affine. Then $F\colon A \to \mathbb{E}_2$ is affine if and only if*

$$F(tx + (1-t)y) = tF(x) + (1-t)F(y) \quad \forall x, y \in A,\; t \in (0, 1). \tag{6}$$

Proof. Assume first that (6) holds. Discriminating the three cases $t \in [0, 1]$, $t > 1$ and $t < 0$, it is straightforward to show that, in fact, we have

$$F(tx + (1-t)y) = tF(x) + (1-t)F(y) \quad \forall x, y \in A,\; t \in \mathbb{R}. \tag{7}$$

Now let $x_0 \in A$, i.e. $\mathrm{par}\,A = A - x_0$, and define $L : \mathrm{par}\,A \to \mathbb{E}_2$ by $L(x) := F(x + x_0) - F(x_0)$. Using (7), we find that $L(tx + (1-t)y) = tL(x) + (1-t)L(y)$ for all $x, y \in \mathrm{par}\,A$ and $t \in \mathbb{R}$. Thus, taking $y = 0$, as $L(0) = 0$, gives $L(tx) = tL(x)$ for all $x \in \mathrm{par}\,A$ and all $t \in \mathbb{R}$. Hence, $L(x + y) = L(\frac{1}{2}(2x) + \frac{1}{2}(2y)) = \frac{1}{2}L(2x) + \frac{1}{2}L(2y) = L(x) + L(y)$, for all $x, y \in \mathrm{par}\,A$. This implies that L is linear. Hence, for all $x \in A$ and $b := F(x_0)$, we have $F(x) = L(x - x_0) + b$. Thus, F is affine.

Conversely, if F is affine, then we can write $F = L((\cdot) - x_0) + b$ for some linear map $L : \operatorname{par} A \to \mathbb{E}_2$, $x_0 \in A$ and $b \in \mathbb{E}_2$. Then

$$F(tx + (1-t)y) = t(L(x - x_0) + b) + (1-t)(L(y - x_0) + b) = tF(x) + (1-t)F(y),$$

for all $x, y \in A$ and $t \in \mathbb{R}$. In particular, this is true for all $t \in (0,1)$. $\square$

Proposition 18. *Let $D \subset \mathbb{E}_1$ be nonempty convex. Then the following are equivalent:*

i) $F\colon D \to \mathbb{E}_2$ is $\{0\}$-convex;

ii) There exists an affine function $G\colon \operatorname{aff} D \to \mathbb{E}_2$ such that $F = G_{|D}$;

iii) There exists an affine function $H\colon \mathbb{E}_1 \to \mathbb{E}_2$ such that $F = H_{|D}$.

Proof. 'i)⇒ii)': Assuming that $F\colon D \to \mathbb{E}_2$ is $\{0\}$-convex and letting $z \in \operatorname{aff} D$. By Lemma 1, we can write $z = \alpha x - \beta y$, for some $\alpha, \beta \geqslant 0$, $\alpha - \beta = 1$, and $x, y \in D$. Suppose z has two representations of this form, i.e. $z = \alpha x - \beta y = \alpha' x' - \beta' y'$, with $\alpha, \beta, \alpha', \beta' \geqslant 0$, $\alpha - \beta = \alpha' - \beta' = 1$, and $x, y, x', y' \in D$. Then $\Delta := \alpha + \beta' (= \alpha' + \beta) = 1 + \beta + \beta' > 0$. By convexity of D, we find that $\frac{\alpha}{\Delta} x + \frac{\beta'}{\Delta} y' = \frac{\alpha'}{\Delta} x' + \frac{\beta}{\Delta} y \in D$. Using the $\{0\}$-convexity of F, we have $\frac{\alpha}{\Delta} F(x) + \frac{\beta'}{\Delta} F(y') = \frac{\alpha'}{\Delta} F(x') + \frac{\beta}{\Delta} F(y)$. Multiplying by Δ and rearranging the above terms, we get

$$\alpha F(x) - \beta F(y) = \alpha' F(x') - \beta' F(y').$$

Therefore, we the function $G : \operatorname{aff} D \to \mathbb{E}_2$, $G(z) = \alpha F(x) - \beta F(y)$ for $z \in \operatorname{aff} D$ given by $z = \alpha x - \beta y$, $\alpha, \beta \geqslant 0$, $\alpha - \beta = 1$, $x, y \in D$ is well-defined.

Now, let z and z' in D given by $z = \alpha x - \beta y$ and $z' = \alpha' x' - \beta y'$, with $\alpha, \beta, \alpha', \beta' \geqslant 0$, $\alpha - \beta = 1$, $\alpha' - \beta' = 1$, and $x, y, x', y' \in D$. Let $t \in (0,1)$ and set $p := (1-t)z + tz'$, as well as $a := (1-t)\alpha + t\alpha'$ and $b := (1-t)\beta + t\beta'$. Then $a, b \geqslant 0$ and $a - b = 1$.

If $b = 0$, then $\beta = \beta' = 0$ and $\alpha = \alpha' = 1$, so $z = x \in D$ and $z' = x' \in D$, and thus, using the $\{0\}$-convexity of F, we have $G(p) = F(p) = F((1-t)z + tz') = (1-t)F(z) + tF(z') = (1-t)G(z) + tG(z')$.

If $b \neq 0$, then $\beta \neq 0$ or $\beta' \neq 0$, hence $a, b > 0$. Then, we have

$$p = a \underbrace{\left[\frac{(1-t)\alpha}{a} x + \frac{t\alpha'}{a} x' \right]}_{\in D} - b \underbrace{\left[\frac{(1-t)\beta}{b} y + \frac{t\beta'}{b} y' \right]}_{\in D}.$$

Using this, and recalling the fact that $a, b \geqslant 0$ and $a - b = 1$, the definition of G yields

$$G(p) = aF\left(\frac{(1-t)\alpha}{a} x + \frac{t\alpha'}{a} x' \right) - bF\left(\frac{(1-t)\beta}{b} y + \frac{t\beta'}{b} y' \right).$$

As F is $\{0\}$-convex, we thus infer

$$\begin{aligned} G(p) &= a\frac{(1-t)\alpha}{a} F(x) + a\frac{t\alpha'}{a} F(x') - b\frac{(1-t)\beta}{b} F(y) - b\frac{t\beta'}{b} F(y') \\ &= (1-t) \underbrace{[\alpha F(x) - \beta F(y)]}_{G(z)} + t \underbrace{[\alpha' F(x') - \beta' F(y')]}_{G(z')} \\ &= (1-t)G(z) + tG(z'). \end{aligned}$$

All in all, by Lemma 17, G is affine.

'ii)⇒iii)': Let $U := \operatorname{par}(\operatorname{aff} D)$, $\bar{x} \in \operatorname{aff} D$ and let $L : U \to \mathbb{E}_2$ be the linear map given by $L(x) := G(x + \bar{x})$ for all $x \in U$. Now define $H : \mathbb{E}_1 \to \mathbb{E}_2$ by $H(u + u') = L(u - \bar{x})$ for all $u \in U, u' \in U^{\perp}$. Then H is affine, and $H(x) = G(x) = F(x)$ for all $x \in D$.

'iii)⇒i)': If there exists $H : \mathbb{E}_1 \to \mathbb{E}_2$ affine such that $F = H_{|D}$, then Lemma 17 in particular yields $G(tx + (1-t)y) = tG(x) + (1-t)G(y)$ for all $t \in (0,1)$ and $x, y \in D$. Hence $F(tx + (1-t)y) = tF(x) + (1-t)F(y)$ for all $x, y \in D$, $t \in (0,1)$, and as D is a convex set, F is $\{0\}$-convex. $\square$

As a simple corollary we get the following result.

Corollary 19. *The $\{0\}$-convex functions $\mathbb{E}_1 \to \mathbb{E}_2$ are exactly the affine functions.*

3.2 Convexity with respect to a (nontrivial) subspace

Using our study on $\{0\}$-convexity above, we are now in a position to investigate the functions which are convex w.r.t. a given (nontrivial) subspace. To this end, observe that for a subspace $U \subset \mathbb{E}$, the dual (and the polar) cone of U equal the orthogonal complement $U^\perp := \{u \in \mathbb{E} \mid \langle u,\, y\rangle = 0 \; \forall y \in U\}$.

Lemma 20. *Let $U \subset \mathbb{E}_2$ be a nontrivial subspace and let $\{e_1, \dots, e_r\}$ be a basis of $U^\perp$. Then for $F : D \subset \mathbb{E}_1 \to \mathbb{E}_2$, with $D \subset \mathbb{E}_1$ convex, the following are equivalent:*

i) F is U-convex;

ii) $\langle u,\, F\rangle$ is $\{0\}$-convex for all $u \in U^\perp$;

iii) $\langle e_i,\, F\rangle$ is $\{0\}$-convex for all $i = 1, \dots, r$.

Proof. 'i)$\Leftrightarrow$ii)': If F is U-convex and since $U^\perp$ is a subspace, by Proposition 14 we find that both $\langle u,\, F\rangle$ and $\langle -u,\, F\rangle$ are convex for every $u \in U^\perp$. Therefore, $\langle u,\, F\rangle$ is $\{0\}$-convex for all $u \in U^\perp$. In turn, if $\langle u,\, F\rangle$ is $\{0\}$-convex for all $u \in U^\perp$, then in particular $\langle u,\, F\rangle$ is ($\mathbb{R}_+$-)convex for all $u \in U^\perp$, so we can use Proposition 14 to conclude that F is U-convex.
'ii)$\Leftrightarrow$iii)': The implication ii)$\Rightarrow$iii) is trivial. In turn, if iii) holds, let $u \in U^\perp$, i.e. $u = \sum_{i=1}^r u_i e_i$ for some $u_i \in \mathbb{R}$. By assumption, $\langle e_i,\, F\rangle$ $(i = 1, \dots, r)$ are $\{0\}$-convex. Since $\langle u,\, F\rangle(x) = \sum_{i=1}^r u_i \langle e_i,\, F\rangle(x)$ for all $x \in \operatorname{dom} F$, we find that $\langle u,\, F\rangle$ is $\{0\}$-convex. $\square$

Proposition 21. *Let $U \subset \mathbb{E}_2$ be a nontrivial subspace and let $F : D \subset \mathbb{E}_1 \to \mathbb{E}_2$ with D convex. Then F is U-convex if and only if there exists an affine map $G\colon \operatorname{aff} D \to \mathbb{E}_2$ and a function $H\colon \operatorname{aff} D \to U$ such that $G_{|D} + H_{|D} = F$.*

Proof. Suppose that F is U-convex and let $p : \mathbb{E}_2 \to U$ be the orthogonal projection onto U. Define

$$H : \operatorname{aff} D \to U, \quad H(x) := \begin{cases} p(F(x)), & x \in D, \\ 0, & \text{else.} \end{cases}$$

Set $\hat{G} := F - H_{|D}$. By Lemma 20 and Proposition 18, $\langle u,\, F\rangle_{|D}$ is the restriction of an affine function $f^u : \operatorname{aff} D \to \mathbb{R}$ for all $u \in U^\perp$. However, as $H(x) \in U$ $(x \in \mathbb{E}_1)$, we have $\left\langle u,\, \hat{G}\right\rangle = \langle u,\, F\rangle$ for all $u \in U^\perp$. Now, let $\{e_1, \dots, e_r\}$ be an orthogonal basis of $U^\perp$. Then $\hat{G} = \sum_{i=1}^r f^{e_i}_{|D} e_i$. Then $G := \sum_{i=1}^r f^{e_i} e_i : \operatorname{aff} D \to \mathbb{E}_2$ is affine and $F = G_{|D} + H_{|D}$.

Conversely, suppose that there exists a function $H\colon \operatorname{aff} D \to U$ and an affine map $G\colon \operatorname{aff} D \to \mathbb{E}_2$ such that $F = G_{|D} + H_{|D}$. Then for $u \in U^\perp$ we have $\langle u,\, F\rangle = \langle u,\, G\rangle_{|D}$, and thus $\langle u,\, F\rangle$ is $\{0\}$-convex, as $\langle u,\, G\rangle$ is affine on $\operatorname{aff} D \supset D$ and D is convex. Thus, by Lemma 20, F is U-convex. $\square$

3.3 Convexity with respect to a half-space and polyhedral cones

Every (proper) half-space $H \subset \mathbb{E}$ that is also a cone is of the form $H = \{x \in \mathbb{E} \mid \langle w,\, x\rangle \geq 0\}$ for some $w \in \mathbb{E} \setminus \{0\}$. Clearly, H is (polyhedral) convex and closed with dual cone $H^+ = \mathbb{R}_+ w$.

Proposition 22. *Let $w \in \mathbb{E}_2 \setminus \{0\}$ and let $H = \{y \in \mathbb{E}_2 \mid \langle w,\, y\rangle \geqslant 0\}$ be the associated half-space. Then for $F : D \to \mathbb{E}_2$ with $D \subset \mathbb{E}_1$ (nonempty) convex we have:*

a) F is H-convex if and only if $\langle w,\, F\rangle$ is convex.

b) F is H-closed if and only if $\langle w,\, F\rangle$ is lower semicontinuous.

Proof. a) By Proposition 14 a), F is H-convex if and only if $\langle u,\, F\rangle$ is convex for all $u \in \operatorname{ri}(H^+) = \mathbb{R}_{++} w$. However, $\langle tw,\, F\rangle$ is convex if and only if $\langle w,\, F\rangle$ is convex for all $t > 0$.
b) If $\langle w,\, F\rangle$ is lower semicontinuous, the conclusion that F is H-closed follows with a similar argument as in a) from Proposition 14 b) observing that $H \neq \mathbb{E}_2$ and $H^+ \setminus \{0\} = \mathbb{R}_{++} w$.

If $\langle w,\, F\rangle$ is not lower semicontinuous there exists $\{x_k \in \operatorname{dom} F\} \to \bar{x}$ such that $\langle w,\, F\rangle(x_k) \to \alpha \in \mathbb{R}$, and $\alpha < \langle w,\, F\rangle(\bar{x})$. Now pick $r \in (\alpha, \langle w,\, F\rangle(\bar{x}))$ and choose $\bar{y} \in \mathbb{E}_2$ such that $\langle w,\, \bar{y}\rangle = r$ (which is possible as $w \neq 0$). Then $(x_k, \bar{y}) \in H\text{-epi}\, F$ for all k sufficiently large, and $(x_k, \bar{y}) \to (\bar{x}, \bar{y}) \notin H\text{-epi}\, F$. Hence, $H\text{-epi}\, F$ is not closed then. $\square$

Proposition 22 combined with Lemma 15 yields the following result on polyhedral cones.

Corollary 23. *Let $w_1, \dots, w_l \in \mathbb{E} \setminus \{0\}$ and let $P = \bigcap_{i=1}^l H_i$ with $H_i = \{x \mid \langle w_i,\, x\rangle \geq 0\}$. Then for $F : D \to \mathbb{E}_2$ with $D \subset \mathbb{E}_1$ (nonempty) convex we have:*

a) F is P-convex if and only if $\langle w_i,\, F\rangle$ is convex for all $i = 1, \dots, l$.

b) F is P-closed if $\langle w_i,\, F\rangle$ is lower semicontinuous for all $i = 1, \dots, l$.

3.4 The smallest closed cone with respect to which F is convex

The following result ensures the existence of a smallest closed cone K_F with respect to which F is convex (hence, K_F is also convex by Proposition 11) and characterizes its dual cone.

Proposition 24 (The cone K_F and its dual)**.** *Let $D \subset \mathbb{E}_1$ be nonempty and convex and let $F\colon D \to \mathbb{E}_2$. Then the following hold:*

a) There exists a smallest closed (and convex) cone $K_F \subset \mathbb{E}_2$ with respect to which F is convex, i.e if F is K-convex and K is closed and convex, then $K \supset K_F$.

b) The dual cone of K_F from a) is given by $K_F^+ = \{u \in \mathbb{E}_2 \mid \langle u, F\rangle$ is convex$\}$.

Proof. a) Define K_F as the intersection of all closed and convex cones which respect to which F is convex. Then K_F is nonempty (as F is $\mathbb{E}_2$-convex and every cone contains 0), closed and convex, and, by Lemma 15, F is K_F-convex. By construction, there is no smaller cone with these properties.

b) See [17, Lemma 6.1]. □

3.5 The smallest (closed) cone with respect to which F is convex *and* closed

We now investigate how the situation changes in comparison to the study in Section 3.4 when we are looking for the smallest (by Proposition 11 necessarily closed and convex) cone with respect to which a function F is convex *and* closed. Such cone does not need to exist, simply because a given function $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ need not be $\mathbb{E}_2$-closed. More concretely, for every cone $K \subset \mathbb{R}$, the indicator $F := \delta_{(0,1]}$ has K-epi $F = (0, 1] \times K$, which is not closed.

Proposition 25 (The cone $\hat{K}_F$)**.** *Let $D \subset \mathbb{E}_1$ be nonempty and convex and let $F\colon D \to \mathbb{E}_2$ such that there exists a (necessarily closed and convex) cone $K \subset \mathbb{E}_2$ with respect to which F is closed and convex. Then there exists a smallest closed and convex cone $\hat{K}_F \subset \mathbb{E}_2$ such that F is $\hat{K}_F$-closed and $\hat{K}_F$-convex.*

Proof. Follows readily from Lemma 15. □

In the spirit of Proposition 24 b), we want to characterize the dual cone of $\hat{K}_F$ (if it exists). To this end, the following lemma is useful.

Lemma 26. *Let $F : D \subset \mathbb{E}_1 \to \mathbb{E}_2$ with D convex (and nonempty). Then the following hold:*

a) The set $\mathrm{cl}\ \{u \in \mathbb{E}_2 \mid \langle u,\, F\rangle \in \Gamma_0(\mathbb{E}_1)\}$ *is either empty or a closed convex cone.*

b) The set $\{u \in \mathbb{E}_2 \mid \langle u,\, F\rangle \in \Gamma_0(D)\}$ is a convex cone (in particular nonempty).

Proof. a) Assume nonemptiness, in which case $S := \{u \in \mathbb{E}_2 \mid \langle u,\, F\rangle \in \Gamma_0(\mathbb{E}_1)\}$ is also nonempty. Then, clearly, S is convex and, consequently, so is cl S. Moreover, for $u \in \mathrm{cl}\ S$ and $\lambda \geq 0$ there exist $\{u_k \in S\} \to u$ and $\{\lambda_k > 0\} \to \lambda$ and with $\lambda_k u_k \in S$, hence $\lambda u \in \mathrm{cl}\ S$, and thus cl S is a closed convex cone.

b) Set $K := \{u \in \mathbb{E}_2 \mid \langle u,\, F\rangle \in \Gamma_0(D)\}$. We first show that K is a cone. To this end, note that $\langle 0,\, F\rangle = \delta_D$, whose epigraph $D \times \mathbb{R}_+$ is clearly closed in the topology induced by $D \times \mathbb{R}$. Now, for $u \in K$ and $\lambda > 0$ observe that epi $\langle \lambda u,\, F\rangle = L_\lambda^{-1}(\mathrm{epi}\ \langle u,\, F\rangle)$ for $L_\lambda : (x, \alpha) \mapsto (x, \alpha/\lambda)$, which is then closed as the linear preimage of a closed set in the topology induced by $D \times \mathbb{R}$. Therefore, K is a cone. In order prove that K is convex, it hence suffices to show that $K + K \subset K$, see [19, Exercise 3.7]. To this end, let $u, v \in K$ and take $\{(x_k, \alpha_k) \in \mathrm{epi}\ \langle u + v,\, F\rangle\} \to (x, \alpha) \in D \times \mathbb{R}$. In particular, we have $(x_k, \alpha_k - \langle v,\, F(x_k)\rangle) \in \mathrm{epi}\ \langle u,\, F\rangle$ for all $k \in \mathbb{N}$. Let $z := \liminf_k \langle v,\, F(x_k)\rangle$. W.l.o.g. $z = \lim_k \langle v,\, F(x_k)\rangle$ (otherwise go to subsequence), and by D-closedness of $\langle v,\, F\rangle$, we find $(x, z) \in \mathrm{epi}\ \langle v,\, F\rangle$, i.e. $\langle v,\, F(x)\rangle \leq z$. Moreover, by D-closedness of $\langle u,\, F\rangle$ we find that $(x, \alpha - z) \in \mathrm{epi}\ \langle u,\, F\rangle$, hence $\alpha \geq \langle u,\, F(x)\rangle + z \geq \langle u,\, F\rangle (x) + \langle v,\, F\rangle (x)$. Therefore, $(x, \alpha) \in \mathrm{epi}\ \langle u + v,\, F\rangle$. □

Proposition 27 (The dual cone of $\hat{K}_F$)**.** *Let $F\colon D \subset \mathbb{E}_1 \to \mathbb{E}_2$ with D nonempty and convex. If $\hat{K}_F$ (in the sense of Proposition 25) exists, we have*

$$\hat{K}_F^+ = \mathrm{cl}\,\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)\} = \mathrm{cl}\,\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}. \tag{8}$$

Proof. Set $Q := \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)\}$ and $R := \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$. From Remark 4, we deduce that $Q \subset R$. Moreover, use [17, Corollary 7.4(ii)], to find $\langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $u \in \mathrm{ri}\,(\hat{K}_F^+)$. Consequently, $\mathrm{ri}\,(\hat{K}_F^+) \subset Q$, hence $\hat{K}_F^+ \subset \mathrm{cl}\,Q \subset \mathrm{cl}\,R$. We thus have

$$R^+ = (\mathrm{cl}\,R)^+ \subset (\mathrm{cl}\,Q)^+ \subset \hat{K}_F. \tag{9}$$

We now show that F is R^+-convex: To this end, first note that $\langle u, F\rangle$ is convex for all $u \in R$. Any $u \in \mathrm{cl}\,R$ is a limit $\{u_k \in R\} \to u$, and therefore $\langle u, F\rangle$ is the pointwise limit of convex functions $\langle u_k, F\rangle$, and hence convex (Lemma 2). Moreover, by Lemma 26 b), R is a convex cone, hence $R^{++} = \mathrm{cl}\,R$. Thus, by Proposition 14 a), F is R^+-convex.

We now prove that F is also R^+-closed: To this end, let $\{(x_k, y_k) \in R^+\text{-epi}\,F\} \to (x, y)$. In particular, there exists $\{v_k \in R^+\}$ such that $F(x_k) + v_k = y_k$ for all $k \in \mathbb{N}$. Moreover, as $R^+ \subset \hat{K}_F$ (see (9)) and $\hat{K}_F$-epi F is closed (by definition), we have $(x, y) \in \hat{K}_F$-epi F, and consequently, $x \in D$. Thus we can use the fact that, by definition of R, $\langle u, F\rangle$ is D-closed for all $u \in R$, and hence

$$\langle u, y\rangle = \lim_{k\to\infty} \langle u, y_k\rangle = \lim_{k\to\infty} \langle u, F(x_k)\rangle + \underbrace{\langle u, v_k\rangle}_{\geqslant 0} \geqslant \liminf_{k\to\infty} \langle u, F\rangle (x_k) \geqslant \langle u, F\rangle (x)$$

for all $u \in R$. Now, every $u \in R^{++} = \mathrm{cl}\,R$ is a limit $\{u_k \in R\} \to u$ with $\langle u_k, y\rangle \geq \langle u_k, F(x)\rangle$, and hence $\langle u, y\rangle \geq \langle u, F(x)\rangle$. As $u \in R^{++}$ was arbitrary, this shows that $y \geq_{R^+} F(x)$, and thus $(x, y) \in R^+$-epi F, which shows that R^+-epi F is closed and hence F is R^+-closed (and R^+-convex as proved earlier). Since $R^+ \subset (\mathrm{cl}\,Q)^+ \subset \hat{K}_F$ it follows that $\hat{K}_F = \mathrm{cl}\,R = \mathrm{cl}\,Q$, which concludes the proof. ☐

The natural question as to when closures in the above result are superfluous is addressed now.

Corollary 28. *Let $F\colon D \subset \mathbb{E}_1 \to \mathbb{E}_2$ with D nonempty and convex and assume that $\hat{K}_F$ exists. Then:*

a) $\hat{K}_F^+ = \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$ if and only if $\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$ is closed.

b) If, for every sequence $\{x_k \in D\} \to x \in D$ (and every $x \in D$), there exists $v \in \mathrm{ri}\,(\hat{K}_F^+)$ such that $\{\langle v, F\rangle (x_k)\}$ does not tend to $+\infty$, then $\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$ is closed.

c) If, for all $x \in D \setminus \mathrm{ri}\,D$, there exists a neighborhood $\mathcal{N}_x$ of x, and a continuous $\hat{K}_F$-majorant[2] *of F on $\mathcal{N}_x \cap D$, then $\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$ is closed.*

In particular, $\hat{K}_F^+ = \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$ if D is relatively open (e.g. affine).

Proof. a) Follows readily from Proposition 27.

b) Denote $K = \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$, which is a convex cone by Lemma 26 b). By Proposition 27, $\hat{K}_F^+ = \mathrm{cl}\,K$. Consider $u \in \mathrm{cl}\,K$. Then $\langle u, F\rangle$ is convex by Lemma 2 and proper with $\mathrm{dom}\,\langle u, F\rangle = D$.

Now let $x \in D$ and $\{x_k \in D\} \to x \in D$. Then, by assumption, there exists $v \in \mathrm{ri}\,(\hat{K}_F^+) = \mathrm{ri}\,K$ such that $\{\langle v, F\rangle (x_k)\}$ is uniformly bounded away from $+\infty$, hence w.l.o.g. we can assume that $\langle v, F\rangle (x_k) \to r < +\infty$. As $\langle v, F\rangle \in \Gamma_0(D)$, we have $-\infty < \langle v, F\rangle (x) \leq r < +\infty$. Using Lemma 5 (and that $\mathrm{ri}\,K$ is a pre-cone), we have $u + tv \in \mathrm{ri}\,K$, and hence $\langle u + tv, F\rangle \in \Gamma_0(D)$ for all $t > 0$. Thus

$$\begin{aligned}\liminf_{k\to\infty} \langle u, F\rangle (x_k) &= \liminf_{k\to\infty} \langle u + tv, F\rangle (x_k) - t\,\langle v, F\rangle (x_k)\\ &= \liminf_{k\to\infty} \langle u + tv, F\rangle (x_k) - tr\\ &\geqslant \langle u + tv, F\rangle (x) - tr.\end{aligned}$$

Letting $t \to 0$ gives $\liminf_{k\to\infty} \langle u, F\rangle (x_k) \geqslant \langle u, F\rangle (x)$, hence $\langle u, F\rangle \in \Gamma_0(D)$ as desired.

c) Let $\{x_k \in D\} \to x \in D$. We will show that $\langle v, F\rangle (x_k)$ does not tend to $+\infty$, for any $v \in \mathrm{ri}\,(\hat{K}_F^+)$, which by b) then gives the desired conclusion.

[2]We refer the reader to Definition 38 for a formal introduction.

First suppose that $x \in \operatorname{ri} D$. Let $v \in \operatorname{ri}(\hat{K}_F^+) = \operatorname{ri}\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$. Then $\langle v, F\rangle$ is convex with $\operatorname{dom}\langle v, F\rangle = D$, and hence $\langle v, F\rangle_{|\operatorname{ri} D}$ is continuous, see e.g. [18, Theorem 10.1]. Now, as $\{x_k \in D\} \to x \in \operatorname{ri} D$, for k sufficiently large, we have $x_k \in \operatorname{ri} D$. Therefore, $\langle v, F\rangle(x_k) \to \langle v, F\rangle(x) < +\infty$ as $k \to \infty$, Hence $\langle v, F\rangle(x_k)$ does not tend to $+\infty$.

In turn, for $x \in D \setminus \operatorname{ri} D$ let G_x be the continuous $\hat{K}_F$-majorant of F on $\mathcal{N}_x \cap D$ and let $v \in \operatorname{ri}(\hat{K}_F^+)$. Then, as $G_x(y) \geqslant_K F(y)$ for all $y \in \mathcal{N}_x \cap D$, hence $\langle v, G_x\rangle(y) \geqslant \langle v, F\rangle(y)$ for all $y \in \mathcal{N}_x \cap D$. Since $\{x_k \in D\} \to x$, we have that $x_k \in \mathcal{N}_x \cap D$ for k sufficiently large, and thus $\langle v, F\rangle(x_k) \leqslant \langle v, G_x\rangle(x_k)$. However, G_x is continuous, so $\langle v, G_x\rangle$ is continuous as well, hence $\langle v, G_x\rangle(x_k) \to \langle v, G_x\rangle(x) < +\infty$, thus $\langle v, F\rangle(x_k)$ does not tend to $+\infty$. □

We close out this section by clarifying the question as to when K_F and $\hat{K}_F$ coincide.

Proposition 29 ($K_F = \hat{K}_F$)**.** *Let $D \subset \mathbb{E}_1$ be nonempty and convex and let $F : D \to \mathbb{E}_2$. Then $K_F = \hat{K}_F$ if and only if F is K_F-closed.*

Proof. By definition of the respective cones, we always have $\hat{K}_F \supset K_F$. But if F is K_F-closed then, $\hat{K}_F \subset K_F$, by definition of $\hat{K}_F$, and hence equality holds.

In turn, if F is not K_F-closed, then $K_F \neq \hat{K}_F$, since F is $\hat{K}_F$-closed by definition. □

4 When is $\overline{\operatorname{conv}}\,(\operatorname{gph} F) = K\text{-}\operatorname{epi} F$?

This section is devoted to our main question as to when the closed convex hull of the graph of a function equals its K-epigraph.

4.1 A characterization via the horizon cone

We commence this subsection with the central link between the graph and the K-epigraph of a function. To obtain an elegant proof we briefly tap into Fenchel conjugacy [18]. To this end, realize that every set $S \subset \mathbb{E}$ is uniquely determined through its indicator function $\delta_S : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ which is paired in duality with the *support function* $\sigma_S : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$, $\sigma_S(x) = \sup_{y \in S}\langle x, y\rangle$ via the conjugacy relations $\delta_S^* = \sigma_S = \sigma_{\overline{\operatorname{conv}} S}$, hence $\sigma_S^* = \delta_{\overline{\operatorname{conv}} S}$, and thus $\overline{\operatorname{conv}}\,\delta_S = \delta_{\overline{\operatorname{conv}} S}$.

Proposition 30. *Let $K \subset \mathbb{E}_2$ be a closed, convex cone and let $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper, K-closed and K-convex. Then*

$$K\text{-}\operatorname{epi} F = \operatorname{cl}\left(\overline{\operatorname{conv}}(\operatorname{gph} F) + \{0\} \times K\right).$$

Proof. We first observe that Using (3) and the set-additivity for support functions we have

$$\sigma_{K\text{-}\operatorname{epi} F} = \sigma_{\operatorname{gph} F + \{0\}\times K} = \sigma_{\operatorname{gph} F} + \sigma_{\{0\}\times K} = \sigma_{\overline{\operatorname{conv}}\,(\operatorname{gph} F)} + \delta_{\mathbb{E}_1 \times K^\circ}.$$

Moreover, the assumptions on F imply that K-epi F is closed and convex, and hence

$$\delta_{K\text{-}\operatorname{epi} F} = \left(\sigma_{\overline{\operatorname{conv}}\,(\operatorname{gph} F)} + \delta_{\mathbb{E}_1 \times K^\circ}\right)^* = \operatorname{cl}\left(\delta_{\overline{\operatorname{conv}}\,(\operatorname{gph} F)} \,\square\, \delta_{\{0\}\times K}\right) = \delta_{\operatorname{cl}\left(\overline{\operatorname{conv}}\,(\operatorname{gph} F) + \{0\}\times K\right)}.$$

Here the second identity uses [18, Theorem 16.4], while the third holds due to the identity $\delta_A \square \delta_B = \delta_{A+B}$ for any two sets. □

Note that Proposition 30 could also be proved without Fenchel conjugacy, using more elementary facts about closures and convex hulls of sums of sets.

We are now in a position to state our first main theorem.

Theorem 31. *Let $K \subset \mathbb{E}_2$ be a closed convex cone and let $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be K-convex and K-closed. Then*

$$K\text{-}\operatorname{epi} F = \overline{\operatorname{conv}}\,(\operatorname{gph} F) \iff \{0\} \times K \subset [\overline{\operatorname{conv}}\,(\operatorname{gph} F)]^\infty.$$

Proof. Suppose that K-epi $F = \overline{\operatorname{conv}}\,(\operatorname{gph} F)$. It follows from Proposition 30 that

$$\overline{\operatorname{conv}}\,(\operatorname{gph} F) + \{0\} \times K \subset \operatorname{cl}\left(\overline{\operatorname{conv}}(\operatorname{gph} F) + \{0\} \times K\right) = \overline{\operatorname{conv}}\,(\operatorname{gph} F).$$

Taking the horizon cone on both sides and using [19, Exercise 3.12], yields $\{0\} \times K \subset [\overline{\operatorname{conv}}\,(\operatorname{gph} F)]^\infty$.

Now suppose that $\{0\} \times K \subset [\overline{\operatorname{conv}}\,(\operatorname{gph} F)]^\infty$. Then

$$\overline{\operatorname{conv}}\,(\operatorname{gph} F) + \{0\} \times K \subset \overline{\operatorname{conv}}\,(\operatorname{gph} F) + [\overline{\operatorname{conv}}\,(\operatorname{gph} F)]^\infty = \overline{\operatorname{conv}}\,(\operatorname{gph} F).$$

where the last identity uses e.g. [19, Theorem 3.6]. Therefore, again using Proposition 30, we obtain

$$K\text{-epi}\, F = \mathrm{cl}\,(\overline{\mathrm{conv}}\,(\mathrm{gph}\, F) + \{0\} \times K) \subset \overline{\mathrm{conv}}\,(\mathrm{gph}\, F) \subset K\text{-epi}\, F.$$

□

We will frequently make use of the following trivial observation.

Remark 32. *We observe that the closure operation in $[\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)]^\infty$ is superfluous, i.e.*

$$[\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)]^\infty = [\mathrm{conv}\,(\mathrm{gph}\, F)]^\infty.$$

We immediately obtain the following sufficient condition.

Corollary 33. *Let $K \subset \mathbb{E}_2$ be a closed, convex cone and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be K-convex and K-closed such that $\{0\} \times K \subset \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$. Then $K\text{-epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$.*

Proof. Combine Theorem 31, the fact that $\{0\} \times K$ is a closed cone, and the fact that the horizon operation preserves inclusion. □

Combining Corollary 33 with Lemma 9 yields the following result.

Corollary 34. *Let K be a cone of $\mathbb{E}_2$ and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper, and define the closed, convex cone $K^F := \{u \in \mathbb{E}_2 \mid (0, u) \in [\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)]^\infty\}$. Then*

$$K\text{-epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F) \iff K = K^F = \hat{K}_F.$$

Proof. First, let $K = K^F = \hat{K}_F$. By definition of $K^F = K$ we hence have $\{0\} \times K \subset [\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)]^\infty$. From Theorem 31 we thus conclude that $K\text{-epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$.

In turn, assume that $K\text{-epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$. Then by Theorem 31 we have $K \subset K^F$, and hence $\overline{\mathrm{conv}}\,(\mathrm{gph}\, F) = K\text{-epi}\, F \subset K^F\text{-epi}\, F$. In addition, $\{0\} \times K^F \subset [\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)]^\infty$, by definition of K^F. Hence, using (3) and the horizon property of convex sets, we have

$$K^F\text{-epi}\, F = \mathrm{gph}\, F + \{0\} \times K^F \subset \overline{\mathrm{conv}}\,(\mathrm{gph}\, F) + [\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)]^\infty = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F).$$

Thus, $K^F\text{-epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F) = K\text{-epi}\, F$ and hence, by Lemma 9, we already have $K^F = K$. Moreover, as F is K-convex and K-closed, we have $\hat{K}_F \subset K$, thus $\hat{K}_F\text{-epi}\, F \subset K\text{-epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$. Using the fact that $\overline{\mathrm{conv}}\,(\mathrm{gph}\, F) \subset \hat{K}_F\text{-epi}\, F$ (as $\hat{K}_F\text{-epi}\, F$ is a closed convex set containing $\mathrm{gph}\, F$), we conclude that $\hat{K}_F\text{-epi}\, F = K\text{-epi}\, F$, hence, again by Lemma 9, $K = \hat{K}_F$. □

4.2 Necessary conditions

In this subsection we discuss necessary conditions for $\overline{\mathrm{conv}}\,(\mathrm{gph}\, F) = K\text{-epi}\, F$.

4.2.1 Necessary conditions on the dual cone

Proposition 35. *Let $K \subset \mathbb{E}_2$ be a cone and $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ proper such that $K\text{-epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$. Then $K \setminus \{0\} \subset \{u \in \mathbb{E}_2 \mid \mathrm{epi}\, \langle u,\, F\rangle = \overline{\mathrm{conv}}\,(\mathrm{gph}\, \langle u,\, F\rangle)\}$. In particular, $K \subset \hat{K}_F^+$.*

Proof. First, note that the conclusions are trivally true for $K = \{0\}$. Otherwise let $u \in K \setminus \{0\}$. By Theorem 31, we have that $(0, u) \in [\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)]^\infty$. By Remark 32 there hence exist $\{(x_k, y_k) \in \mathrm{conv}\,(\mathrm{gph}\, F)\}$ and $\{\lambda_k\} \downarrow 0$ such that $\lambda_k(x_k, y_k) \to (0, u)$. With $\kappa := \dim \mathbb{E}_1 \times \mathbb{E}_2$ and Carathéodory's theorem [18, Theorem 17.1], for $i = 1, \ldots, \kappa + 1$, we find sequences $\{x_k^i \in \mathrm{dom}\, F\}_k$ and $\{\alpha_k^i\}_k$ such that $\sum_{i=1}^{\kappa+1} \alpha_k^i = 1$ for all $k \in \mathbb{N}$ as well as

$$x_k = \sum_{i=1}^{\kappa+1} \alpha_k^i x_k^i \quad \text{and} \quad y_k = \sum_{i=1}^{\kappa+1} \alpha_k^i F(x_k^i) \quad \forall k \in \mathbb{N}.$$

Now let $t \geq 0$. Then $t_k := t\frac{\lambda_k}{\|u\|^2} \downarrow 0$ and

$$t_k x_k \to 0 \quad \text{and} \quad t_k \sum_{i=1}^{\kappa+1} \alpha_k^i \,\langle u,\, F\rangle\,(x_k^i) = t\frac{\langle \lambda_k y_k,\, u\rangle}{\|u\|^2} \to t.$$

Thus, for $t \geq 0$, we have $(0, t) \in [\mathrm{conv}\,(\mathrm{gph}\, \langle u, F\rangle)]^\infty = [\overline{\mathrm{conv}}\,(\mathrm{gph}\, \langle u, F\rangle)]^\infty$. Hence $\{0\} \times \mathbb{R}_+ \subset [\overline{\mathrm{conv}}\,(\mathrm{gph}\, \langle u, F\rangle)]^\infty$, so by Theorem 31, $\mathrm{epi}\, \langle u, F\rangle = \overline{\mathrm{conv}}\,(\mathrm{gph}\, \langle u, F\rangle)$. This shows

$$K \setminus \{0\} \subset \{u \in \mathbb{E}_2 \mid \mathrm{epi}\, \langle u,\, F\rangle = \overline{\mathrm{conv}}\,(\mathrm{gph}\, \langle u,\, F\rangle)\} \subset \{u \in \mathbb{E}_2 \mid \langle u,\, F\rangle \in \Gamma_0(\mathbb{E}_1)\},$$

the first inclusion of which is the first claim of the proposition. The second claim now follows as $K \subset \mathrm{cl}\,(\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)\}) = \hat{K}_F^+$, by Proposition 27. □

We readily derive the following necessary condition on the dual cone.

Corollary 36. *Under the assumptions of Proposition 35, we have $K \subset K^+$.*

Proof. By Corollary 34, $K = \hat{K}_F$, and by Proposition 35, $K \subset \hat{K}_F^+ = K^+$. □

4.2.2 Affine majorization and minorization

For motivational purposes, we start this subsection with the scalar case ($K = \mathbb{R}_+$), where the question whether the closed convex hull of the graph of a function equals its K-epigraph can be fully answered via affine majorization. The proof relies, in essence, on a standard separation argument.

Theorem 37 (The scalar case)**.** *Let $f \in \Gamma_0(\mathbb{E})$. Then* $\operatorname{epi} f = \overline{\operatorname{conv}}\,(\operatorname{gph} f)$ *if and only if f does not have an affine majorant on its domain.*

Proof. Suppose that there exists $(\bar{x}, \bar{t}) \in \operatorname{epi} f \backslash \overline{\operatorname{conv}}\,(\operatorname{gph} f)$. In particular, $\bar{x} \in \operatorname{dom} f$, and by strong separation [18, Corollary 11.4.2], there exists $(s, \alpha) \in \mathbb{E}_1 \times \mathbb{R}$ such that

$$\langle s,\, \bar{x}\rangle + \alpha\bar{t} > \sup_{(x,t)\in\overline{\operatorname{conv}}\,(\operatorname{gph} f)} \langle s,\, x\rangle + \alpha t. \tag{10}$$

Choosing $(x, t) := (\bar{x}, f(\bar{x}))$, we find that $\alpha(\bar{t} - f(\bar{x})) > 0$, and hence, $\alpha > 0$. It then follows from (10) with $x \in \operatorname{dom} f$ and $t = f(x)$ that $\langle s/\alpha,\, \bar{x} - x\rangle + \bar{t} > f(x)$. Thus f is majorized on its domain by the affine map $x \mapsto -\langle s/\alpha,\, x\rangle + \langle s/\alpha,\, \bar{x}\rangle + \bar{t}$, which proves one direction.

To prove the converse implication, suppose now that f has an affine majorant on its domain, i.e. there exists $(a, \beta) \in \mathbb{E} \times \mathbb{R}$, such that $f(x) \le \langle a,\, x\rangle + \beta =: g(x)$ for all $x \in \operatorname{dom} f$. Now pick $\bar{x} \in \operatorname{dom} f$. Then $(\bar{x}, g(\bar{x})+1) \in \operatorname{epi} f$, and it hence suffices to show that $(\bar{x}, g(\bar{x})+1) \notin \overline{\operatorname{conv}}\,(\operatorname{gph} F)$. Assume, by contradiction, that $(\bar{x}, g(\bar{x}) + 1) \in \overline{\operatorname{conv}}\,(\operatorname{gph} F)$. Then with $\kappa := \dim \mathbb{E} \times \mathbb{R}$, by Carathéodory's theorem [18, Theorem 17.1], for $i = 1, \ldots, \kappa+1$ there exist sequences $\{t_{i,k} \ge 0\}_{k\in\mathbb{N}}$ and $\{x_i^k \in \operatorname{dom} f\}_{k\in\mathbb{N}}$ such that $\sum_{i=1}^{\kappa+1} t_{i,k} = 1$ and $\sum_{i=1}^{\kappa+1} t_{i,k}(x_i^k, f(x_i^k)) \to_{k\to\infty} (\bar{x}, g(\bar{x}) + 1)$. Consequently

$$g(\bar{x}) + 1 = \lim_{k\to\infty} \sum_{i=1}^{\kappa+1} t_{i,k} f(x_i^k) \le \lim_{k\to\infty} \sum_{i=1}^{\kappa+1} t_{i,k} g(x_i^k) = \lim_{k\to\infty} \left\langle a,\, \sum_{i=1}^{\kappa+1} t_{i,k} x_i^k \right\rangle + \beta = g(\bar{x}),$$

which is the desired contradiction and thus concludes the proof. □

The questions as to what can be said when f in the above result is only proper and convex, but not necessarily closed is answered as the opening to Section 4.3.

To start our analysis of the vector-valued case we now formally introduce the notion of *K-minorants and -majorants*, respectively.

Definition 38 (K-minorants/-majorants)**.** *Let $K \subset \mathbb{E}_1$ be a cone, and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper. A function $G : \mathbb{E}_1 \to \mathbb{E}_2$ is said to be:*

- *a* K-majorant *of F on $S \subset \operatorname{dom} F$ if*

$$G(x) - F(x) \in K \quad \forall x \in S,$$

- *a* K-minorant *of F on $S \subset \operatorname{dom} F$ if*

$$F(x) - G(x) \in K \quad \forall x \in S.$$

For $S = \operatorname{dom} F$, we say that G is a K-minorant of F.

Naturally, in view of the scalar case from Theorem 37, we are mainly interested in the case where G is an affine function.

For a function $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$, we record that, for a pointed, closed, convex cone $K \neq \{0\}$ such that $K\text{-}\operatorname{epi} F = \overline{\operatorname{conv}}\,(\operatorname{gph} F)$, there cannot exist an affine K-majorant on the domain F.

Proposition 39. *Let $\{0\} \subsetneq K \subset \mathbb{E}_2$ be a closed, convex, pointed cone and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper. If $K\text{-}\operatorname{epi} F = \overline{\operatorname{conv}}\,(\operatorname{gph} F)$, then F cannot have an affine K-majorant on its domain.*

Proof. By assumption, K-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$. Now, pick $u \in K \setminus \{0\}$ and define $f := \langle u, F\rangle$. Then, by Proposition 35, epi $f = \overline{\text{conv}}\,(\text{gph}\, f)$. Now, assume that F has an affine K-majorant T on dom F. As $u \in K^+$ (by Corollary 36), it follows that $\langle u, T\rangle$ is an affine majorant of f on dom F = dom f. This contradicts Theorem 37, which proves the desired result. $\square$

It is well known that a proper, convex function posseses an affine ($\mathbb{R}_+$-)minorant [1, Theorem 9.20]. In the vector-valued setting, we can fall back on this result to get affine K-minorants for proper K-convex functions when K is a particular polyhedral cone.

Proposition 40. *Let $K = \{x \in \mathbb{E}_2 \mid \langle b_i, x\rangle \geq 0 \quad \forall i = 1, \dots, m\}$ with $b_1, \dots, b_m$ linearly independent. If $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ is K-convex and proper, then F has an affine K-minorant.*

Proof. It holds that $K^+ = \text{cone}\,\{b_1, \dots, b_m\}$, see e.g. [19, Lemma 6.45]. This cone is pointed by linear independence of $\{b_1, \dots, b_m\}$, hence $-K$ has nonempty interior, see [19, Exercise 6.22]. Now, in view of Proposition 14 a), for all $i = 1, \dots, m$, the functions $\langle b_i, F\rangle : \mathbb{E}_1 \to \mathbb{R}\cup\{+\infty\}$ are proper, convex and hence, see e.g. [1, Theorem 9.20], there exist $(c_i, \delta_i) \in \mathbb{E}_1 \times \mathbb{R}$ $(i = 1, \dots, m)$ such that

$$\langle b_i, F(x)\rangle \geq \langle c_i, x\rangle + \delta_i \quad \forall x \in \mathbb{E}_1, i = 1, \dots, m. \tag{11}$$

Now, let $A : \mathbb{E}_1 \to \mathbb{E}_2$ be linear such that $A(b_i) = c_i$ for all $i = 1, \dots, m$ and let $w \in -\text{int}\, K$. Then $\langle w, b_i\rangle < 0$ for all $i = 1, \dots, m$, cf. [19, Exercise 6.22]. By positive homogeneity (and since int K is a pre-cone), there hence exists $\bar{w} \in -\text{int}\, K$ with $\langle \bar{w}, b_i\rangle < \delta_i$ for all $i = 1, \dots, m$. Finally, with $\bar{L} := A^*$ it hence follows

$$\langle b_i, F(x)\rangle \geq \langle c_i, x\rangle + \delta_i \geq \left\langle b_i, \bar{L}(x)\right\rangle + \langle \bar{w}, b_i\rangle \quad \forall x \in \mathbb{E}_1, i = 1, \dots, m.$$

Therefore, for all $x \in \text{dom}\, F$, we have $F(x) - (\bar{L}(x) + \bar{w}) \in K$, and $x \mapsto \bar{L}(x) + \bar{w}$ is the desired affine K-minorant. $\square$

We point out that, with slightly more effort, the following stronger conclusion of Proposition 40 can be proven: *For every $x_0 \in \text{ri}\,(\text{dom}\, F)$ there exists a linear operator $T : \mathbb{E}_1 \to \mathbb{E}_2$ such that $T(x) - T(x_0) \leq_K F(x) - F(x_0)$ for all $x \in \text{dom}\, F$.*

4.3 Sufficient conditions

In this subsection we are primarily concerned with sufficient conditions. We start with some considerations in the scalar case.

Lemma 41. *Let $f \in \Gamma(\mathbb{E})$. Then the following hold:*

a) $\overline{\text{conv}}\,(\text{gph}\,\text{cl}\, f) = \overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)})$.

b) $\phi : \mathbb{E} \to \mathbb{R}$ *is an affine majorant of* cl f *on* dom (cl f) *if and only if ϕ is affine majorant of f on* ri (dom f).

c) If f is (dom f)*-closed (hence $f \in \Gamma_0(\text{dom}\, f)$), then $\phi : \mathbb{E} \to \mathbb{R}$ is an affine majorant of* cl f *on* dom (cl f) *if and only if ϕ is affine majorant of f on* dom f.

Proof. a) As $f(x) = \text{cl}\ f(x)$ for all $x \in \text{ri}\,(\text{dom}\, f)$ ([18, Theorem 7.4]), we have $\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)} \subset \text{gph}\,\text{cl}\ f$, and hence $\overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)}) \subset \overline{\text{conv}}\,(\text{gph}\, f)$. To prove the converse inclusion let $(x, \text{cl}\ f(x)) \in \text{gph}\,\text{cl}\ f$. Invoking [18, Theorem 7.5] (and [18, Theorem 6.1]), we find a sequence $\{x_k \in \text{ri}\,(\text{dom}\, f)\} \to x$ with $f(x_k) \to \text{cl}\ f(x)$. Therefore, $\text{gph}\,\text{cl}\ f \subset \text{cl}\ (\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)}) \subset \overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)})$, and hence, the desired inclusion follows by applying the $\overline{\text{conv}}$-operator on both sides.

b) If ϕ is an affine majorant of cl f on dom (cl f), then ϕ is an affine majorant of cl f on ri (dom f) $\subset$ dom (cl f), and hence an affine majorant of f on ri (dom f), since f and cl f coincide on ri (dom f). In turn, if ϕ is an affine majorant of f on ri (dom f), then for all $x \in$ dom (cl f), since ri (dom (cl f)) = ri (dom f) (see [18, Corollary 7.4.1]), by [18, Theorem 7.5] (and [18, Theorem 6.1]), there exists $\{x_k \in \text{ri}\,(\text{dom}\,(\text{cl}\ f))\} \to x$ with $\lim_k f(x_k) = \text{cl}\ f(x)$. However $\phi(x_k) \geqslant f(x_k)$ and ϕ is continuous so $\phi(x) \geqslant \text{cl}\ f(x)$, thus ϕ is an affine majorant of cl f on dom cl f.

c) By Lemma 3, $f(x) = \text{cl}\ f(x)$ for all $x \in \text{dom}\, f$. Therefore, if $\phi(x) \geq \text{cl}\ f(x)$ for all $x \in$ dom (cl f) $\supset$ dom f, then $\phi(x) \geq f(x)$ for all $x \in \text{dom}\, f$. In turn, if ϕ is an affine majorant of f on dom f $\supset$ ri (dom f), then b) shows that ϕ is an affine majorant of cl f on dom (cl f). $\square$

We record some immediate consequences of the foregoing result.

Corollary 42. *Let $f \in \Gamma(\mathbb{E})$. Then the following are equivalent:*

i) $\overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)}) = \text{cl}\,(\text{epi}\, f)$*;*

ii) f *has no affine majorant on* $\text{ri}\,(\text{dom}\, f)$*;*

iii) $\{0\} \times \mathbb{R}_+ \subset [\overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)})]^\infty$.

Proof. Observe that $\text{epi}\,(\text{cl}\, f) = \text{cl}\,(\text{epi}\, f)$, hence by Lemma 41 a) we have

$$i) \iff \overline{\text{conv}}\,(\text{gph}\,\text{cl}\, f) = \text{epi}\,(\text{cl}\, f). \tag{12}$$

'i)⇔ii)': By Lemma 41 b), we have that ii) is equivalent to saying that $\text{cl}\, f$ has no affine majorant on its domain. Therefore, the desired equivalence follows with (12) from Theorem 37 applied to $\text{cl}\, f \in \Gamma_0(\mathbb{E})$.

'i)⇔iii)': Apply Theorem 31 to $\text{cl}\, f$ and use (12). □

Corollary 43. *Let $f \in \Gamma_0(\text{dom}\, f)$. Then the following are equivalent:*

i) $\overline{\text{conv}}\,(\text{gph}\, f) = \text{cl}\,(\text{epi}\, f)$*;*

ii) f *has no affine majorants on its domain;*

iii) $\{0\} \times \mathbb{R}_+ \subset [\overline{\text{conv}}\,(\text{gph}\, f)]^\infty$.

Proof. We observe that $f \in \Gamma(\mathbb{E})$ (by definition of $\Gamma_0(\text{dom}\, f)$), and that $f(x) = \text{cl}\, f(x)$ for all $x \in \text{dom}\, f$, by Lemma 3. In addition, by Lemma 41, we have $\overline{\text{conv}}\,(\text{gph}\, f|_{\text{ri}\,(\text{dom}\, f)}) = \overline{\text{conv}}\,(\text{gph}\,\text{cl}\, f)$. Thus, we have

$$\overline{\text{conv}}\,(\text{gph}\, f) \subset \overline{\text{conv}}\,(\text{gph}\,\text{cl}\, f) = \overline{\text{conv}}\,(\text{gph}\, f|_{\text{ri}\,(\text{dom}\, f)}) \subset \overline{\text{conv}}\,(\text{gph}\, f),$$

and hence $\overline{\text{conv}}\,(\text{gph}\, f) = \overline{\text{conv}}\,(\text{gph}\, f|_{\text{ri}\,(\text{dom}\, f)})$. Consequently

$$i) \iff \overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)}) = \text{cl}\,(\text{epi}\, f),$$

and

$$iii) \iff \{0\} \times \mathbb{R}_+ \subset [\overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)})]^\infty.$$

Moreover, with Lemma 41 we find that

$$ii) \iff f \text{ has no affine majorant on } \text{ri}\,(\text{dom}\, f).$$

Therefore, the claimed equivalences follow from Corollary 42. □

We now establish sufficient conditions in the vector-valued case, building on the results provided above. We start with the most general result, and then successively tighten the assumptions to obtain (weaker but) more handy conditions.

Lemma 44. *Let $K \subset \mathbb{E}_2$ be a (nontrivial) closed, convex cone, and let $F \colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper, K-closed and K-convex. Assume that the following hold:*

i) There is a nonempty set $L \subset K \cap \text{rge}\, F$ such that $\overline{\text{cone}}\, L = K$;

ii) For every $u \in L$, there exists a (nonempty) convex set $C^u \subset F^{-1}(\mathbb{R}_+ u)$ such that $f_u := \langle u, F\rangle + \delta_{C^u} \in \Gamma_0(C^u)$ and f_u has no affine majorant on its domain.

Then K-$\text{epi}\, F = \overline{\text{conv}}\,(\text{gph}\, F)$. In particular, $K = \hat{K}_F$.

Proof. Let $u \in L \setminus \{0\}$, and let us prove that $(0, u) \in [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty$. With the set C^u from ii), we have $f_u \in \Gamma_0(C^u)$, hence Corollary 43 yields that $\{0\} \times \mathbb{R}_+ \subset [\overline{\text{conv}}\,(\text{gph}\, f_u)]^\infty$. As $\text{conv}\,(\text{gph}\, f_u)$ is convex, we hence have $(0, t) + \text{conv}\,(\text{gph}\, f_u) \subset \overline{\text{conv}}\,(\text{gph}\, f_u)$ for all $t \geq 0$.

Now, let $(x, r) \in \text{conv}\,(\text{gph}\, f_u)$. Then $(x, r + t) \in \overline{\text{conv}}\,(\text{gph}\, f_u)$ for all $t \geq 0$, and hence, with $\kappa := \dim \mathbb{E} \times \mathbb{R}$, we have

$$x = \lim_{n\to\infty} \sum_{i=1}^{\kappa+1} \alpha_t^{i,n} x_t^{i,n} \quad \text{and} \quad r + t = \lim_{n\to\infty} \sum_{i=1}^{\kappa+1} \alpha_t^{i,n} \langle u, F\rangle (x_t^{i,n})$$

for certain $x_t^{i,n} \in C^u$ and $\alpha_t^{i,n} \geq 0$ $(i = 1, \ldots, \kappa + 1)$ with $\sum_{i=1}^{\kappa+1} \alpha_t^{i,n} = 1$ for all $n \in \mathbb{N}$. However, by ii), there exist $\gamma_t^n \geqslant 0$ $(n \in \mathbb{N})$ such that

$$\sum_{i=1}^{\kappa+1} \alpha_t^{i,n} F(x_t^{i,n}) = \gamma_t^n u \quad \forall n \in \mathbb{N}.$$

Thus, as $u \neq 0$, we find that

$$\gamma_t^n = \frac{\langle u,\, \gamma_t^n u\rangle}{\|u\|^2} = \frac{1}{\|u\|^2}\left\langle u,\, \sum_{i=1}^{\kappa+1} \alpha_t^{i,n} F(x_t^{i,n})\right\rangle = \frac{1}{\|u\|^2}\sum_{i=1}^{\kappa+1} \alpha_t^{i,n} \langle u,\, F\rangle (x_t^{i,n}) \to \frac{r+t}{\|u\|^2}.$$

Thus, $(x, \frac{ru+tu}{\|u\|^2}) = (x, \frac{r+t}{\|u\|^2}u) \in \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$ for all $t \geqslant 0$, i.e. $(x, \frac{r}{\|u\|^2}u) + \mathbb{R}_+(0,u) \subset \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$. Thus $\{0\} \times \mathbb{R}_+ u = \mathbb{R}_+(0,u) = [(x, \frac{r}{|u|^2}u) + \mathbb{R}_+(0,u)]^\infty \subset [\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)]^\infty$ for every $u \in L \setminus \{0\}$. And consequently, by i), we have $\{0\} \times K = \{0\} \times \overline{\mathrm{cone}}\, L \subset [\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)]^\infty$. Thus, by Theorem 31, we have $K\text{-epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$. □

We record an immediate consequence.

Proposition 45. *Let $K \neq \{0\}$ be a closed, convex cone such that $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ is proper, K-convex and K-closed. Assume that $K = \mathrm{cone}\,(b_1, \dots, b_N)$ for $b_1, \dots, b_N \in \mathrm{rge}\, F$, and that, for any $i = 1, \dots, N$, there exists a nonempty convex set $C^{b_i} \subset F^{-1}(\mathbb{R}_+ b_i)$ such that, for all $i = 1, \dots, N$, the function $f_{b_i} := \langle b_i, F\rangle + \delta_{C^{b_i}}$ is C^{b_i}-closed and does not have any affine majorant on C^{b_i}. Then $K\text{-epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$.*

Proof. Apply Lemma 44, with $L = \{b_1, \dots, b_N\}$. □

To wrap up this section we want to provide a simplified version of Lemma 44 with more restrictive, but less arduous assumptions. To this end, we need the following lemma.

Lemma 46. *Let $K \subset \mathbb{E}_2$ be a closed, convex cone with $K \subset K^+$, and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper, K-convex and K-closed. Then $\langle u,\, F\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $u \in \mathrm{ri}\, K$.*

Proof. We observe from [17, Corollary 7.4(ii)] that $\langle u,\, F\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $u \in \mathrm{ri}\,(K^+)$. Thus if $\mathrm{ri}\, K \subset \mathrm{ri}\,(K^+)$ there is nothing to prove.

Hence, we only need to consider the case $\mathrm{ri}\, K \not\subset \mathrm{ri}\,(K^+)$. Since we assume that $K \subset K^+$, by the definition of the relative topology, this can only hold, if $\mathrm{aff}\, K \subsetneq \mathrm{aff}\,(K^+)$. We note that both of these sets contain 0, and hence are subspaces of $\mathbb{E}_2$. In particular, the orthogonal projection $p : \mathbb{E}_2 \to \mathbb{E}_2$ onto $\mathrm{aff}\, K$ (which is ordered by K) is a linear self-adjoint operator. We define $G : \mathbb{E}_1 \to (\mathrm{aff}\, K)^\bullet$ by

$$G(x) := \begin{cases} p(F(x)), & x \in \mathrm{dom}\, F, \\ +\infty_\bullet, & \text{else.} \end{cases}$$

Note that for all $\alpha \in (0,1)$ and $x, y \in \mathrm{dom}\, F$, we have $\alpha F(x) + (1-\alpha)F(y) - F(\alpha x + (1-\alpha)y) \in K$. Hence, as $K \subset \mathrm{aff}\, K$ and by linearity of p, for all $\alpha \in (0,1)$ and $x, y \in \mathrm{dom}\, F = \mathrm{dom}\, G$, we have

$$\alpha F(x) + (1-\alpha)F(y) - F(\alpha x + (1-\alpha)y) = \alpha G(x) + (1-\alpha)G(y) - G(\alpha x + (1-\alpha)y).$$

Therefore, G is K-convex. Moreover, if we denote $D := \mathrm{dom}\, G$ and $H := F - G : D \to \mathrm{aff}\,(K)$, then

$$\alpha H(x) + (1-\alpha)H(y) - H(\alpha x + (1-\alpha)y) = 0 \quad \forall x, y \in D,\ \alpha \in (0,1).$$

Hence, H is $\{0\}$-convex, and by Proposition 18, there exists an affine function $\hat{H} : \mathbb{E}_1 \to \mathrm{aff}\, K$ such that $\hat{H}_{|D} = H$.

Now, let $\{(x_k, z_k) \in K\text{-epi}\, G]\} \to (x, z) \in \mathbb{E}_1 \times \mathrm{aff}\, K$. Then, for all $k \in \mathbb{N}$, $x_k \in \mathrm{dom}\, G$, and there exists $v_k \in K$ such that

$$z_k = G(x_k) + v_k = F(x_k) - H(x_k) + v_k = F(x_k) - \hat{H}(x_k) + v_k.$$

As $\hat{H} : \mathbb{E}_1 \to \mathrm{aff}\, K$ is affine, it is continuous, so $\hat{H}(x_k) \to \hat{H}(x)$, and thus $F(x_k) + v_k \to z + \hat{H}(x)$. Therefore $\{(x_k, F(x_k) + v_k) \in K\text{-epi}\, F\} \to (x, z + \hat{H}(x))$. As F is K-closed, we have $(x, z + \hat{H}(x)) \in K\text{-epi}\, F$, thus $x \in \mathrm{dom}\, F = \mathrm{dom}\, G$, $\hat{H}(x) = H(x)$, and $z + H(x) - F(x) \in K$, so $z - G(x) \in K$ and $(x, z) \in K\text{-epi}\, G$. This proves that G is K-closed.

Let K' be the dual cone of K in $\mathrm{aff}\, K$. As $K \subset K^+$ by assumption, we consequently obtain $K \subset K^+ \cap \mathrm{aff}\, K = K' \subset \mathrm{aff}\, K$. Hence, $\mathrm{ri}\, K \subset \mathrm{ri}\, K'$. Moreover, as $G : \mathbb{E}_1 \to (\mathrm{aff}\, K)^\bullet$ is proper, K-closed and K-convex, by [17, Corollary 7.4(ii)], we have $\langle u,\, G\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $u \in \mathrm{ri}\, K'$. But for any $u \in \mathrm{ri}\, K \subset \mathrm{aff}\, K$, as p is self-adjoint, we have $\langle u,\, G\rangle = \langle u,\, p(F)\rangle = \langle u,\, F\rangle$. Thus, for any $u \in \mathrm{ri}\, K$ we have $\langle u,\, F\rangle \in \Gamma_0(\mathbb{E}_1)$. □

Proposition 47. *Let $K \subset \mathbb{E}_2$ be a proper, closed, convex cone such that $K \subset K^+$ and let $F \colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper, K-convex and K-closed with $\mathrm{ri}\, K \subset \mathrm{rge}\, F$. Moreover, assume that, for any $u \in \mathrm{ri}\, K$, there exists a nonempty convex set $C^u \subset F^{-1}(\mathbb{R}_+ u)$ such that $f_u := \langle u, F\rangle + \delta_{C^u}$ does not have any affine majorant on its domain. Then, $K\text{-epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$.*

Proof. By Lemma 46, for all $u \in \mathrm{ri}\, K$, we have $\langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)$, hence $f_u \in \Gamma_0(C^u)$. Applying Lemma 44 with $L = \mathrm{ri}\, K$ yields the desired result. □

4.4 Examples

In this section we put our findings from the previous sections to the test on various examples of K-convex functions. Throughout, we equip the matrix space $\mathbb{R}^{n\times m}$ with the *Frobenius* inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \to \mathbb{R}$, $\langle X, Y\rangle = \mathrm{tr}\,(X^T Y)$. In particular, on the space of symmetric matrices $\mathbb{S}^n$, the transposition is superfluous.

4.4.1 $F : X \mapsto \frac{1}{2} X X^T$

We consider the function

$$F : \mathbb{R}^{n\times m} \to \mathbb{S}^n, \quad F(X) = \frac{1}{2} X X^T. \tag{13}$$

It plays a central role in study of *matrix-fractional* [7–9] and *variational Gram functions* [10, 13].

Proposition 48. *Let F be given by* (13)*. Then the following hold:*

a) $\hat{K}_F = K_F = \mathbb{S}^n_+ = \mathrm{conv}\,(\mathrm{rge}\, F)$. For $m \geq n$, the convex hull is superfluous.

b) F is $\mathbb{S}^n_+$-closed and -convex.

c) $\overline{\mathrm{conv}}\,(\mathrm{gph}\, F) = \mathbb{S}^n_+\text{-epi}\, F$.

Proof. a) We know from [10, Lemma 8] that $K_F = \mathbb{S}^n_+$. But as F is continuous and K_F is closed, we have that F is K_F-closed, and hence $K_F = \hat{K}_F$, which shows the first identity. For the third, observe that, clearly $\mathbb{S}^n_+ \supset \mathrm{conv}\,(\mathrm{rge}\, F)$. On the other hand for $V \in \mathbb{S}^n_+$, there exists $L \in \mathbb{R}^{n\times n}$ such that $\frac{1}{2} L L^T = V$. This already shows that $\mathbb{S}^n_+ = \mathrm{rge}\, F$ if $m \geq n$ (for $m > n$, 0 columns can be added to L). If not, we denote the columns of L by $\ell_1, \ldots, \ell_n$ and set $x_i := \sqrt{n}\ell_i$ for all $i = 1, \ldots, n$, and $X_i := [x_i, 0, \ldots, 0] \in \mathbb{R}^{n\times m}$. Then

$$V = \frac{1}{2}\sum_{i=1}^n \ell_i \ell_i^T = \frac{1}{2}\sum_{i=1}^n \left(\frac{x_i}{\sqrt{n}}\right)\left(\frac{x_i}{\sqrt{n}}\right)^T = \sum_{i=1}^n \frac{1}{n} F(X_i) \in \mathrm{conv}\,(\mathrm{rge}\, F),$$

which gives the desired inclusion.

b) Follows from a).

c) We prove that $\{0\} \times \mathbb{S}^n_+ \subset \mathrm{conv}\,(\mathrm{gph}\, F)$ which then gives the desired result via b) and Corollary 33. To this end, let $(0, V) \in \{0\} \times \mathbb{S}^n_+$. Hence, by a), there exist $\alpha_1, \ldots, \alpha_r \geqslant 0$ and $X_1, \ldots X_r \in \mathbb{R}^{n\times m}$ such that $\sum_{i=1}^r \alpha_i = 1$ and $V = \sum_{i=1}^r \alpha_i F(X_i)$. However, $F(-X_i) = F(X_i)$. Hence, we also have $V = \sum_{i=1}^r \frac{\alpha_i}{2} F(X_i) + \frac{\alpha_i}{2} F(-X_i)$. As $\sum_{i=1}^r \frac{\alpha_i}{2} X_i + \frac{\alpha_i}{2}(-X_i) = 0$, we then have $(0, V) \in \mathrm{conv}\,(\mathrm{gph}\, F)$. □

4.4.2 The squared matrix mapping

We consider the map

$$F : \mathbb{S}^n \to \mathbb{S}^n, \quad F(X) = X^2. \tag{14}$$

Proposition 49. *Let F be given by* (14)*. Then the following hold:*

a) $\hat{K}_F = K_F = \mathbb{S}^n_+ = \mathrm{rge}\, F$.

b) F is $\mathbb{S}^n_+$-closed and -convex.

c) $\overline{\mathrm{conv}}\,(\mathrm{gph}\, F) = \mathbb{S}^n_+\text{-epi}\, F$.

Proof. a) Using Proposition 24 b) we know that $K_F^+ = \{V \in \mathbb{S}^n \mid \langle V, F\rangle \text{ convex}\}$. Now for any $V \in \mathbb{S}^n$, we have $\nabla \langle V, F\rangle (X) = VX + XV$. Therefore, for $X, Y \in \mathbb{S}^n$, we find that

$$\langle V, F\rangle (X) - \langle V, F\rangle (Y) + \langle \nabla \langle V, F\rangle (X), Y - X\rangle = -\mathrm{tr}\,((X - Y)V(X - Y)).$$

For $\langle V, F\rangle$ to be convex, by the gradient inequality, it is therefore necessary and sufficient that

$$\mathrm{tr}\,((X-Y)V(X-Y)) \geq 0 \quad \forall X, Y \in \mathbb{S}^n,$$

which is equivalent to saying that $V \succeq 0$. Therefore $K_F^+ = \mathbb{S}_+^n$, and by bipolarity, we obtain $K_F = \mathbb{S}_+^n$. Since F is continuous, we have $K_F = \hat{K}_F$, and the fact that $\mathrm{rge}\, F = \mathbb{S}_+^n$ is obvious.

b) Follows from a).

c) Use the same reasoning as in the proof of Proposition 48 c). $\square$

4.4.3 The inverse matrix mapping

We consider the map

$$F : \mathbb{S}_{++}^n \to \mathbb{S}^n, \quad F(X) = X^{-1}. \tag{15}$$

Proposition 50. *Let F be given by* (15)*. Then the following hold:*

a) $\hat{K}_F = K_F = \mathbb{S}_+^n$.

b) *F is $\mathbb{S}_+^n$-convex and -closed.*

c) $\overline{\mathrm{conv}}\,(\mathrm{gph}\, F) = \mathbb{S}_+^n\text{-}\mathrm{epi}\, F$.

Proof. a) By Proposition 24 b), we know that $K_F^+ = \{V \in \mathbb{S}^n \mid \langle V, F\rangle \text{ convex}\}$. Now let $V \in \mathbb{S}^n$ and observe that $\nabla \langle V, F\rangle (X) = -X^{-1}VX^{-1}$ for all $X \succ 0$. Therefore, for all $X, Y \succ 0$

$$\begin{aligned}
&\langle V, F\rangle (X) - \langle V, F\rangle (Y) + \langle \nabla \langle V, F\rangle (X), Y - X\rangle \\
&\quad = \mathrm{tr}\,(VX^{-1}) - \mathrm{tr}\,(VY^{-1}) - \mathrm{tr}\,(X^{-1}VX^{-1}(Y-X)) \\
&\quad = -\mathrm{tr}\,\left(V[Y^{-1} - 2X^{-1} + X^{-1}YX^{-1}]\right) \\
&\quad = -\mathrm{tr}\,\left([Y^{-1/2} - X^{-1}Y^{1/2}]^T V [Y^{-1/2} - X^{-1}Y^{1/2}]\right).
\end{aligned}$$

For $\langle V, F\rangle$ to be convex, by the gradient inequality, it is therefore necessary and sufficient that

$$\mathrm{tr}\,\left([Y^{-1/2} - X^{-1}Y^{1/2}]^T V [Y^{-1/2} - X^{-1}Y^{1/2}]\right) \geq 0 \quad \forall X, Y \succ 0.$$

This holds if and only if $V \succeq 0$.

b) Follows from a).

c) First note that $\mathrm{rge}\, F = \mathbb{S}_{++}^n = \mathrm{ri}\,\mathbb{S}_+^n$ and that $\mathbb{S}_+^n$ is self-dual, i.e. $\mathbb{S}_+^n = (\mathbb{S}_+^n)^+$. Moreover, for every $U \in \mathrm{ri}\,\mathbb{S}_+^n$, we have that $C^U := F^{-1}(\mathbb{R}_+ U) = \left\{X \,\middle|\, \exists t > 0 : \; X = \frac{1}{t}U^{-1}\right\} = \mathbb{R}_{++}U^{-1}$ is convex and nonempty. The desired statement will follow from Proposition 47, once we show that $\langle U, F\rangle$ has no affine majorant on C^U. To this end, let $V_t = tU^{-1} \in C^U$ for $t > 0$. Then $\langle U, F\rangle (V_t) = \frac{\|U\|^2}{t}$. Since $0 < t \mapsto 1/t$ has no affine majorant on $\mathbb{R}_{++}$, then $\langle U, F\rangle$ cannot have an affine majorant on C^U. $\square$

4.4.4 Entry-wise convex functions

It is well known [10] that a component-wise convex function $D \subset \mathbb{E}_1 \to \mathbb{R}^n$ is $\mathbb{R}_+^n$-convex. This can be slightly generalized.

Proposition 51. *Let $\{b_1, \ldots, b_n\} \subset \mathbb{E}_2$ and let $f_i \in \Gamma(\mathbb{E}_1)$ for all $i = 1, \ldots, n$ such that $D := \bigcap_{i=1}^n \mathrm{dom}\, f_i \neq \emptyset$. Define $F : D \to \mathbb{E}_2$ by $F(x) = \sum_{i=1}^n f_i(x) b_i$ and let $K := \mathrm{cone}\,\{b_1, \ldots, b_n\}$. Then the following hold:*

a) *F is K-convex.*

b) *We have $K \subset K^+$ if and only if $\langle b_i, b_j\rangle \geq 0$ for all $i, j = 1, \ldots, n$.*

c) *Assume that $\mathrm{dom}\, f_i = \mathbb{E}_1$ for all $i = 1, \ldots, n$. Then,*

  I.) *F is K-convex and K-closed.*

  II.) *Suppose that $K \subset K^+$ and that, for all $i = 1, \ldots, n$, we have $C_i := \bigcap_{i \neq j} \mathrm{argmin}\, f_j \neq \emptyset$ and that f_i has no affine majorant on C_i. Then $K\text{-}\mathrm{epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$.*

Proof. a) Observe that $K^+ = \{y \mid \langle b_i, y\rangle \geq 0 \ (i = 1, \dots, n)\}$. Therefore for all $z \in K^+$ we have $\langle z, F\rangle = \sum_{i=1}^n \langle b_i, z\rangle f_i \in \Gamma(\mathbb{E}_1)$. Hence Proposition 14 yields the desired statement.

b) K is closed and convex, hence, by the bipolar theorem [18, Corollary 6.21], $K^{++} = K$.

c) I.) As $\operatorname{dom} f_i = \mathbb{E}_1$, then all f_i are continuous. Thus $\langle z, F\rangle = \sum_{i=1}^n \langle b_i, z\rangle f_i$ is convex and continuous for all $z \in K^+$, hence $\langle z, F\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $z \in K^+$. Hence Proposition 14 yields the desired statement.

II.) Define $m_i := \min f_i$ and $g_i = f_i - m_i$ for all $i = 1, \dots, n$, and set $c := \sum_{i=1}^n m_i b_i$. Let $G\colon D \to \mathbb{E}_2$ be defined by $G = F - c$. Then, for all $x \in D$, we have $G(x) = \sum_{i=1}^n g_i(x) b_i$, and $\langle b_i, G\rangle + \delta_{C^i} = \langle b_i, F\rangle + \delta_{C^i} - \langle c, b_i\rangle \in \Gamma_0(C^i)$, as $b_i \in K \subset K^+$. Now, let $x \in C^i$. Then for all $j \neq i$, $f_j(x) = m_j$, hence $g_j(x) = f_j(x) - m_j = 0$. Thus, $G(x) = g_i(x) b_i$. Moreover, $g_i(x) = f_i(x) - m_i \geqslant m_i - m_i = 0$, thus $G(x) \in \mathbb{R}_+\{b_i\}$, and so $C^i \subset G^{-1}(\mathbb{R}_+\{b_i\})$. By Proposition 45, $\overline{\operatorname{conv}}\,(\operatorname{gph} G) = K\text{-epi}\, G$.

Furthermore, as $F = G + c$, then $\operatorname{gph} F = \operatorname{gph} G + (0, c)$, thus $\overline{\operatorname{conv}}\,(\operatorname{gph} F) = \overline{\operatorname{conv}}\,(\operatorname{gph} G) + (0, c)$, and $K\text{-epi}\, F = K\text{-epi}\, G + (0, c)$. We deduce then that $\overline{\operatorname{conv}}\,(\operatorname{gph} F) = K\text{-epi}\, F$. □

We point out that, with a more refined topological argument in the proof of c), we could replace the assumption $\operatorname{dom} f_i = \mathbb{E}_1$ by the weaker condition $f_i \in \Gamma_0(\mathbb{E}_1)$, but since this result is not essential to our further study we confine ourselves with the current version.

4.4.5 The spectral function

The *spectral function* [10,15,17] is the map $\lambda\colon \mathbb{S}^n \to \mathbb{R}^n$, $\lambda(A) = [\lambda_1(A), \dots \lambda_n(A)]^T$ where $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ are the ordered eigenvalues of A (with multiplicity). Define the cone

$$K_n = \left\{ v \in \mathbb{R}^n \mid \sum_{i=1}^k v_i \geq 0, k = 1, \dots, n-1, \sum_{i=1}^n v_i = 0 \right\}. \tag{16}$$

The following result clarifies the convexity properties of λ w.r.t K_n and shows, based on Proposition 45 and Corollary 36, respectively, that the question whether $K_n\text{-epi}\, \lambda = \overline{\operatorname{conv}}\,(\operatorname{gph} \lambda)$ depends on n.

Proposition 52 (Spectral map)**.** *Let K_n be given by* (16)*. Then the following hold:*

a) K_n is closed, convex and pointed with $K_n^+ = \{w \in \mathbb{R}^n \mid w_1 \leq \cdots \leq w_n\}$.

b) λ is K_n-convex and K_n-closed.

c) The following are equivalent:

i) $K_n \subset K_n^+$;

ii) $n \leq 2$;

iii) $K_n\text{-epi}\, \lambda = \overline{\operatorname{conv}}\,(\operatorname{gph} \lambda)$.

Proof. a) The properties of K_n are straightforward. The formula for K_n^+ can be found in e.g. [10, 15, 17].

b) See e.g. [10, 15, 17] for K_n-convexity. The K_n-closedness follows because λ is continuous on $\mathbb{S}^n$ and K_n is closed.

c) Consider the following implications:

'i)⇒ii)': For $n > 2$ we have $[0, \dots, 0, 1, -1]^T \in K_n \setminus K_n^+$, see a).

'ii)⇒iii)': For $n = 1$ there's nothing to prove. For $n = 2$ set $b_1 := [1; -1]^T$ so that $K_n = \operatorname{cone}\{b_1\}$ and define $C^{b_1} := \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} \mid \alpha \in \mathbb{R} \right\}$ which is a subspace, hence nonempty and closed, and convex. Then $C^{b_1} \subset \lambda^{-1}(\mathbb{R}_+ b_1)$ and we have $\langle b_1, \lambda\rangle \left(\begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} \right) = 2|\alpha|$ for all $\alpha \in \mathbb{R}$. Therefore $\langle b_1, \lambda\rangle + \delta_{C^{b_1}} \in \Gamma_0(\mathbb{S}^n)$ and has no affine majorant on its domain C^{b_1}. Therefore Proposition 45 yields the desired implication.

'iii)⇒i)': Corollary 36. □

5 Convex convex-composite functions

We start this section with the definition of K-increasing functions.

Definition 53 (K-increasing functions)**.** *Let $K \subset \mathbb{E}$ be a cone. The function $g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is said to be* K-increasing *if*

$$y \geq_K x \quad \Longrightarrow \quad g(y) \geq g(x) \quad \forall x, y \in \mathbb{E}.$$

It is well known and explored extensively in the literature [4–6,10] that, given $K \subset \mathbb{E}$, a K-increasing function $g \in \Gamma(\mathbb{E}_2)$ and a K-convex function $F : D \subset \mathbb{E}_1 \to \mathbb{E}_2$, the composition

$$g \circ F : \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}, \quad (g \circ F)(x) := \begin{cases} g(F(x)), & x \in D, \\ +\infty, & \text{else} \end{cases} \tag{17}$$

is convex (and proper if and only if $F(D) \cap \operatorname{dom} g \neq \emptyset$). One of the questions we address in this section is the following: given $g \in \Gamma(\mathbb{E}_2)$ and $F : D \subset \mathbb{E}_1 \to \mathbb{E}_2$ such that $g \circ F$ is convex, under which conditions does there exist a (closed) cone K such that F is K-convex and g is K-increasing?

5.1 The horizon cone of a closed, proper, convex function

For a proper function $f : \mathbb{E} \to \overline{\mathbb{R}}$, its *horizon function* $f^\infty : \mathbb{E} \to \overline{\mathbb{R}}$ is defined via $\operatorname{epi} f^\infty = (\operatorname{epi} f)^\infty$. The *horizon cone* $\operatorname{hzn} f$ of f is the level set $\operatorname{hzn} f := \{x \in \mathbb{E} \mid f^\infty(x) \leq 0\}$. For $f \in \Gamma_0(\mathbb{E})$ the horizon function and horizon cone of f coincide with the respective *recession* objects [18, Chapter 8]. We summarize some fundamental properties of the horizon cone of a closed, proper, convex function.

Proposition 54. *Let $g \in \Gamma_0(\mathbb{E})$. Then the following hold:*

a) g^∞ is closed, proper, convex and positively homogenous.

b) We have

$$g^\infty(u) = \sup_{t>0} \frac{g(x + tu) - g(x)}{t} \quad \forall x \in \operatorname{dom} g.$$

c) $\operatorname{hzn} g$ is a closed convex cone.

d) g is $(-\operatorname{hzn} g)$-increasing.

e) K is a cone with respect to which g is increasing if and only if $K \subset -\operatorname{hzn} g$.

Proof. a),b) See [19, Theorem 3.21].

c) From a) and the definition of $\operatorname{hzn} g$.

d) See [10, Lemma 7] or [16, Corollary 3.1].

e) See [16, Proposition 3.2]. □

The next example shows that the convexity in part b) is essential, which also shows that the horizon function is not the recession function (see [16]) without convexity.

Example 55. *Consider $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} 1 + x, & x < -1, \\ 0, & x \in [-1, 1], \\ 1 - x, & x > 1. \end{cases}$$

Then f is continuous (hence proper and lsc), but not convex, and it holds that $f^\infty(u) = -|u|$. Moreover, for any $u \in \mathbb{R}$, $\sup_{x \in \mathbb{R}, t>0} \frac{f(x+tu)-f(x)}{t} = |u|$. Thus $f^\infty(u) \neq \sup_{x \in \mathbb{R}, t>0} \frac{f(x+tu)-f(x)}{t}$.

5.2 The K-increasing case

The next proposition characterizes the situation where there exists a cone with respect which $F : D \to \mathbb{E}_2$ is convex and $g \in \Gamma_0(\mathbb{E}_2)$ is increasing. At this, the cone K_F, the smallest closed cone with respect to which F is convex, comes in to play, which ties our study here to our results from Section 3.

Proposition 56. *Let $g \in \Gamma_0(\mathbb{E}_2)$ and $F : D \to \mathbb{E}_2^\bullet$ with $D \subset \mathbb{E}_1$ convex such that $g \circ F$ is proper. Then the following statements are equivalent.*

i) There exists a cone K such that g is K-increasing and F is K-convex;

ii) g is K_F-increasing;

iii) $K_F \subset -\operatorname{hzn} g$;

iv) $(\operatorname{hzn} g)^\circ \subset K_F^+$.

Proof. We only (need to) show that i), ii) and iii) are equivalent. The equivalence of iii) and iv) follows from (bi)polarization and the fact that both cones in play are closed and convex.

i)$\Rightarrow$ii) : Let $K \subset \mathbb{E}_2$ such that F is K-convex and g is K-increasing. In particular, by Proposition 54 e), we have $K \subset -\operatorname{hzn} g$, and thus F is $(-\operatorname{hzn} g)$-convex and g is $(-\operatorname{hzn} g)$-increasing, by Proposition 54 d). As $(-\operatorname{hzn} g)$ is closed and convex, see Proposition 54 c), by definition of K_F we have $K_F \subset -\operatorname{hzn} g$. By Proposition 54 e) we find that g is K_F-increasing.

ii)$\Rightarrow$iii): From Proposition 54 e).

iii)$\Rightarrow$i): Let $K := K_F$. Clearly, F is K-convex and, by Proposition 54 e), g is K-increasing. □

Proposition 56 yields the following concrete example.

Example 57. *Consider $g\colon (x,y) \in \mathbb{R}^2 \mapsto |x|$ and $F\colon (x,y) \in \mathbb{R}^2 \mapsto (x^2, y)$. Hence $g \in \Gamma_0(\mathbb{E}_2)$ and $g \circ F\colon (x,y) \mapsto x^2 \in \Gamma_0(\mathbb{E}_1)$. Using Proposition 24 b), we find that $K_F = \mathbb{R}_+ \times \{0\}$. However, $(-1,0) \leqslant_{K_F} (0,0)$ and $g((-1,0)) = 1 > 0 = g((0,0))$. Thus, g is not K_F-increasing, and consequently, by Proposition 56, there is no closed cone K such that F is K-convex and g is K-increasing.*

We close out by remarking that, if $\phi\colon \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}$ is proper convex, there always exists a decomposition $\phi = g \circ F$ with $g \in \Gamma_0(\mathbb{E}_2)$, $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ proper with g K_F-increasing: for instance, define $F(x) := (\Phi(x), 0, \dots, 0) \in \mathbb{E}_2$ with $\operatorname{dom} F = \operatorname{dom} \Phi$, and $g(y) = y_1$. Then, $g \in \Gamma_0(\mathbb{E}_2)$, F is $(\mathbb{R}_+ \times \{0\} \times \cdots \times \{0\})$-convex and g is $(\mathbb{R}_+ \times \{0\} \times \cdots \times \{0\})$-increasing.

5.3 Beyond K-monotonicity

It was already observed by Pennanen [17] and Burke et al. [10] that, in order to obtain convexity of the composition $g \circ F$ in (17), the assumption that g be K-increasing can be weakened to

$$g(F(x)) \le g(y) \quad \forall (x,y) \in K\text{-epi}\, F, \tag{18}$$

in which case we call g *increasing w.r.t* K-epi F. Concretely, the following result holds.

Proposition 58 ([10, Proposition 1])**.** *Let $K \subset \mathbb{E}_2$ be a convex cone such that that $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ is K-convex and such that $g \in \Gamma(\mathbb{E}_1)$ is increasing w.r.t K-epi F in the sense of (18). Then $g \circ F$ is convex.*

The next proposition gives a characterization of the situation where there exists a closed convex cone K such that $g \in \Gamma_0(\mathbb{E}_1)$ is increasing w.r.t K-epi F and F is K-convex.

Proposition 59. *Let $g \in \Gamma_0(\mathbb{E}_2)$ and $F : D \to \mathbb{E}_2$ for $D \subset \mathbb{E}_1$ (nonempty convex) such that $g \circ F$ is proper. Then there exists a closed (convex) cone K such that g is increasing w.r.t K-epi F (in the sense of (18)) and F is K-convex if and only if g is increasing w.r.t K_F-epi F.*

Proof. Suppose that g is increasing w.r.t K_F-epi F and set $K := K_F$. Then K is closed and convex and F is K-convex and g is increasing w.r.t K-epi F (by assumption).

Conversely, suppose now that there exists a closed (convex) cone K such that g is increasing w.r.t K-epi F and F is K-convex. As K is closed and F is K-convex, we have$K_F \subset K$, by definition of K_F. Therefore $K_F\text{-epi}\, F = \operatorname{gph} F + \{0\} \times K_F \subset \operatorname{gph} F + \{0\} \times K = K_F\text{-epi}\, F$, and thus g is increasing w.r.t K_F-epi F.

□

It turns out that, in Proposition 58, the assumption that g be increasing w.r.t. to K-epi F can even be further weakened substituting $\operatorname{conv}(\operatorname{gph} F)$ for K-epi F by which, again, ties our considerations here to our previous study.

Proposition 60. *Let $D \subset \mathbb{E}_1$ be (nonempty) convex, $F : D \to \mathbb{E}_2$, and let $g \in \Gamma(\mathbb{E}_2)$ be increasing w.r.t. $\operatorname{conv}(\operatorname{gph} F)$, i.e.*

$$g(F(x)) \le g(y) \quad \forall (x,y) \in \operatorname{conv}(\operatorname{gph} F). \tag{19}$$

Then $g \circ F$ is convex.

Proof. Let $x, y \in \operatorname{dom}(g \circ F)$ and $\alpha \in (0,1)$. Then $(\alpha x + (1-\alpha)y, \alpha F(x) + (1-\alpha)F(y)) \in \operatorname{conv}(\operatorname{gph} F)$. By (19),and the convexity of g we find

$$g(F(\alpha x + (1-\alpha)y)) \leq g(\alpha F(x) + (1-\alpha)F(y)) \leq \alpha g(F(x)) + (1-\alpha)g(F(y)).$$

Hence $g \circ F$ is convex. $\square$

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