# A note on the $K$-epigraph

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**Abstract**

We study the question as to when the closed convex hull of a $K$-convex map equals its $K$-epigraph. In particular, we shed light onto the smallest cone $K$ such that a given map has convex and closed $K$-epigraph, respectively. We apply our findings to several examples in matrix space as well as to convex composite functions.

**Keywords:** Cone-induced ordering, $K$-convexity, $K$-epigraph, convex hull, Fenchel conjugate, horizon cone, spectral function, convex-composite function, matrix analysis.

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## 1 Introduction

**Motivation** In a recent paper, Burke et al. [8, Corollary 9] show that the closed convex hull of the set $\mathcal{D} := \left\{(X, \frac{1}{2}XX^T) \mid X \in \mathbb{R}^{n\times m}\right\}$ is given by

$$\overline{\mathrm{conv}}\,\mathcal{D} = \left\{(X,Y) \in \mathbb{R}^{n\times m} \times \mathbb{S}^n \;\middle|\; Y \succeq \frac{1}{2}XX^T\right\}.$$

Here, '$\succeq$' is the *Löwner partial ordering* [12] on the symmetric matrices $\mathbb{S}^n$ induced by the positive semidefinite cone $\mathbb{S}^n_+$ via '$A \succeq B$ if and only if $A-B \in \mathbb{S}^n_+$'. At second glance, the set $\mathcal{D} \in \mathbb{R}^{n\times m}\times\mathbb{S}^n$ is simply the graph of the matrix-valued map $F : X \in \mathbb{R}^{n\times m} \mapsto \frac{1}{2}XX^T \in \mathbb{S}^n$; and $\overline{\mathrm{conv}}\,\mathcal{D}$ in (1) then appears to be a 'generalized epigraph' of $F$ where the partial ordering on the image space $\mathbb{S}^n$ (induced by $\mathbb{S}^n_+$) plays the role of the ordinary ordering of $\mathbb{R}$ (induced by $\mathbb{R}_+$) for scalar-valued functions.

More generally, given a map $F : \mathbb{E}_1 \to \mathbb{E}_2$ between two (Euclidean) spaces $\mathbb{E}_1$ and $\mathbb{E}_2$ and a cone $K \subset \mathbb{E}_2$, we can order $\mathbb{E}_2$ via '$y \geq_K z$ if and only if $y - z \in K$'. In view of the above identity, the natural question that arises is the following: *When is*

$$\overline{\mathrm{conv}}\,(\mathrm{gph}\, F) = \{(x, y) \mid y \geq_K F(x)\} \tag{1}$$

*valid?* Clearly, this can only hold if the set on the right, which will later be called the *$K$-epigraph of* $F$, is itself closed and convex, in which case we say that $F$ is *$K$-closed* and *$K$-convex*, respectively, or *closed* and *convex, with respect to (w.r.t.)* $K$, respectively.

**Related work** The study of $K$-convexity has a long tradition in convex analysis and is now part of many textbooks, e.g. [2,19]: Borwein [3] pursued an ambitious program of extending most of convex analysis to cone convex functions including conjugacy, subdifferential analysis, and duality, laying out much of the groundwork. Kusraev and Kutateladze [14] take this idea to an even more general setting by considering *convex operators* with values in arbitrary ordered vector spaces. Pennanen [17] develops a deep theory of generalized differentiation for *graph-convex mappings* (these are called *convex correspondences* in [14]) which contains some results on $K$-convexity, highly relevant to our study. One of the most important features of a $K$-convex map $F$ is the fact that the composition $g \circ F$ with a convex function $g$, which is increasing with respect to the ordering induced by $K$, is convex; a fact that has been well observed and utilized widely in the literature [4–6, 10, 11, 17].

**Road map and contributions** We start our study in Section 2 with the necessary tools from convex and variational analysis. In Section 3, we formally introduce and expand on the central notions of $K$-convexity and $K$-closedness. In particular, in Sections 3.1-3.3 we characterize the functions which are convex w.r.t. a given subspace, half-space and polyhedral cone respectively. In Section 3.4, we elaborate on Pennanen's characterization of the dual cone of the smallest closed, necessarily convex (Proposition 11) cone with respect to which a given $F$ is convex. We extend this in Section 3.5 to study the smallest, necessarily closed and convex (Proposition 11) cone with respect to which $F$ is convex *and* closed. Section 4 is fully devoted to the question as to when (1) holds. Theorem 34 in Section 4.1 provides a characterization for (1), which consitutes one of the main workhorses for the the rest of Section 4. Section 4.2 is mainly devoted to necessary conditions for (1). Partly mimicking the scalar case (Theorem 38), in Section 4.2.2 we present necessary conditions based on affine $K$-minorization and $K$-majorization. Section 4.3, in turn, provides sufficient conditions. Section 4.4 presents different examples of $K$-convex maps by which we illustrate the theory developed in Section 3 and, more importantly, Section 4. In particular, we apply our findings to the following maps:

- $F : X \in \mathbb{R}^{n\times m} \mapsto \frac{1}{2}XX^T \in \mathbb{S}^n$;
- $F : X \in \mathbb{S}^n_{++} \to X^{-1} \in \mathbb{S}^n$ (inverse matrix);
- $F : X \in \mathbb{S}^n \mapsto \lambda(X)^1 \in \mathbb{R}^n$ (spectral map);
- $F : \mathbb{E} \to \mathbb{R}^m$ where $F_i$ is convex for all $i = 1, \dots, m$ (component-wise convex).

Most of the criteria worked out in the previous sections for the validity of (1) are brought to bear directly or indirectly here.

Section 5 taps into the composite framework alluded to above, where, primarily, we study the following question: given a vector-valued map $F$ and a (closed, proper) convex function $g$ such that $g \circ F$ is convex, does there exist a cone $K$ such that $F$ is $K$-convex and $g$ is increasing in a $K$-related ordering?

*Notation:* In what follows, $\mathbb{E}$ denotes a Euclidean space, i.e. a finite-dimensional real inner product space with inner product denoted by $\langle\cdot,\cdot\rangle$. Given a set $S \subset \mathbb{E}$, we denote its closure, convex hull, closed convex hull, and convex conical hull by $\mathrm{cl}\, S$, $\mathrm{conv}\, S$, $\overline{\mathrm{conv}}\, S$ and $\mathrm{cone}\, S$, respectively. For a vector $u \in \mathbb{E}$, we denote its (convex) conical hull by $\mathbb{R}_+ u$, and $\mathbb{R}_{++}u = \{\lambda u \mid \lambda > 0\}$. The indicator function $\delta_S : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ of $S \subset \mathbb{E}$ is given by $\delta_S(x) = 0$ if $x \in S$ and $\delta_S(x) = +\infty$ otherwise.

# 2 Preliminaries

Throughout we make use of the *relative topology* for convex sets [18, §6]. The *relative interior* $\mathrm{ri}\, C$ of a convex set $C \subset \mathbb{E}$ is its interior in the subspace topology induced by its *affine hull* $\mathrm{aff}\, C := \{\lambda x + (1-\lambda)y \mid \lambda \in \mathbb{R},\ x, y \in C\}$. For a convex set $C \subset \mathbb{E}$ and (any) $x_0 \in C$, the *subspace parallel to* $C$ is $\mathrm{par}\, C := \mathrm{aff}\, C - x_0$. For convex sets, we have a handy description of the affine hull.

**Lemma 1.** *Let $C \subset \mathbb{E}$ be a convex. Then* $\mathrm{aff}\, C = \{\alpha x - \beta y \mid \alpha, \beta \geqslant 0, \alpha - \beta = 1, x, y \in C\}$.

*Proof.* Set $A := \{\alpha x - \beta y \mid \alpha, \beta \geqslant 0, \alpha - \beta = 1, x, y \in C\}$. Thus $\mathrm{aff}\, C \supset A$. Conversely, for $z \in \mathrm{aff}\, C$, there exist $\lambda \in \mathbb{R}$ and $x, y \in C$ such that $z = \lambda x + (1-\lambda)y$. If $\lambda \in (0, 1)$, then $z \in C$ by convexity, and hence $z = 1 \cdot z - 0 \cdot z \in A$. If $\lambda > 1$, set $\alpha := \lambda \geqslant 0$, $\beta := \lambda - 1 \geqslant 0$, and we get $\alpha - \beta = 1$ and $z = \alpha x - \beta y \in A$. Finally, if $\lambda < 0$, then $1 - \lambda \geqslant 0$, and thus $(\beta := \lambda, \alpha := 1 - \lambda)$ $z \in A$. □

Let $f : \mathbb{E} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. We call $f$ *proper* if its *domain* $\mathrm{dom}\, f := \{x \in \mathbb{E} \mid f(x) < +\infty\}$ is nonempty and $f$ doesn't take the value $-\infty$. We say that $f$ is *convex* if its *epigraph* $\mathrm{epi}\, f := \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid f(x) < \alpha\}$ is convex which coincides with the usual definition via a secant condition

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \quad \forall x, y \in \mathrm{dom}\, f,\ \alpha \in (0, 1),$$

if $f$ does not take the value $-\infty$. Although pointwise convergence is not a suitable for preservation of many variational properties, see e.g. [19, Chapter 7], it still preserves convexity in the limit.

**Lemma 2.** *Let $\{f_k : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}\}$ converge pointwise to $f : \mathbb{E} \to \overline{\mathbb{R}}$, i.e. $f_k(x) \to f(x)$ for all $x \in \mathbb{E}$. If $f_k$ is convex for all $k \in \mathbb{E}$ (sufficiently large), then so is $f$.*

[1] Here $\lambda(X) = (\lambda_1, \dots, \lambda_n)^T$ is the vector of eigenvalues of $X \in \mathbb{S}^n$ in decreasing order.

*Proof.* For $\alpha \in (0,1)$ and $x, y \in \mathbb{E}$, the convexity of $f_k$ yields $\alpha f_k(x) + (1-\alpha) f_k(y) \geqslant f_k(\alpha x + (1-\alpha)y)$. Passing to the limit $k \to \infty$ on both sides gives $\alpha f + (1-\alpha) f(y) \geqslant f(\alpha x + (1-\alpha)y)$. □

We call $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ *closed* or *lower semicontinuous (lsc)* if epi $f$ is closed. We set

$$\begin{aligned} \Gamma(\mathbb{E}) &:= \{ f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\} \mid f \text{ proper, convex} \}, \\ \Gamma_0(\mathbb{E}) &:= \{ f \in \Gamma(\mathbb{E}) \mid f \text{ closed} \}. \end{aligned}$$

For $f \in \Gamma(\mathbb{E})$, its *closure* cl $f \in \Gamma_0(\mathbb{E})$ is defined via cl (epi $f$) = epi (cl $f$). More generally, given a convex subset $D \subset \mathbb{E}$, we call $f$ *$D$-closed* if epi $f$ is closed in the subspace topology induced by $D \times \mathbb{R}$. We note that $D \times \mathbb{R}$ is a metric space, hence closedness is sequential closedness and, in particular, $f$ is $D$-closed if and only if

$$\liminf_{k \to \infty} f(x_k) \geq f(\bar{x}) \quad \forall \{x_k \in D\} \to \bar{x} \in D.$$

We define $\Gamma_0(D) := \{ f \in \Gamma(\mathbb{E}) \mid f \ D\text{-closed} \}$.

**Lemma 3.** *Let $f \in \Gamma(\mathbb{E})$. Then the following are equivalent:*

*i)* $f \in \Gamma_0(\operatorname{dom} f)$*;*

*ii)* $f(x) = (\operatorname{cl} f)(x)$ *for all* $x \in \operatorname{dom} f$.

*Proof.* Since $f \in \Gamma_0(\mathbb{E})$, we have

$$\begin{aligned} f \in \Gamma_0(\operatorname{dom} f) &\iff \operatorname{epi} f \text{ is closed in } \operatorname{dom} f \times \mathbb{R} \\ &\iff \operatorname{epi} f = \operatorname{cl}(\operatorname{epi} f) \cap \operatorname{dom} f \times \mathbb{R} \\ &\iff \operatorname{epi} f = \operatorname{epi}(\operatorname{cl} f) \cap \operatorname{dom} f \times \mathbb{R} \\ &\iff f(x) = \operatorname{cl} f(x) \quad \forall x \in \operatorname{dom} f. \end{aligned}$$

Here the first equivalence is simply the definition of $\Gamma_0(\operatorname{dom} f)$. The second is due to the fact that the closed sets in the $\operatorname{dom} f \times \mathbb{R}$ subspace topology are exactly the intersections of closed sets (in $\mathbb{E} \times \mathbb{R}$) with $\operatorname{dom} f \times \mathbb{R}$. The third one is clear as epi (cl $f$) = cl (epi $f$), and the fourth one follows from elementary considerations. □

**Remark 4.** *We point out that $f \in \Gamma_0(\mathbb{E})$ implies that $f \in \Gamma_0(\operatorname{dom} f)$, since the closed set* epi $f \subset \mathbb{E} \times \mathbb{R}$ *intersected with* $\operatorname{dom} f \times \mathbb{R}$ *is (trivially) closed in the subspace topology induced by* $\operatorname{dom} f \times \mathbb{R}$*. However, the converse statement is not true. Consider for instance $\delta_{(0,1)} \in \Gamma_0((0,1)) \setminus \Gamma_0(\mathbb{R})$, as $(0,1) \times \mathbb{R}_+$ is a closed set in the topology induced by $(0,1) \times \mathbb{R}$, but is not a closed set in $\mathbb{R} \times \mathbb{R}$.*

A nonempty subset $K \subset \mathbb{E}$ is called a *cone* if $\lambda x \in K$ for all $\lambda \geq 0$ and $x \in K$. If the latter only holds for all $\lambda > 0$, we call $K$ a *pre-cone*. For instance if $K$ is a cone, then ri $K$ is a pre-cone, use e.g. [18, Corollary 6.6.1]. Combining this with the *line segment principle* [18, Theorem 6.1] and [18, Theorem 6.3], we find the following result.

**Lemma 5.** *Let $K \subset \mathbb{E}$ be a convex cone. Then* $\operatorname{cl} K + \operatorname{ri} K \subset \operatorname{ri} K$.

The *polar cone* of a (pre-)cone $K$ is given by $K^\circ := \{v \in \mathbb{E}_2 \mid \forall u \in K \colon \langle u, v \rangle \leqslant 0\}$, and $-K^\circ$ is referred to as the *dual cone*. Recall that $\overline{\operatorname{conv}}\, K = (K^\circ)^\circ =: K^{\circ\circ}$ by the bipolar theorem [19, Corollary 6.21], and that the polarity operation is order reversing. The *horizon cone* of $C \subset \mathbb{E}$ is given by $C^\infty := \{u \in \mathbb{E} \mid \exists \{t_k\} \downarrow 0, \{x_k \in C\} : \lim_{k \to \infty} t_k x_k = u\}$. If $C$ is a nonempty convex set, then $C + C^\infty = \operatorname{cl} C$ and for a cone $K$, we have $K^\infty = \operatorname{cl} K$.

A cone $K \subset \mathbb{E}$ induces and ordering on $\mathbb{E}$ via

$$y \geq_K x :\iff y - x \in K \quad \forall x, y \in \mathbb{E}.$$

**Lemma 6.** *If $K$ is a closed and convex cone of $\mathbb{E}_2$, then*

$$y \geq_K x \quad \iff \quad \langle u, x \rangle \geqslant \langle u, y \rangle \quad \forall u \in -K^\circ.$$

*Proof.* By the bipolar theorem [19, Corollary 6.21], we have $K^{\circ\circ} = K$ and hence

$$x \geqslant_K y \iff x - y \in K = K^{\circ\circ} \iff \langle u, x - y \rangle \geqslant 0 \quad \forall u \in -K^\circ.$$

□

A cone $K \subset \mathbb{E}$ is said to be *pointed* if $K \cap (-K) = \{0\}$. Such a cone induces a partial ordering.

**Lemma 7** (Ordering induced by a pointed cone)**.** *Let $K \subset \mathbb{E}$ be pointed. Then*

$$x = y \iff x \geq_K y \text{ and } y \geq_K x.$$

*Proof.* Straightforward. □

## 3 $K$-convexity and $K$-closedness

We commence this section with the central definitions of this paper.

**Definition 8** ($K$-epigraphs, $K$-convexity and $K$-closedness)**.** *Let $K \subset \mathbb{E}_2$ be a cone and let $F\colon D \subset \mathbb{E}_1 \to \mathbb{E}_2$. Then the* $K$-epigraph *of $F$ is given by*

$$K\text{-epi}\, F = \{(x, y) \in D \times \mathbb{E}_2 \mid F(x) \leq_K y\} \subset \mathbb{E}_1 \times \mathbb{E}_2. \tag{2}$$

*We say that $F$ is*

*i)* proper *if $K$-epi $F \neq \emptyset$ (i.e. $D \neq \emptyset$);*

*ii)* $K$-convex *if $K$-epi $F$ is convex;*

*iii)* $K$-closed *if $K$-epi $F$ is closed.*

For $D \subset \mathbb{E}_1$ convex and $K \subset \mathbb{E}_2$ a cone, we point out that $F : D \to \mathbb{E}_2$ is $K$-convex if and only if $K$ is convex and

$$\alpha F(x) + (1 - \alpha)F(y) \geqslant_K F(\alpha x + (1 - \alpha)y) \quad \forall x, y \in D, \alpha \in (0, 1).$$

Moreover, we always have

$$K\text{-epi}\, F = \text{gph}\, F + \{0\} \times K. \tag{3}$$

This has, in particular, the following immediate consequence.

**Lemma 9.** *Let $F\colon D \subset \mathbb{E}_1 \to \mathbb{E}_2$ be proper, and $K_1 \subsetneq K_2 \subset \mathbb{E}_2$ be cones. Then $K_1$-epi $F \subsetneq K_2$-epi $F$. In particular, there is at most one cone $K \subset \mathbb{E}_2$ such that $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$.*

In the convex case we can extract the following.

**Lemma 10.** *Let $F\colon D \subset \mathbb{E}_1 \to \mathbb{E}_2$ be proper, and let $K_1 \subset K_2 \subset \mathbb{E}_2$ be convex cones. Then $K_2$-epi $F = K_1$-epi $F + \{0\} \times K_2$. In particular, if $F$ is $K_1$-convex, then $F$ is $K_2$-convex.*

*Proof.* This is due to (3) combined with the fact that $K_1 + K_2 = K_2$ because $K_1$ and $K_2$ are convex cones. □

Given a cone $K \subset \mathbb{E}$ and its induced ordering, we attach to $\mathbb{E}$ a formal largest element $+\infty_\bullet$ with respect to that ordering, and set $\mathbb{E}^\bullet := \mathbb{E} \cup \{+\infty_\bullet\}$. For $G\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ its *domain* is $\text{dom}\, G := \{x \in \mathbb{E}_1 : G(x) \in \mathbb{E}_2\}$. The *graph* of $G$ is given by $\text{gph}\, G := \{(x, F(x)) \mid x \in \text{dom}\, F\}$. We adopt the notions in Definition 8 via the restriction $F := G_{|\text{dom}\, G}$. We record in the next result that $K$-closedness and $K$-convexity requires certain conditions about the underlying cone $K$.

**Proposition 11.** *Let $K \subset \mathbb{E}_2$ be a cone, and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper. Then the following hold:*

*a) If $F$ is $K$-closed, then $K$ is closed.*

*b) If $F$ is $K$-convex, then $K$ is convex.*

*Proof.* a) Let $\{y_k \in K\} \to y$ and pick $x \in \text{dom}\, F$. Then $(x, F(x) + y_n) \in K$-epi $F$ for all $k \in \mathbb{N}$ and $(x, F(x) + y_k) \to (x, F(x) + y) \in K$-epi $F$ as $K$-epi $F$ is closed. Thus, $y \in F(x) + y - F(x) = y \in K$.

b) Let $y_1, y_2 \in K, \alpha \in (0, 1)$. For $x \in \text{dom}\, F$ we hence find $(x, F(x) + y_1) \in K$-epi $F$, and $(x, F(x) + y_2) \in K$-epi $F$. As $K$-epi $F$ is a convex, we have $(x, F(x) + \alpha y_1 + (1 - \alpha)y_2) \in K$-epi $F$, and consequently $\alpha y_1 + (1 - \alpha)y_2 \in K$. Thus, $K$ is convex. □

The following proposition shows that a function $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ is fully determined by its $K$-epigraph when $K$ is a pointed cone.

**Proposition 12.** *Let $K \subset \mathbb{E}_2$ be a pointed cone, and let $F, G\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper. Then*

$$K\text{-epi}\, F = K\text{-epi}\, G \iff F = G.$$

*Proof.* Suppose that $K$-epi $F = K$-epi $G$. In particular, for all $x \in \text{dom}\, F$, $(x, F(x)) \in K$-epi $F$, so $(x, F(x)) \in K$-epi $G$, hence $x \in \text{dom}\, G$ and $F(x) \geqslant_K G(x)$. Likewise, for all $x \in \text{dom}\, G$, we have $x \in \text{dom}\, F$ and $G(x) \geqslant_K F(x)$. Thus $\text{dom}\, F = \text{dom}\, G$ and for any $x \in \text{dom}\, F = \text{dom}\, G$ we have $F(x) = G(x)$ by Lemma 7. □

Given $F \subset \mathbb{E}_1 \colon D \to \mathbb{E}_2$ and $u \in \mathbb{E}_2$, we define the *scalarization* $\langle u, F\rangle \colon \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}$ by

$$\langle u, F\rangle(x) = \begin{cases} \langle u, F(x)\rangle, & x \in D, \\ +\infty & \text{else.} \end{cases} \tag{4}$$

We adapt this notion for $D = \operatorname{dom} F$ if $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ where $\mathbb{E}_2$ is ordered by some cone $K$. Equipped with this concept, the following proposition gives a characterization of $K$-epi $F$ (and gph $F$) via the epigraphs (and graphs) of the scalarizations $\langle u, F\rangle$ for $u \in$ -$K^\circ$.

**Proposition 13.** *Let $K \subset \mathbb{E}_2$ be a closed and convex cone, and let $F \colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper. Then:*

*a)* $K\text{-epi}\, F = \bigcap_{u \in \text{-}K^\circ} (\mathrm{id}, \langle u, \cdot\rangle)^{-1}\big(\mathrm{epi}\ \langle u, F\rangle\big)$*;*

*b) If $K$ is pointed, then* $\mathrm{gph}\, F = \bigcap_{u \in \text{-}K^\circ} (\mathrm{id}, \langle u, \cdot\rangle)^{-1}\big(\mathrm{gph}\ \langle u, F\rangle\big)$.

*Proof.* We deduce from Lemma 6 that

$$\begin{aligned} K\text{-epi}\, F &= \{(x, v) \mid v \geq_K F(x)\} \\ &= \{(x, v) \mid \langle u, v\rangle \geq \langle u, F(x)\rangle\ \ \forall u \in \text{-}K^\circ\} \\ &= \bigcap_{u \in \text{-}K^\circ} (\mathrm{id}, \langle u, \cdot\rangle)^{-1}\big(\mathrm{epi}\ \langle u, F\rangle\big). \end{aligned}$$

Similarly, if $K$ is pointed, we obtain

$$\begin{aligned} \mathrm{gph}\, F &= \{(x, v) \mid x \in \mathbb{E}_1,\ v = F(x)\} \\ &= \{(x, v) \mid x \in \mathbb{E}_1,\ v - F(x) \in K \cap (-K)\} \\ &= \{(x, v) \mid x \in \mathbb{E}_1,\ v \geq_K F(x)\} \bigcap \{(x, v) \mid x \in \mathbb{E}_1,\ F(x) \geq_K v\} \\ &= \bigcap_{u \in \text{-}K^\circ} (\mathrm{id}, \langle u, \cdot\rangle)^{-1}\big(\mathrm{gph}\ \langle u, F\rangle\big). \end{aligned}$$

□

As an immediate consequence of the latter theorem, one obtains Pennanen's sufficient condition for $K$-closedness [17, Lemma 6.2], which unfortunately excludes functions with domains that are not closed. We therefore provide the following stronger version whose proof is simply a refinement of Pennanen's in the next result's part b). Part a) is a refinement of the scalarization characterization of $K$-convexity.

**Proposition 14.** *Let $K \subset \mathbb{E}_2$ be a closed, convex cone, let $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper. Then:*

*a) The following are equivalent:*

*i) $\langle u, F\rangle$ is convex for all $u \in \mathrm{ri}\,(-K^\circ)$;*

*ii) $\langle u, F\rangle$ is convex for all $u \in -K^\circ$;*

*iii) $F$ is $K$-convex.*

*b) $F$ is $K$-closed if $\langle u, F\rangle$ is lower semicontinuous for all $u \in -K^\circ \setminus \{0\}$ and $K \neq \mathbb{E}_2$.*

*In particular, if $K \neq \mathbb{E}_2$ and $\langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $u \in -K^\circ \setminus \{0\}$, then $F$ is $K$-closed and $K$-convex.*

*Proof.* a) 'i)⇒ ii)': Let $u \in -K^\circ \setminus \{0\}$. Then $u$ is a limit $\{u_k \in \mathrm{ri}\,(-K^\circ)\} \to u$, and hence $\langle u, F\rangle$ is a pointwise limit of conxex functions $\langle u_k, F\rangle$, hence convex by Lemma 2.
'ii)⇒ iii)': Follows from Proposition 13 a).
'iii)⇒ i)': Follows from Lemma 6.

b) Assume that $F$ is not $K$-closed i.e. there exists $\{(x_k, y_k) \in K\text{-epi}\, F\} \to (x, y) \notin K\text{-epi}\, F$. Then $x \notin \operatorname{dom} F$ or $y - F(x) \notin K = K^{\circ\circ}$. In the latter case there exists $u^* \in -K^\circ$ such that

$$\langle u^*, y\rangle < \langle u^*, F(x)\rangle \quad \text{and} \quad \langle u^*, y_k\rangle \geq \langle u^*, F(x_k)\rangle \quad \forall k \in \mathbb{N}. \tag{5}$$

If $x \in \operatorname{dom} F$, then necessarily $u^* \neq 0$. On other hand, if $x \notin \operatorname{dom} F$, since $-K^\circ \supsetneq \{0\}$ by assumption, we can choose $u^* \neq 0$. All in all, there exists $u^* \in -K^\circ \setminus \{0\}$ such that (5) holds. We hence obtain

$$\langle u^*, F\rangle\,(x) > \langle u^*, y\rangle = \liminf_{k\to\infty} \langle u^*, y_k\rangle \geq \liminf_{k\to\infty} \langle u^*, F\rangle\,(x_k),$$

and, consequently, $\langle u^*, F\rangle$ is not lsc, which concludes the proof of part b). □

We close out this preparatory paragraph with the following useful result.

**Lemma 15.** *Let $D \subset \mathbb{E}_1$ be nonempty, let $F\colon D \to \mathbb{E}_2$, and let $(K_i)_{i\in I}$ be a family of cones of $K_i \subset \mathbb{E}_2$. Then*

$$\left(\bigcap_{i\in I} K_i\right)\text{-epi}\, F = \bigcap_{i\in I} (K_i\text{-epi}\, F)\,.$$

*In particular, if $F$ is $K_i$-closed for all $i \in I$, then $F$ is $(\cap_{i\in I} K_i)$-closed. Moreover, if $F$ is $K_i$-convex for all $i \in I$ then $F$ is $(\cap_{i\in I} K_i)$-convex. The latter is an equivalence if $K_i$ is convex for all $i \in I$.*

*Proof.* For any $x \in \operatorname{dom} F$ and $y \in \mathbb{E}_2$, we have

$$\begin{aligned}
(x,y) \in \left(\bigcap_{i\in I} K_i\right)\text{-epi}\, F \quad &\iff \quad y - F(x) \in \bigcap_{i\in I} K_i \\
&\iff \quad \forall i \in I:\ y - F(x) \in K_i \\
&\iff \quad \forall i \in I:\ (x,y) \in K_i\text{-epi}\, F \\
&\iff \quad (x,y) \in \bigcap_{i\in I} K_i\text{-epi}\, F.
\end{aligned}$$

The addendum follows from the fact that intersection preserves closedness and convexity, and that $\bigcap_{i\in I} K_i \subset K_i$, so $\bigcap_{i\in I} K_i$-convexity implies $K_i$-convexity for all $i \in I$ if these are convex, see Lemma 10. □

## 3.1 Affine and {0}-convex functions

We extend the notion of affine functions to affine subsets of $\mathbb{E}$, see, e.g., Rockafellar [18, §1] for the standard case.

**Definition 16.** *Let $A \subset \mathbb{E}$ be an affine set and let $x_0 \in A$. Then a function $F\colon A \to \mathbb{E}_2$ is said to be affine if there exists a linear map $L : \operatorname{par} A \mapsto \mathbb{E}_2$ and a vector $b \in \mathbb{E}_2$ such that, for all $x \in A$, we have $F(x) = L(x - x_0) + b$ for all $x \in A$.*

**Lemma 17.** *Let $A \subset \mathbb{E}_1$ be affine. Then $F\colon A \to \mathbb{E}_2$ is affine if and only if*

$$F(tx + (1-t)y) = tF(x) + (1-t)F(y) \quad \forall x, y \in A,\ t \in (0,1). \tag{6}$$

*Proof.* Assume first that (6) holds. Discriminating the three cases $t \in [0,1]$, $t > 1$ and $t < 0$, it is straightforward to show that, in fact, we have

$$F(tx + (1-t)y) = tF(x) + (1-t)F(y) \quad \forall x, y \in A,\ t \in \mathbb{R}. \tag{7}$$

Now let $x_0 \in A$, i.e. $\operatorname{par} A = A - x_0$, and define $L : \operatorname{par} A \to \mathbb{E}_2$ by $L(x) := F(x + x_0) - F(x_0)$. Using (7), we find that $L(tx + (1-t)y) = tL(x) + (1-t)L(y)$ for all $x, y \in \operatorname{par} A$ and $t \in \mathbb{R}$. Thus, taking $y = 0$, as $L(0) = 0$, gives $L(tx) = tL(x)$ for all $x \in \operatorname{par} A$ and all $t \in \mathbb{R}$. Hence, $L(x+y) = L(\frac{1}{2}(2x) + \frac{1}{2}(2y)) = \frac{1}{2}L(2x) + \frac{1}{2}L(2y) = L(x) + L(y)$, for all $x, y \in \operatorname{par} A$. This implies that $L$ is linear. Hence, for all $x \in A$ and $b := F(x_0)$, we have $F(x) = L(x - x_0) + b$. Thus, $F$ is affine.

Conversely, if $F$ is affine, then we can write $F = L((\cdot) - x_0) + b$ for some linear map $L : \operatorname{par} A \to \mathbb{E}_2$, $x_0 \in A$ and $b \in \mathbb{E}_2$. Then

$$F(tx + (1-t)y) = t(L(x - x_0) + b) + (1-t)(L(y - x_0) + b) = tF(x) + (1-t)F(y),$$

for all $x, y \in A$ and $t \in \mathbb{R}$. In particular, this is true for all $t \in (0,1)$. □

**Proposition 18.** *Let $D \subset \mathbb{E}_1$ be nonempty convex. Then $F\colon D \to \mathbb{E}_2$ is $\{0\}$-convex if and only if there exists an affine function $G\colon \operatorname{aff} D \to \mathbb{E}_2$ such that $F = G_{|D}$.*

*Proof.* First, assume that $F\colon D \to \mathbb{E}_2$ is $\{0\}$-convex and $z \in \operatorname{aff} D$. By Lemma 1, we can write $z = \alpha x - \beta y$, for some $\alpha, \beta \geqslant 0$, $\alpha - \beta = 1$, and $x, y \in D$. Suppose $z$ has two representations of this form, i.e. $z = \alpha x - \beta y = \alpha' x' - \beta' y'$, with $\alpha, \beta, \alpha', \beta' \geqslant 0$, $\alpha - \beta = \alpha' - \beta' = 1$, and $x, y, x', y' \in D$. Then $\Delta := \alpha + \beta' (= \alpha' + \beta) = 1 + \beta + \beta' > 0$. By convexity of $D$, we find that $\frac{\alpha}{\Delta}x + \frac{\beta'}{\Delta}y' = \frac{\alpha'}{\Delta}x' + \frac{\beta}{\Delta}y \in D$. Using the $\{0\}$-convexity of $F$, we have $\frac{\alpha}{\Delta}F(x) + \frac{\beta'}{\Delta}F(y') = \frac{\alpha'}{\Delta}F(x') + \frac{\beta}{\Delta}F(y)$. Multiplying by $\Delta$ and rearranging the above terms, we get

$$\alpha F(x) - \beta F(y) = \alpha' F(x') - \beta' F(y').$$

Therefore, we the function $G : \operatorname{aff} D \to \mathbb{E}_2$, $G(z) = \alpha F(x) - \beta F(y)$ for $z \in \operatorname{aff} D$ given by $z = \alpha x - \beta y$, $\alpha, \beta \geqslant 0$, $\alpha - \beta = 1$, $x, y \in D$ is well-defined.

Now, let $z$ and $z'$ in $D$ given by $z = \alpha x - \beta y$ and $z' = \alpha' x' - \beta y'$, with $\alpha, \beta, \alpha', \beta' \geqslant 0$, $\alpha - \beta = 1$, $\alpha' - \beta' = 1$, and $x, y, x', y' \in D$. Let $t \in (0,1)$ and set $p := (1-t)z + tz'$, as well as $a := (1-t)\alpha + t\alpha'$ and $b := (1-t)\beta + t\beta'$. Then $a, b \geqslant 0$ and $a - b = 1$.

If $b = 0$, then $\beta = \beta' = 0$ and $\alpha = \alpha' = 1$, so $z = x \in D$ and $z' = x' \in D$, and thus, using the $\{0\}$-convexity of $F$, we have $G(p) = F(p) = F((1-t)z + tz') = (1-t)F(z) + tF(z') = (1-t)G(z) + tG(z')$.

If $b \neq 0$, then $\beta \neq 0$ or $\beta' \neq 0$, hence $a, b > 0$. Then, we have

$$p = a \underbrace{\left[\frac{(1-t)\alpha}{a} x + \frac{t\alpha'}{a} x'\right]}_{\in D} - b \underbrace{\left[\frac{(1-t)\beta}{b} y + \frac{t\beta'}{b} y'\right]}_{\in D}.$$

Using this, and recalling the fact that $a, b \geqslant 0$ and $a - b = 1$, the definition of $G$ yields

$$G(p) = aF\left(\frac{(1-t)\alpha}{a} x + \frac{t\alpha'}{a} x'\right) - bF\left(\frac{(1-t)\beta}{b} y + \frac{t\beta'}{b} y'\right).$$

As $F$ is $\{0\}$-convex, we thus infer

$$\begin{aligned} G(p) &= a\frac{(1-t)\alpha}{a} F(x) + a\frac{t\alpha'}{a} F(x') - b\frac{(1-t)\beta}{b} F(y) - b\frac{t\beta'}{b} F(y') \\ &= (1-t) \underbrace{[\alpha F(x) - \beta F(y)]}_{G(z)} + t \underbrace{[\alpha' F(x') - \beta' F(y')]}_{G(z')} \\ &= (1-t)G(z) + tG(z'). \end{aligned}$$

All in all, by Lemma 17, $G$ is affine.

Conversely, if there exists $G : \operatorname{aff} D \to \mathbb{E}_2$ affine such that $F = G_{|D}$, then Lemma 17 yields $G(tx + (1-t)y) = tG(x) + (1-t)G(y)$ for all $t \in (0,1)$ and $x, y \in D$. Hence $F(tx + (1-t)y) = tF(x) + (1-t)F(y)$ for all $x, y \in D$, $t \in (0,1)$, and as $D$ is a convex set, $F$ is $\{0\}$-convex. □

As a simple corollary we get the following result.

**Corollary 19.** *The $\{0\}$-convex functions $\mathbb{E}_1 \to \mathbb{E}_2$ are exactly the affine functions.*

## 3.2 Convexity with respect to a (nontrivial) subspace

Using our study on $\{0\}$-convexity above, we are now in a position to investigate the functions which are convex w.r.t. a given (nontrivial) subspace. To this end, observe that for a subspace $U \subset \mathbb{E}$, the polar (and the dual) cone of $U$ equal the orthogonal complement $U^\perp := \{u \in \mathbb{E} \mid \langle u,\, y\rangle = 0 \;\forall y \in U\}$.

**Lemma 20.** *Let $U \subset \mathbb{E}_2$ be a nontrivial subspace and let $\{e_1, \ldots, e_r\}$ be a basis of $U^\perp$. Then for $F : D \subset \mathbb{E}_1 \to \mathbb{E}_2$, with $D \subset \mathbb{E}_1$ convex, the following are equivalent:*

*i) $F$ is $U$-convex;*

*ii) $\langle u,\, F\rangle$ is $\{0\}$-convex for all $u \in U^\perp$;*

*iii) $\langle e_i,\, F\rangle$ is $\{0\}$-convex for all $i = 1, \ldots, r$.*

*Proof.* 'i)⇔ii)': If $F$ is $U$-convex and since $U^\perp$ is a subspace, by Proposition 14 we find that both $\langle u,\, F\rangle$ and $\langle -u,\, F\rangle$ are convex for every $u \in U^\perp$. Therefore, $\langle u,\, F\rangle$ is $\{0\}$-convex for all $u \in U^\perp$. In turn, if $\langle u,\, F\rangle$ is $\{0\}$-convex for all $u \in U^\perp$, then in particular $\langle u,\, F\rangle$ is ($\mathbb{R}_+$-)convex for all $u \in U^\perp$, so we can use Proposition 14 to conclude that $F$ is $U$-convex.

'ii)⇔iii)': The implication ii)⇒iii) is trivial. In turn, if iii) holds, let $u \in U^\perp$, i.e. $u = \sum_{i=1}^r u_i e_i$ for some $u_i \in \mathbb{R}$. By assumption, $\langle e_i,\, F\rangle$ $(i = 1, \ldots, r)$ are $\{0\}$-convex. Since $\langle u,\, F\rangle(x) = \sum_{i=1}^r u_i \langle e_i,\, F\rangle(x)$ for all $x \in \operatorname{dom} F$, we find that $\langle u,\, F\rangle$ is $\{0\}$-convex. □

**Proposition 21.** *Let $U \subset \mathbb{E}_2$ be a nontrivial subspace and let $F : D \subset \mathbb{E}_1 \to \mathbb{E}_2$ with $D$ convex. Then $F$ is $U$-convex if and only if there exists an affine map $G\colon \operatorname{aff} D \to \mathbb{E}_2$ and a function $H\colon \operatorname{aff} D \to U$ such that $G_{|D} + H_{|D} = F$.*

*Proof.* Suppose that $F$ is $U$-convex and let $p : \mathbb{E}_2 \to U$ be the orthogonal projection onto $U$. Define

$$H : \operatorname{aff} D \to U, \quad H(x) := \begin{cases} p(F(x)), & x \in D, \\ 0, & \text{else.} \end{cases}$$

Set $\hat{G} := F - H_{|D}$. By Lemma 20 and Proposition 18, $\langle u, F\rangle_{|D}$ is the restriction of an affine function $f^u : \operatorname{aff} D \to \mathbb{R}$ for all $u \in U^\perp$. However, as $H(x) \in U$ $(x \in \mathbb{E}_1)$, we have $\left\langle u, \hat{G}\right\rangle = \langle u, F\rangle$ for all $u \in U^\perp$. Now, let $\{e_1, \dots, e_r\}$ be an orthogonal basis of $U^\perp$. Then $\hat{G} = \sum_{i=1}^r f^{e_i}_{|D} e_i$. Then $G := \sum_{i=1}^r f^{e_i} e_i : \operatorname{aff} D \to \mathbb{E}_2$ is affine and $F = G_{|D} + H_{|D}$.

Conversely, suppose that there exists a function $H : \operatorname{aff} D \to U$ and an affine map $G\colon \operatorname{aff} D \to \mathbb{E}_2$ such that $F = G_{|D} + H_{|D}$. Then for $u \in U^\perp$ we have $\langle u, F\rangle = \langle u, G\rangle_{|D}$, and thus $\langle u, F\rangle$ is $\{0\}$-convex, as $\langle u, G\rangle$ is affine on $\operatorname{aff} D \supset D$ and $D$ is convex. Thus, by Lemma 20, $F$ is $U$-convex. □

## 3.3 Convexity with respect to a half-space and polyhedral cones

Every (proper) half-space $H \subset \mathbb{E}$ that is also a cone is of the form $H = \{x \in \mathbb{E} \mid \langle w, x\rangle \geq 0\}$ for some $w \in \mathbb{E} \setminus \{0\}$. Clearly, $H$ is (polyhedral) convex and closed with dual cone $-H^\circ = \mathbb{R}_+ w$.

**Proposition 22.** *Let $w \in \mathbb{E}_2 \setminus \{0\}$ and let $H = \{x \in \mathbb{E}_2 \mid \langle w, x\rangle \geqslant 0\}$ be the associated half-space. Then for $F : D \to \mathbb{E}_2$ with $D \subset \mathbb{E}_1$ (nonempty) convex we have:*

*a) $F$ is $H$-convex if and only if $\langle w, F\rangle$ is convex.*

*b) $F$ is $H$-closed if $\langle w, F\rangle$ is lower semicontinuous.*

*Proof.* a) By Proposition 14 a), $F$ is $H$-convex if and only if $\langle u, F\rangle$ is convex for all $u \in \operatorname{ri}(-H^\circ) = \mathbb{R}_{++} w$. However, $\langle tw, F\rangle$ is convex if and only if $\langle w, F\rangle$ is convex for all $t > 0$.
b) This follows with a similar argument as in a) from Proposition 14 b) observing that $H \neq \mathbb{E}_2$ and $-H^\circ \setminus \{0\} = \mathbb{R}_{++} w$. □

Proposition 22 combined with Lemma 15 yields the following result on polyhedral cones.

**Corollary 23.** *Let $w_1, \dots, w_l \in \mathbb{E} \setminus \{0\}$ and let $P = \bigcap_{i=1}^l H_i$ with $H_i = \{x \mid \langle w_i, x\rangle \geq 0\}$. Then for $F : D \to \mathbb{E}_2$ with $D \subset \mathbb{E}_1$ (nonempty) convex we have:*

*a) $F$ is $P$-convex if and only if $\langle w_i, F\rangle$ is convex for all $i = 1, \dots, l$.*

*b) $F$ is $P$-closed if $\langle w_i, F\rangle$ is lower semicontinuous for all $i = 1, \dots, l$.*

## 3.4 The smallest closed cone with respect to which $F$ is convex

The following result ensures the existence of a smallest closed cone $K_F$ with respect to which $F$ is convex (hence, $K_F$ is also convex by Proposition 11) and characterizes its dual cone.

**Proposition 24** (The cone $K_F$ and its dual)**.** *Let $D \subset \mathbb{E}_1$ be nonempty and convex and let $F\colon D \to \mathbb{E}_2$. Then the following hold:*

*a) There exists a smallest (nonempty) closed (and convex) cone $K_F \subset \mathbb{E}_2$ with respect to which $F$ is convex, i.e if $F$ is $K$-convex and $K$ is closed and convex, then $K \supset K_F$.*

*b) The dual cone of $K_F$ from a) is given by $-K_F^\circ = \{u \in \mathbb{E}_2 \mid \langle u, F\rangle$ is convex$\}$.*

*Proof.* a) Define $K_F$ as the intersection of all closed and convex cones which respect to which $F$ is convex. Then $K_F$ is nonempty (as $F$ is $\mathbb{E}_2$-convex and every cone contains 0), closed and convex, and, by Lemma 15, $F$ is $K_F$-convex. By construction, there is no smaller cone with these properties.
b) See [17, Lemma 6.1]. □

## 3.5 The smallest (closed) cone with respect to which $F$ is convex *and* closed

We now investigate how the situation changes in comparison to the study in Section 3.4 when we are looking for the smallest (by Proposition 11 necessarily closed and convex) cone with respect to which a function $F$ is convex *and* closed. Such cone does not need to exist, simply because a given function $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ need not be $\mathbb{E}_2$-closed. More concretely, for every cone $K \subset \mathbb{R}$, the indicator $F := \delta_{(0,1]}$ has $K$-epi $F = (0, 1] \times K$, which is not closed.

**Proposition 25** (The cone $\hat{K}_F$)**.** *Let $D \subset \mathbb{E}_1$ be nonempty and convex and let $F\colon D \to \mathbb{E}_2$ such that there exists a (necessarily closed and convex) cone $K \subset \mathbb{E}_2$ with respect to which $F$ is closed and convex. Then there exists a smallest closed and convex cone $\hat{K}_F \subset \mathbb{E}_2$ such that $F$ is $\hat{K}_F$-closed and $\hat{K}_F$-convex.*

*Proof.* Follows readily from Lemma 15. □

In the spirit of Proposition 24 b), we want to characterize the dual cone of $\hat{K}_F$ (if it exists). To this end, the following lemma is useful.

**Lemma 26.** *Let $F : D \subset \mathbb{E}_1 \to \mathbb{E}_2$ with $D$ convex (and nonempty). Then the following hold:*

*a) The set* $\mathrm{cl}\ \{u \in \mathbb{E}_2 \mid \langle u,\, F\rangle \in \Gamma_0(\mathbb{E}_1)\}$ *is either empty or a closed convex cone.*

*b) The set $\{u \in \mathbb{E}_2 \mid \langle u,\, F\rangle \in \Gamma_0(D)\}$ is a convex cone (in particular nonempty).*

*Proof.* a) Assume nonemptiness, in which case $S := \{u \in \mathbb{E}_2 \mid \langle u,\, F\rangle \in \Gamma_0(\mathbb{E}_1)\}$ is also nonempty. Then, clearly, $S$ is convex and, consequently, so is $\mathrm{cl}\ S$. Moreover, for $u \in \mathrm{cl}\ S$ and $\lambda \geq 0$ there exist $\{u_k \in S\} \to u$ and $\{\lambda_k > 0\} \to \lambda$ and with $\lambda_k u_k \in S$, hence $\lambda u \in \mathrm{cl}\ S$, and thus $\mathrm{cl}\ S$ is a closed convex cone.

b) Set $K := \{u \in \mathbb{E}_2 \mid \langle u,\, F\rangle \in \Gamma_0(D)\}$. We first show that $K$ is a cone. To this end, note that $\langle 0,\, F\rangle = \delta_D$, whose epigraph $D \times \mathbb{R}_+$ is clearly closed in the topology induced by $D \times \mathbb{R}$. Now, for $u \in K$ and $\lambda > 0$ observe that $\mathrm{epi}\ \langle \lambda u,\, F\rangle = L_\lambda^{-1}(\mathrm{epi}\ \langle u,\, F\rangle)$ for $L_\lambda : (x, \alpha) \mapsto (x, \alpha/\lambda)$, which is then closed as the linear preimage of a closed set in the topology induced by $D \times \mathbb{R}$. Therefore, $K$ is a cone. In order prove that $K$ is convex, it hence suffices to show that $K + K \subset K$, see [19, Exercise 3.7]. To this end, let $u, v \in K$ and take $\{(x_k, \alpha_k) \in \mathrm{epi}\ \langle u + v,\, F\rangle\} \to (x, \alpha) \in D \times \mathbb{R}$. In particular, we have $(x_k, \alpha_k - \langle v,\, F(x_k)\rangle) \in \mathrm{epi}\ \langle u,\, F\rangle$ for all $k \in \mathbb{N}$. Let $z := \liminf_k \langle v,\, F(x_k)\rangle$. W.l.o.g. $z = \lim_k \langle v,\, F(x_k)\rangle$ (otherwise go to subsequence), and by $D$-closedness of $\langle v,\, F\rangle$, we find $(x, z) \in \mathrm{epi}\ \langle v,\, F\rangle$, i.e. $\langle v,\, F(x)\rangle \leq z$. Moreover, by $D$-closedness of $\langle u,\, F\rangle$ we find that $(x, \alpha - z) \in \mathrm{epi}\ \langle u,\, F\rangle$, hence $\alpha \geq \langle u,\, F(x)\rangle + z \geq \langle u,\, F\rangle(x) + \langle v,\, F\rangle(x)$. Therefore, $(x, \alpha) \in \mathrm{epi}\ \langle u + v,\, F\rangle$. □

**Proposition 27** (The dual cone of $\hat{K}_F$)**.** *Let $F\colon D \subset \mathbb{E}_1 \to \mathbb{E}_2$ with $D$ nonempty and convex. If $\hat{K}_F$ (in the sense of Proposition 25) exists, we have*

$$-\hat{K}_F^\circ = \mathrm{cl}\ \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)\} = \mathrm{cl}\ \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}. \tag{8}$$

*Proof.* Set $\bar{K} := \mathrm{cl}\ (\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)\})$. To prove the first identity, use [17, Corollary 7.4(ii)], to find $\langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)$ for all all $u \in \mathrm{ri}\,(-\hat{K}_F^\circ)$. Consequently, also using [18, Theorem 6.2], we have $-\hat{K}_F^\circ \subset \bar{K}$. Now, assume that the converse inclusion were false, and consequently (by what was already proved) $-\hat{K}_F^\circ \subsetneq \bar{K}$. Then, by definition of $\bar{K}$, there must acutally exist $u_0 \in \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)\} \setminus (-\hat{K}_F^\circ)$ (since otherwise $-\hat{K}_F^\circ = \bar{K}$, in contradiction to the assumption). In particular, $u_0 \neq 0$ and, by positive homogeneity, we have $\langle tu_0,\, F\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $t > 0$, thus $\langle u,\, F\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $u \in \mathbb{R}_+ u_0 \setminus \{0\}$. Therefore, by Proposition 14, with $L := -(\mathbb{R}_+ u_0)^\circ \subsetneq \mathbb{E}_2$, we find that $F$ is $L$-convex and $L$-closed. Thus, by Lemma 15, we find that $F$ is $(\hat{K}_F \cap L)$-closed and -convex. However, since $u_0 \notin -\hat{K}_F^\circ$, we cannot have $L \supset \hat{K}_F$, and hence $\hat{K}_F \cap L \subsetneq \hat{K}_F$. This contradicts the definition of $\hat{K}_F$ as the smallest closed, convex cone with respect to which $F$ is convex and closed, and thus we have proved the first identity.

To prove the second identity we first note that, from what was proved above and by Remark 4, we have $-\hat{K}_F^\circ = \mathrm{cl}\ (\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)\}) \subset \mathrm{cl}\ \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$. In order to prove the converse inclusion, set $K := -\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}^\circ$, and observe that by Lemma 26 b) and [18, Theorem 6.3], respectively, we have

$$-K^\circ = \mathrm{cl}\ \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\} \ \text{ and } \ \mathrm{ri}\,(-K^\circ) \subset \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}. \tag{9}$$

We now show that $F$ is $K$-convex: To this end, first note that $\langle u,\, F\rangle$ is convex for all $u \in \mathrm{ri}\,(-K^\circ)$, by (9). Any $u \in -K^\circ$ is a limit $\{u_k \in \mathrm{ri}\,(-K^\circ)\} \to u$, and therefore $\langle u,\, F\rangle$ is the pointwise limit of convex functions $\langle u_k,\, F\rangle$ , and hence convex (Lemma 2). Thus, by Proposition 14 a), $F$ is $K$-convex.

We now prove that $F$ is also $K$-closed: To this end, let $\{(x_k, y_k) \in K\text{-epi}\,F\} \to (x, y)$. In particular, there exists $\{v_k \in K\}$ such that $F(x_k) + v_k = y_k$ for all $k \in \mathbb{N}$. Moreover, as $K \subset \hat{K}_F$

(see above) and $\hat{K}_F$-epi $F$ is closed (by definition), we have $(x, y) \in \hat{K}_F$-epi $F$, and consequently, $x \in D$. Thus we can use the fact that, by (9), $\langle u, F\rangle$ is $D$-closed for all $u \in \operatorname{ri}(-K^\circ)$, and hence

$$\langle u, y\rangle = \lim_{k\to\infty} \langle u, y_k\rangle = \lim_{k\to\infty} \langle u, F(x_k)\rangle + \underbrace{\langle u, v_k\rangle}_{\geqslant 0} \geqslant \liminf_{k\to\infty} \langle u, F\rangle (x_k) \geqslant \langle u, F\rangle (x)$$

for all $u \in \operatorname{ri}(-K^\circ)$. Now, every $u \in -K^\circ$ is a limit $\{u_k \in -\operatorname{ri}(K^\circ)\} \to u$ with $\langle u_k, y\rangle \geq \langle u_k, F(x)\rangle$, and hence $\langle u, y\rangle \geq \langle u, F(x)\rangle$. As $u \in -K^\circ$ was arbitrary, this shows that $y \geq_K F(x)$, and thus $(x, y) \in K$-epi $F$, which shows that $K$-epi $F$ is closed and hence $F$ is $K$-closed (and $K$-convex as proved earlier). Since $K \subset \hat{K}_F$ it follows that $\hat{K}_F = K$, which concludes the proof. □

The natural question as to when closures in the above result are superfluous is addressed after the following auxiliary result.

**Lemma 28.** *Let $\{e_1, \ldots, e_n\} \subset \mathbb{E}$ be an orthonormal system and $A \subset \mathbb{E}$ an affine set such that $\operatorname{par} A = \operatorname{span}\{e_1, \ldots, e_n\}$. Then, for all $r > 0$ and all $x \in A$, we have $B_{\frac{r}{n}}(x) \cap A \subset \operatorname{conv}\{x \pm re_i \mid i = 1, \ldots, n\}$.*

*Proof.* Let $y \in B_{\frac{r}{n}}(x) \cap A$, i.e. $y = x + \sum_{i=1}^n y_i e_i$ for some $y_1, \ldots, y_n \in \mathbb{R}$ $(i = 1, \ldots, n)$ and $\|y - x\|^2 \leqslant \frac{r^2}{n^2}$. As $e_1, \ldots, e_n$ is an orthonormal system, we have

$$\sum_{i=1}^n y_i^2 = \left\| \sum_{i=1}^n y_i e_i \right\|^2 = \|y - x\|^2 \leq \frac{r^2}{n^2}. \tag{10}$$

On the other hand, we have

$$\begin{aligned} y &= x + \sum_{i=1}^n y_i e_i \\ &= x + \frac{1}{2}\left(1 - \sum_{i=1}^n \frac{|y_i|}{r}\right) \underbrace{(re_1 - re_1)}_{=0} + \sum_{i=1}^n \frac{|y_i|}{r} (\operatorname{sgn}(y_i) re_i) \\ &= \frac{1}{2}\left(1 - \sum_{i=1}^n \frac{|y_i|}{r}\right)(x + re_1) + \frac{1}{2}\left(1 - \sum_{i=1}^n \frac{|y_i|}{r}\right)(x - re_1) + \sum_{i=1}^n \frac{|y_i|}{r}(x + \operatorname{sgn}(y_i) re_i). \end{aligned}$$

In view of (10), $|y_i| \leqslant \frac{r}{n}$ for all $i = 1, \ldots, n$, and thus $y \in \operatorname{conv}\{x \pm re_i \mid i = 1, \ldots, n\}$, which concludes the proof. □

**Corollary 29.** *Let $F\colon D \subset \mathbb{E}_1 \to \mathbb{E}_2$ with $D$ nonempty and convex and assume that $\hat{K}_F$ exists. Then:*

*a) $-\hat{K}_F^\circ = \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$ if and only if $\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$ is closed.*

*b) If, for every sequence $\{x_k \in D\} \to x \in D$ (and every $x \in D$), there exists $v \in \operatorname{ri}(-\hat{K}_F^\circ)$ for which $\{\langle v, F\rangle (x_k)\}$ does not tend to $+\infty$, then $\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$ is closed.*

*c) If, for all $x \in D \setminus \operatorname{ri} D$, there exists a neighborhood $\mathcal{N}_x$ of $x$, and a continuous $\hat{K}_F$-majorant[2] of $F$ on $\mathcal{N}_x \cap D$, then $\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$ is closed.*

*In particular, $-\hat{K}_F^\circ = \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$ if $D$ is relatively open (e.g. affine).*

*Proof.* a) Follows readily from Proposition 27.

b) Denote $K = \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$, which is a convex cone by Lemma 26 b). By Proposition 27, $-\hat{K}_F^\circ = \operatorname{cl} K$. Consider $u \in \operatorname{cl} K$. Then $\langle u, F\rangle$ is convex by Lemma 2 and proper with $\operatorname{dom}\langle u, F\rangle = D$.

Now let $x \in D$ and $\{x_k \in D\} \to x \in D$. Then, by assumption, there exists $v \in \operatorname{ri}(-\hat{K}_F^\circ) = \operatorname{ri} K$ such that $\{\langle v, F\rangle (x_k)\}$ is uniformly bounded away from $+\infty$, hence w.l.o.g. we can assume that $\langle v, F\rangle (x_k) \to r < +\infty$. As $\langle v, F\rangle \in \Gamma_0(D)$, we have $-\infty < \langle v, F\rangle (x) \leq r < +\infty$. Using Lemma 5 (and that $\operatorname{ri} K$ is a pre-cone), we have $u + tv \in \operatorname{ri} K$, and hence $\langle u + v, F\rangle \in \Gamma_0(D)$ for all $t > 0$. Thus

$$\begin{aligned} \liminf_{k\to\infty} \langle u, F\rangle (x_k) &= \liminf_{k\to\infty} \langle u + tv, F\rangle (x_k) - t\langle v, F\rangle (x_k) \\ &= \liminf_{k\to\infty} \langle u + tv, F\rangle (x_k) - tr \\ &\geqslant \langle u + tv, F\rangle (x) - tr. \end{aligned}$$

[2]We refer the reader to Definition 39 for a formal introduction.

Letting $t \to 0$ gives $\liminf_{k\to\infty} \langle u, F\rangle (x_k) \geqslant \langle u, F\rangle (x)$, hence $\langle u, F\rangle \in \Gamma_0(D)$ as desired.

c) Let $\{x_k \in D\} \to x \in D$. We will show that $\langle v, F\rangle (x_k)$ does not tend to $+\infty$, for any $v \in \mathrm{ri}\,(-\hat{K}_F^\circ)$, which by b) then gives the desired conclusion.

First, suppose that $x \in \mathrm{ri}\, D$ and set $A := \mathrm{aff}\, D$. Let $\{e_1, \dots, e_n\} \subset \mathbb{E}_1$ be an orthonormal system such that $A = x + \mathrm{span}\,\{e_1, \dots, e_n\}$. Now, $x \in \mathrm{ri}\, D$ implies that there exists $\varepsilon > 0$ such that $x \pm \varepsilon e_i \in D$ for all $i = 1, \dots, n$. As $D$ is convex, we have $\mathrm{conv}\,\{x \pm \varepsilon e_i \mid i = 1, \dots, n\} \subset D$. As $x_k \to x$, then $x_k \in B_{\frac{\varepsilon}{n}}(x)$ for $k \in \mathbb{N}$ large enough. As $x_k \in D$, we have $x_k \in A$ for all $k \in \mathbb{N}$. Hence, , by Lemma 28, we have $x_k \in B_{\frac{\varepsilon}{n}}(x) \cap A \subset \mathrm{conv}\,\{x \pm \varepsilon e_i \mid i = 1, \dots, n\} \subset D$ for $k \in \mathbb{N}$ large enough. Now let $v \in \mathrm{ri}\,(-\hat{K}_F^\circ) = \mathrm{ri}\,\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(D)\}$. Then, $\langle v, F\rangle$ is convex, so for $k$ large enough, we have $\langle v, F\rangle (x_k) \leqslant \max\{\langle v, F\rangle (x \pm \varepsilon e_i) \mid i = 1, \dots, n\} < +\infty$. Hence $\langle v, F\rangle (x_k)$ does not tend to $+\infty$.

In turn, for $x \in D\backslash \mathrm{ri}\, D$ let $G_x$ be the continuous $\hat{K}_F$-majorant of $F$ on $\mathcal{N}_x$ and let $v \in \mathrm{ri}\,(-\hat{K}_F^\circ)$. Then, as $G_x(y) \geqslant_K F(y)$ for all $y \in \mathcal{N}_x$, hence $\langle v, G_x\rangle (y) \geqslant \langle v, F\rangle (y)$ for all $y \in \mathcal{N}_x$. Since $\{x_k \in D\} \to x$, we have that $x_k \in \mathcal{N}_x$ for $k$ sufficiently large, and thus $\langle v, F\rangle (x_k) \leqslant \langle v, G_x\rangle (x_k)$. However, $G_x$ is continuous, so $\langle v, G_x\rangle$ is continuous as well, hence $\langle v, G_x\rangle (x_k) \to \langle v, G_x\rangle (x) < +\infty$, thus $\langle v, F\rangle (x_k)$ does not tend to $+\infty$. □

We close out this section by clarifying the question as to when $K_F$ and $\hat{K}_F$ coincide.

**Proposition 30** ($K_F = \hat{K}_F$)**.** *Let $D \subset \mathbb{E}_1$ be nonempty and convex and let $F : D \to \mathbb{E}_2$. Then $K_F = \hat{K}_F$ if and only if $F$ is $K_F$-closed.*

*Proof.* By definition of the respective cones, we always have $\hat{K}_F \supset K_F$. But if $F$ is $K_F$-closed then, $\hat{K}_F \subset K_F$, by definition of $\hat{K}_F$, and hence equality holds.

In turn, if $F$ is not $K_F$-closed, then $K_F \neq \hat{K}_F$, since $F$ is $\hat{K}_F$-closed by definition. □

# 4 When is $\overline{\mathrm{conv}}\,(\mathrm{gph}\, F) = K\text{-epi}\, F$?

This section is devoted to our main question as to when the closed convex hull of the graph of a function equals its $K$-epigraph.

## 4.1 A characterization via the horizon cone

We commence this subsection with the central link between the graph and the $K$-epigraph of a function. To obtain an elegant proof we briefly tap into Fenchel conjugacy [18]. To this end, realize that every set $S \subset \mathbb{E}$ is uniquely determined through its indicator function $\delta_S : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ which is paired in duality with the *support function* $\sigma_S : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$, $\sigma_S(x) = \sup_{y\in S} \langle x, y\rangle$ via the conjugacy relations $\delta_S^* = \sigma_S = \sigma_{\overline{\mathrm{conv}}\, S}$, hence $\sigma_S^* = \delta_{\overline{\mathrm{conv}}\, S}$, and thus $\overline{\mathrm{conv}}\, \delta_S = \delta_{\overline{\mathrm{conv}}\, S}$.

**Proposition 31.** *Let $K \subset \mathbb{E}_2$ be a closed, convex cone and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper, $K$-closed and $K$-convex. Then*

$$K\text{-epi}\, F = \mathrm{cl}\,\big(\overline{\mathrm{conv}}(\mathrm{gph}\, F) + \{0\} \times K\big).$$

*Proof.* Using (3) and the set-additivity for support functions we have

$$\sigma_{K\text{-epi}\, F} = \sigma_{\mathrm{gph}\, F + \{0\}\times K} = \sigma_{\mathrm{gph}\, F} + \sigma_{\{0\}\times K} = \sigma_{\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)} + \delta_{\mathbb{E}_1\times K^\circ}.$$

Moreover, the assumptions on $F$ imply that $K$-epi $F$ is closed and convex, and hence

$$\delta_{K\text{-epi}\, F} = \big(\sigma_{\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)} + \delta_{\mathbb{E}_1\times K^\circ}\big)^* = \mathrm{cl}\,\big(\delta_{\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)} \,\square\, \delta_{\{0\}\times K}\big) = \delta_{\mathrm{cl}\,\left(\overline{\mathrm{conv}}\,(\mathrm{gph}\, F) + \{0\}\times K\right)}.$$

Here the second identity uses [18, Theorem 16.4], while the third holds due to the identity $\delta_A \square \delta_B = \delta_{A+B}$ for any two sets. □

We are now in a position to state our first main theorem.

**Theorem 32.** *Let $K \subset \mathbb{E}_2$ be a closed convex cone and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be $K$-convex and $K$-closed. Then*

$$K\text{-epi}\, F = \overline{\mathrm{conv}}\,(\mathrm{gph}\, F) \iff \{0\} \times K \subset [\overline{\mathrm{conv}}\,(\mathrm{gph}\, F)]^\infty.$$

*Proof.* Suppose that $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$. It follows from Proposition 31 that

$$\overline{\text{conv}}\,(\text{gph}\, F) + \{0\} \times K \subset \text{cl}\,\left(\overline{\text{conv}}(\text{gph}\, F) + \{0\} \times K\right) = \overline{\text{conv}}\,(\text{gph}\, F).$$

Taking the horizon cone on both sides and using [19, Exercise 3.12], yields $\{0\}\times K \subset [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty$.

Now suppose that $\{0\} \times K \subset [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty$. Then

$$\overline{\text{conv}}\,(\text{gph}\, F) + \{0\} \times K \subset \overline{\text{conv}}\,(\text{gph}\, F) + [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty = \overline{\text{conv}}\,(\text{gph}\, F).$$

where the last identity uses e.g. [19, Theorem 3.6]. Therefore, again using Proposition 31, we obtain

$$K\text{-epi}\, F = \text{cl}\,(\overline{\text{conv}}\,(\text{gph}\, F) + \{0\} \times K) \subset \overline{\text{conv}}\,(\text{gph}\, F) \subset K\text{-epi}\, F.$$

□

We will frequently make us of the following trivial observation.

**Remark 33.** *We observe that the closure operation in $[\overline{\text{conv}}\,(\text{gph}\, F)]^\infty$ is superfluous, i.e.*

$$[\overline{\text{conv}}\,(\text{gph}\, F)]^\infty = [\text{conv}\,(\text{gph}\, F)]^\infty.$$

We immediately obtain the following sufficient condition.

**Corollary 34.** *Let $K \subset \mathbb{E}_2$ be a closed, convex cone and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be $K$-convex and $K$-closed such that $\{0\} \times K \subset \overline{\text{conv}}\,(\text{gph}\, F)$. Then $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$.*

*Proof.* Combine Theorem 34 with [19, Exercise 3.11] and the fact that the horizon operation preserves inclusion. □

Combining Theorem 34 with Lemma 9 yields the following result.

**Corollary 35.** *Let $K$ be a cone of $\mathbb{E}_2$ and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper, and define the closed, convex cone $K^F := \{u \in \mathbb{E}_2 \mid (0, u) \in [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty\}$. Then*

$$K\text{-epi}\, F = \overline{\text{conv}}\,(\text{gph}\, F) \iff K = K^F = \hat{K}_F.$$

*Proof.* First, let $K = K^F = \hat{K}_F$. By definition of $K^F = K$ we hence have $\{0\}\times K \subset [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty$. From Theorem 34 we thus conclude that $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$.

In turn, assume that $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$. Then by Theorem 34 we have $K \subset K^F$, and hence $\overline{\text{conv}}\,(\text{gph}\, F) = K$-epi $F \subset K^F$-epi $F$. In addition, $\{0\} \times K^F \subset [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty$, by definition of $K^F$. Hence, using (3) and the horizon property of convex sets, we have

$$K^F\text{-epi}\, F = \text{gph}\, F + \{0\} \times K^F \subset \overline{\text{conv}}\,(\text{gph}\, F) + [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty = \overline{\text{conv}}\,(\text{gph}\, F).$$

Thus, $K^F$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F) = K$-epi $F$ and hence, by Lemma 9, we already have $K^F = K$. Moreover, as $F$ is $K$-convex and $K$-closed, we have $\hat{K}_F \subset K$, thus $\hat{K}_F$-epi $F \subset K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$. Using the fact that $\overline{\text{conv}}\,(\text{gph}\, F) \subset \hat{K}_F$-epi $F$ (as $\hat{K}_F$-epi $F$ is a closed convex set containing gph $F$), we conclude that $\hat{K}_F$-epi $F = K$-epi $F$, hence, again by Lemma 9, $K = \hat{K}_F$. □

## 4.2 Necessary conditions

In this subsection we discuss necessary conditions for $\overline{\text{conv}}\,(\text{gph}\, F) = K$-epi $F$.

### 4.2.1 Necessary conditions on the dual cone

**Proposition 36.** *Let $K \subset \mathbb{E}_2$ be a cone and $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ proper such that $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$. Then $K \subset -\hat{K}_F^\circ$.*

*Proof.* Let $u \in K \setminus \{0\}$. By Theorem 34, we have that $(0, u) \in [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty$. By Remark 33 there hence exist $\{(x_k, y_k) \in \text{conv}\,(\text{gph}\, F)\}$ and $\{\lambda_k\} \downarrow 0$ such that $\lambda_k(x_k, y_k) \to (0, u)$. With $\kappa := \dim \mathbb{E}_1 \times \mathbb{E}_2$ and Carathéodory's theorem [18, Theorem 17.1], for $i = 1, \ldots, \kappa + 1$, we find sequences $\{x_k^i \in \text{dom}\, F\}_k$ and $\{\alpha_k^i\}_k$ such that $\sum_{i=1}^{\kappa+1} \alpha_k^i = 1$ for all $k \in \mathbb{N}$ as well as

$$x_k = \sum_{i=1}^{\kappa+1} \alpha_k^i x_k^i \quad \text{and} \quad y_k = \sum_{i=1}^{\kappa+1} \alpha_{k,j}^i F(x_{k,j}^i) \quad \forall k \in \mathbb{N}.$$

Now let $t \geq 0$. Then $t_k := t\frac{\lambda_k}{\|u\|^2} \downarrow 0$ and

$$t_k x_k \to 0 \quad \text{and} \quad t_k \sum_{i=1}^{\kappa+1} \alpha_k^i \langle u, F\rangle (x_k^i) = t\frac{\langle \lambda_k y_k, u\rangle}{\|u\|^2} \to t.$$

Thus, for $t \geq 0$, we have $(0,t) \in [\mathrm{conv}\,(\mathrm{gph}\,\langle u, F\rangle)]^\infty = [\overline{\mathrm{conv}}\,(\mathrm{gph}\,\langle u, F\rangle)]^\infty$. Hence $\{0\} \times \mathbb{R}_+ \subset [\overline{\mathrm{conv}}\,(\mathrm{gph}\,\langle u, F\rangle)]^\infty$, so by Theorem 34, $\mathrm{epi}\,\langle u, F\rangle = \overline{\mathrm{conv}}\,(\mathrm{gph}\,\langle u, F\rangle)$. This means that $\mathrm{epi}\,\langle u, F\rangle$ is closed and convex so that $\langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)$. Therefore $K \setminus \{0\} \in \{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)\}$, hence $K \subset \mathrm{cl}\,(\{u \in \mathbb{E}_2 \mid \langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)\}) = -\hat{K}_F^\circ$, by Proposition 27. □

We readily derive the following necessary condition on the dual cone.

**Corollary 37.** *Under the assumptions of Proposition 36, we have* $K \subset -K^\circ$.

*Proof.* By Corollary 35, $K = \hat{K}_F$, and by Proposition 36, $K \subset -\hat{K}_F^\circ = -K^\circ$. □

### 4.2.2 Affine majorization and minorization

For motivational purposes, we start this subsection with the scalar case ($K = \mathbb{R}_+$), where the question whether the closed convex hull of the graph of a function equals its $K$-epigraph can be fully answered via affine majorization. The proof relies, in essence, on a standard separation argument.

**Theorem 38** (The scalar case)**.** *Let* $f \in \Gamma_0(\mathbb{E})$*. Then* $\mathrm{epi}\, f = \overline{\mathrm{conv}}\,(\mathrm{gph}\, f)$ *if and only if* $f$ *does not have an affine majorant on its domain.*

*Proof.* Suppose that there exists $(\bar{x}, \bar{t}) \in \mathrm{epi}\, f \backslash \overline{\mathrm{conv}}\,(\mathrm{gph}\, f)$. In particular, $\bar{x} \in \mathrm{dom}\, f$, and by strong separation [18, Corollary 11.4.2], there exists $(s, \alpha) \in \mathbb{E}_1 \times \mathbb{R}$ such that

$$\langle s, \bar{x}\rangle + \alpha\bar{t} > \sup_{(x,t) \in \overline{\mathrm{conv}}\,(\mathrm{gph}\, f)} \langle s, x\rangle + \alpha t. \tag{11}$$

Choosing $(x,t) := (\bar{x}, f(\bar{x}))$, we find that $\alpha(\bar{t} - f(\bar{x})) > 0$, and hence, $\alpha > 0$. It then follows from (11) with $x \in \mathrm{dom}\, f$ and $t = f(x)$ that $\langle s/\alpha, \bar{x} - x\rangle + \bar{t} > f(x)$. Thus $f$ is majorized on its domain by the affine map $x \mapsto -\langle s/\alpha, x\rangle + \langle s/\alpha, \bar{x}\rangle + \bar{t}$, which proves one direction.

To prove the converse implication, suppose now that $f$ has an affine majorant on its domain, i.e. there exists $(a, \beta) \in \mathbb{E} \times \mathbb{R}$, such that $f(x) \leq \langle a, x\rangle + \beta =: g(x)$ for all $x \in \mathrm{dom}\, f$. Now pick $\bar{x} \in \mathrm{dom}\, f$. Then $(\bar{x}, g(\bar{x})+1) \in \mathrm{epi}\, f$, and it hence suffices to show that $(\bar{x}, g(\bar{x})+1) \notin \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$. Assume, by contradiction, that $(\bar{x}, g(\bar{x}) + 1) \in \overline{\mathrm{conv}}\,(\mathrm{gph}\, F)$. Then with $\kappa := \dim \mathbb{E} \times \mathbb{R}$, by Carathéodory's theorem [18, Theorem 17.1], for $i = 1, \ldots, \kappa+1$ there exist sequences $\{t_{i,k} \geq 0\}_{k\in\mathbb{N}}$ and $\{x_i^k \in \mathrm{dom}\, f\}_{k\in\mathbb{N}}$ such that $\sum_{i=1}^{\kappa+1} t_{i,k} = 1$ and $\sum_{i=1}^{\kappa+1} t_{i,k}(x_i^k, f(x_i^k)) \to_{k\to\infty} (\bar{x}, g(\bar{x}) + 1)$. Consequently

$$g(\bar{x}) + 1 = \lim_{k\to\infty} \sum_{i=1}^{\kappa+1} t_{i,k} f(x_i^k) \leq \lim_{k\to\infty} \sum_{i=1}^{\kappa+1} t_{i,k} g(x_i^k) = \lim_{k\to\infty} \left\langle a, \sum_{i=1}^{\kappa+1} t_{i,k} x_i^k \right\rangle + \beta = g(\bar{x}),$$

which is the desired contradiction and thus concludes the proof. □

The questions as to what can be said when $f$ in the above result is only proper and convex, but not necessarily closed is answered as the opening to Section 4.3.

To start our analysis of the vector-valued case we now formally introduce the notion of *$K$-minorants and -majorants*, respectively.

**Definition 39** ($K$-minorants/-majorants)**.** *Let* $K \subset \mathbb{E}_1$ *be a cone, and let* $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ *be proper. A function* $G : \mathbb{E}_1 \to \mathbb{E}_2$ *is said to be:*

- *a* $K$-majorant *of* $F$ *on* $S \subset \mathrm{dom}\, F$ *if*

$$G(x) - F(x) \in K \quad \forall x \in S,$$

- *a* $K$-minorant *of* $F$ *on* $S \subset \mathrm{dom}\, F$ *if*

$$F(x) - G(x) \in K \quad \forall x \in S.$$

*For* $S = \mathrm{dom}\, F$*, we say that* $G$ *is a* $K$*-minorant of* $F$*.*

Naturally, in view of the scalar case from Theorem 38, we are mainly interested in the case where $G$ is an affine function.

For a function $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$, we record that, for a pointed, closed, convex cone $K$ such that $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$, there cannot exist both an affine $K$-majorant on the domain and an affine $K$-minorant of $F$.

**Proposition 40.** *Let $\{0\} \subsetneq K \subset \mathbb{E}_2$ be a closed, convex, pointed cone and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper. If $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$, then $F$ cannot have both an affine $K$-majorant on its domain and an affine $K$-minorant.*

*Proof.* Assume, by contradiction, that there exist $T, L : \mathbb{E}_1 \to \mathbb{E}_2$ be linear and $u, w \in \mathbb{E}_2$ be such that

$$T(x) + u \geqslant_K F(x) \geqslant_K L(x) + w, \quad \forall x \in \text{dom}\, F.$$

We thus find that $\text{gph}\, F \subset \{(x, y) \in \text{dom}\, F \times \mathbb{E}_2 \mid T(x) - y \in -u + K,\ y - L(x) \in w + K\} := C$. As $K$ is closed and convex, so is $C$, and we hence deduce that $\overline{\text{conv}}\,(\text{gph}\, F) \subset C$. Now pick $x \in \text{dom}\, F$ and $v \in K \setminus \{0\}$. Then $(x, F(x) + tv) \in K$-epi $F$ for all $t > 0$. Since $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F) \subset C$, it follows that

$$T(x) - F(x) - tv \in -u + K \quad \text{and} \quad F(x) + tv - L(x) \in w + K \quad \forall t > 0.$$

Dividing by $t$ and letting $t \to +\infty$ we get $v \in K \cap (-K) = \{0\}$, which contradicts the choice of $v$, and therefore proves the statement. □

It is well known that a proper, convex function posseses an affine ($\mathbb{R}_+$-)minorant [1, Theorem 9.20]. In the vector-valued setting, we can fall back on this result to get affine $K$-minorants for proper $K$-convex functions when $K$ is a particular polyhedral cone.

**Proposition 41.** *Let $K = \{x \in \mathbb{E}_2 \mid \langle b_i, x\rangle \geq 0 \quad \forall i = 1, \dots, m\}$ with $b_1, \dots, b_m$ linearly independent. If $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ is $K$-convex and proper, then $F$ has an affine $K$-minorant.*

*Proof.* It holds that $-K^\circ = \text{cone}\,\{b_1, \dots, b_m\}$, see e.g. [19, Lemma 6.45]. This cone is pointed by linear independence of $\{b_1, \dots, b_m\}$, hence $-K$ has nonempty interior, see [19, Exercise 6.22]. Now, in view of Proposition 14 a), for all $i = 1, \dots, m$, the functions $\langle b_i, F\rangle : \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}$ are proper, convex and hence, see e.g. [1, Theorem 9.20], there exist $(c_i, \delta_i) \in \mathbb{E}_1 \times \mathbb{R}$ $(i = 1, \dots, m)$ such that

$$\langle b_i, F(x)\rangle \geq \langle c_i, x\rangle + \delta_i \quad \forall x \in \mathbb{E}_1, i = 1, \dots, m. \tag{12}$$

Now, let $A : \mathbb{E}_1 \to \mathbb{E}_2$ be linear such that $A(b_i) = c_i$ for all $i = 1, \dots, m$ and let $w \in -\text{int}\, K$. Then $\langle w, b_i\rangle < 0$ for all $i = 1, \dots, m$, cf. [19, Exercise 6.22]. By positive homogeneity (and since $\text{int}\, K$ is a pre-cone), there hence exists $\bar{w} \in -\text{int}\, K$ with $\langle \bar{w}, b_i\rangle < \delta_i$ for all $i = 1, \dots, m$. Finally, with $\bar{L} := A^*$ it hence follows

$$\langle b_i, F(x)\rangle \geq \langle c_i, x\rangle + \delta_i \geq \langle b_i, \bar{L}(x)\rangle + \langle \bar{w}, b_i\rangle \quad \forall x \in \mathbb{E}_1, i = 1, \dots, m.$$

Therefore, for all $x \in \text{dom}\, F$, we have $F(x) - (\bar{L}(x) + \bar{w}) \in K$, and $x \mapsto \bar{L}(x) + \bar{w}$ is the desired affine $K$-minorant. □

## 4.3 Sufficient conditions

In this subsection we are primarily concerned with sufficient conditions. We start with some considerations in the scalar case.

**Lemma 42.** *Let $f \in \Gamma(\mathbb{E})$. Then the following hold:*

*a)* $\overline{\text{conv}}\,(\text{gph}\,\text{cl}\, f) = \overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)})$.

*b)* *$\phi : \mathbb{E} \to \mathbb{R}$ is an affine majorant of* $\text{cl}\, f$ *on* $\text{dom}\,(\text{cl}\, f)$ *if and only if $\phi$ is affine majorant of $f$ on* $\text{ri}\,(\text{dom}\, f)$.

*c)* *If $f$ is* $(\text{dom}\, f)$*-closed (hence $f \in \Gamma_0(\text{dom}\, f)$), then $\phi : \mathbb{E} \to \mathbb{R}$ is an affine majorant of* $\text{cl}\, f$ *on* $\text{dom}\,(\text{cl}\, f)$ *if and only if $\phi$ is affine majorant of $f$ on* $\text{dom}\, f$.

*Proof.* a) As $f(x) = \text{cl}\, f(x)$ for all $x \in \text{ri}\,(\text{dom}\, f)$ ( [18, Theorem 7.4]), we have $\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)} \subset \text{gph}\,\text{cl}\, f$, and hence $\overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)}) \subset \overline{\text{conv}}\,(\text{gph}\, f)$. To prove the converse inclusion let $(x, \text{cl}\, f(x)) \in \text{gph}\,\text{cl}\, f$. Invoking [18, Theorem 7.5] (and [18, Theorem 6.1]), we find a sequence $\{x_k \in \text{ri}\,(\text{dom}\, f)\} \to x$ with $f(x_k) \to \text{cl}\, f(x)$. Therefore, $\text{gph}\,\text{cl}\, f \subset \text{cl}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)}) \subset$

$\overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)})$, and hence, the desired inclusion follows by applying the $\overline{\text{conv}}$-operator on both sides.

b) If $\phi$ is an affine majorant of cl $f$ on dom (cl $f$), then $\phi$ is an affine majorant of cl $f$ on ri (dom $f$) $\subset$ dom (cl $f$), and hence an affine majorant of $f$ on ri (dom $f$), since $f$ and cl $f$ coincide on ri (dom $f$). In turn, if $\phi$ is an affine majorant of $f$ on ri (dom $f$), then for all $x \in$ dom (cl $f$), since ri (dom (cl $f$)) = ri (dom $f$) (see [18, Corollary 7.4.1]), by [18, Theorem 7.5] (and [18, Theorem 6.1]), there exists $\{x_k \in \text{ri}\,(\text{dom}\,(\text{cl}\, f))\} \to x$ with $\lim_k f(x_k) = \text{cl}\, f(x_k)$. However $\phi(x_k) \geqslant f(x_k)$ and $\phi$ is continuous so $\phi(x) \geqslant \text{cl}\, f(x)$, thus $\phi$ is an affine majorant of cl $f$ on dom cl $f$.

c) By Lemma 3, $f(x) = \text{cl}\, f(x)$ for all $x \in \text{dom}\, f$. Therefore, if $\phi(x) \geq \text{cl}\, f(x)$ for all $x \in$ dom (cl $f$) $\supset$ dom $f$, then $\phi(x) \geq f(x)$ for all $x \in \text{dom}\, f$. In turn, if $\phi$ is an affine minorant of $f$ on dom $f \supset$ ri (dom $f$), then b) shows that $\phi$ is an affine minorant of cl $f$ on dom (cl $f$).

□

We record some immediate consequences of the foregoing result.

**Corollary 43.** *Let $f \in \Gamma(\mathbb{E})$. Then the following are equivalent:*

*i)* $\overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)}) = \text{cl}\,(\text{epi}\, f)$*;*

*ii) $f$ has no affine majorant on* ri (dom $f$)*;*

*iii)* $\{0\} \times \mathbb{R}_+ \subset [\overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)})]^\infty$.

*Proof.* Observe that epi (cl $f$) = cl (epi $f$), hence by Lemma 42 a) we have

$$i) \iff \overline{\text{conv}}\,(\text{gph}\,\text{cl}\, f) = \text{epi}\,(\text{cl}\, f). \tag{13}$$

'i)⇔ii)': By Lemma 42 b), we have that ii) is equivalent to saying that cl $f$ has no affine minorant on its domain. Therefore, the desired equivalence follows with (13) from Proposition 38 applied to cl $f \in \Gamma_0(\mathbb{E})$.

'i)⇔iii)': Apply Theorem 34 to cl $f$ and use (13). □

**Corollary 44.** *Let $f \in \Gamma_0(\text{dom}\, f)$. Then the following are equivalent:*

*i)* $\overline{\text{conv}}\,(\text{gph}\, f) = \text{cl}\,(\text{epi}\, f)$*;*

*ii) $f$ has no affine majorants on its domain;*

*iii)* $\{0\} \times \mathbb{R}_+ \subset [\overline{\text{conv}}\,(\text{gph}\, f)]^\infty$.

*Proof.* We observe that $f \in \Gamma(\mathbb{E})$ (by definition of $\Gamma_0(\text{dom}\, f)$), and that $f(x) = \text{cl}\, f(x)$ for all $x \in$ dom $f$, by Lemma 3. In addition, by Lemma 42, we have $\overline{\text{conv}}\,(\text{gph}\, f|_{\text{ri}\,(\text{dom}\, f)}) = \overline{\text{conv}}\,(\text{gph}\,\text{cl}\, f)$. Thus, we have

$$\overline{\text{conv}}\,(\text{gph}\, f) \subset \overline{\text{conv}}\,(\text{gph}\,\text{cl}\, f) = \overline{\text{conv}}\,(\text{gph}\, f|_{\text{ri}\,(\text{dom}\, f)}) \subset \overline{\text{conv}}\,(\text{gph}\, f),$$

and hence $\overline{\text{conv}}\,(\text{gph}\, f) = \overline{\text{conv}}\,(\text{gph}\, f|_{\text{ri}\,(\text{dom}\, f)})$. Consequently

$$i) \iff \overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)}) = \text{cl}\,(\text{epi}\, f),$$

and

$$iii) \iff \{0\} \times \mathbb{R}_+ \subset [\overline{\text{conv}}\,(\text{gph}\, f_{|\text{ri}\,(\text{dom}\, f)})]^\infty.$$

Moreover, with Lemma 42 we find that

$$ii) \iff f \text{ has no affine majorant on ri}\,(\text{dom}\, f).$$

Therefore, the claimed equivalences follow from Corollary 43.

□

We now establish sufficient conditions in the vector-valued case, building on the results provided above. We start with the most general result, and then successively tighten the assumptions to obtain (weaker but) more handy conditions.

**Lemma 45.** *Let $K \subset \mathbb{E}_2$ be a (nontrivial) closed, convex cone with $K \subset -K^\circ$, and let $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper, $K$-closed and $K$-convex. Assume that the following hold:*

*i) There is a nonempty set $L \subset K \cap \text{rge}\, F$ such that $\overline{\text{cone}}\, L = K$;*

*ii) For every $u \in L$, there exists a convex set $C^u \subset \text{dom}\, F$ such that $F(C^u) \subset \mathbb{R}_+ u$;*

*iii) For all $u \in L$ we have $f_u := \langle u, F\rangle + \delta_{C^u} \in \Gamma_0(C^u)$ and it has no affine majorant on its domain.*

*Then* $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$. *In particular,* $K = \hat{K}_F$.

*Proof.* Let $u \in L \setminus \{0\}$, and let us prove that $(0, u) \in [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty$. As the latter is a cone, we may assume w.l.o.g that $\|u\| = 1$. By iii), $f_u \in \Gamma_0(C^u)$, hence Corollary 44 yields that $\{0\} \times \mathbb{R}_+ \subset [\overline{\text{conv}}\,(\text{gph}\, f_u)]^\infty$. As conv (gph $f_u$) is convex, we hence have $(0, t) + \text{conv}\,(\text{gph}\, f_u) \subset \overline{\text{conv}}\,(\text{gph}\, f_u)$ for all $t \geq 0$.

Now, let $(x, r) \in \text{conv}\,(\text{gph}\, f_u)$. Hence, with $\kappa := \dim \mathbb{E} \times \mathbb{R}$, there there exist convex combinations $x = \sum_{i=1}^{\kappa+1} \alpha_i x^i$ and $r = \sum_{u=1}^{\kappa+1} \alpha_i \langle u, F\rangle (x^i)$ with $x^i \in C^u$ $(i = 1, \ldots, \kappa + 1)$. By ii), there exists $\gamma \geqslant 0$ such that $\sum_{i=1}^{\kappa+1} \alpha^i F(x^i) = \gamma u$, and thus $\gamma = \langle u, \gamma u\rangle = r$. Consequently, $(x, ru) \in \text{conv}\,(\text{gph}\, F)$.

Moreover, we have $(x, r + t) \in \overline{\text{conv}}\,(\text{gph}\, f_u)$ for all $t \geq 0$, hence

$$x = \lim_{n\to\infty} \sum_{i=1}^{\kappa+1} \alpha_t^{i,n} x_t^{i,n} \quad \text{and} \quad r + t = \lim_{n\to\infty} \sum_{i=1}^{\kappa+1} \alpha_t^{i,n} \langle u, F\rangle (x_t^{i,n})$$

for certain $x_t^{i,n} \in C^u$ and $\alpha_t^{i,n} \geq 0$ $(i = 1, \ldots, \kappa + 1)$ with $\sum_{i=1}^{\kappa+1} \alpha^{i,n} = 1$ for all $n \in \mathbb{N}$. However, by ii), there exist $\gamma_t^n \geqslant 0$ $(n \in \mathbb{N})$ such that

$$\sum_{i=1}^{\kappa+1} \alpha_t^{i,n} F(x_t^{i,n}) = \gamma_t^n u \quad \forall n \in \mathbb{N}.$$

Thus

$$\gamma_t^n = \left\langle u, \sum_{i=1}^{\kappa+1} \alpha_t^{i,n} F(x_t^{i,n}) \right\rangle = \sum_{i=1}^{\kappa+1} \alpha_t^{i,n} \langle u, F\rangle (x_t^{i,n}) \to r + t.$$

Thus, $(x, ru + tu) = (x, (r + t)u) \in \overline{\text{conv}}\,(\text{gph}\, F)$ for all $t \geqslant 0$. Now, for all $(\bar{x}, \bar{y}) \in \text{conv}\,(\text{gph}\, F)$, and for all $s \geqslant 0$, we have

$$(\bar{x}, \bar{y} + su) = \lim_{\varepsilon\downarrow 0} (1 - \varepsilon)(\bar{x}, \bar{y}) + \varepsilon(x, ru + (s/\varepsilon)u).$$

However, by what was argued above, $(x, ru + (s/\varepsilon)u) \in \overline{\text{conv}}\,(\text{gph}\, F)$ for all $\varepsilon > 0$, and we picked $(\bar{x}, \bar{y}) \in \text{conv}\,(\text{gph}\, F) \subset \overline{\text{conv}}\,(\text{gph}\, F)$. By convexity and closedness , it follows that

$$(\bar{x}, \bar{y} + su) \in \overline{\text{conv}}\,(\text{gph}\, F) \quad \forall (\bar{x}, \bar{y}) \in \text{conv}\,(\text{gph}\, F),\ s \geq 0.$$

Thus $(0, u) \in [\text{conv}\,(\text{gph}\, F)]^\infty$ for all $u \in L \setminus \{0\}$ as desired. For $u = 0$ this is trivially true, hence, altogether we find that $\{0\} \times L \subset [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty$. But as $[\overline{\text{conv}}\,(\text{gph}\, F)]^\infty$ is a closed convex cone, we have then $\overline{\text{cone}}\,(\{0\} \times L) = \{0\} \times K \subset [\overline{\text{conv}}\,(\text{gph}\, F)]^\infty$, cf. i). Thus, by Theorem 34, we have $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$. □

We record an immediate consequence.

**Proposition 46.** *Let $K \neq \{0\}$ be a closed, convex cone such that $K \subset -K^\circ$ and such that $F \colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ is proper, $K$-convex and $K$-closed. Assume that $K = \text{cone}\,(b_1, \ldots, b_N)$ for $b_1, \ldots, b_n \in \text{rge}\, F$, and that, for any $i = 1, \ldots, N$, there exists a nonempty convex set $C^i \subset F^{-1}(\mathbb{R}_+ b_i)$ such that, for all $i = 1, \ldots, N$, the function $f_{b_i} := \langle b_i, F\rangle + \delta_{C^{b_i}}$ is $C^{b_i}$-closed and does not have any affine majorant on $C^{b_i}$. Then $K$-epi $F = \overline{\text{conv}}\,(\text{gph}\, F)$.*

*Proof.* Apply Lemma 45, with $L = \{b_1, \ldots, b_N\}$. □

To wrap up this section we want to provide a simplified version of Lemma 45 with more restrictive, but less arduous assumptions. To this end, we need the following lemma.

**Lemma 47.** *Let $K \subset \mathbb{E}_2$ be a closed, convex cone with $K \subset -K^\circ$, and let $F \colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper, $K$-convex and $K$-closed. Then $\langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $u \in \text{ri}\, K$.*

*Proof.* We observe from [17, Corollary 7.4(ii)] that $\langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $u \in \text{ri}\,(-K^\circ)$. Thus if $\text{ri}\, K \subset \text{ri}\,(-K^\circ)$ there is nothing to prove.

Hence, we only need to consider the case $\text{ri}\, K \not\subset \text{ri}\,(-K^\circ)$. Since we assume that $K \subset -K^\circ$, by the definition of the relative topology, this can only hold, if $\text{aff}\, K \subsetneq \text{aff}\,(K^\circ)$. We note that both of these sets contain 0, and hence are subspaces of $\mathbb{E}_2$. In particular, the orthogonal projection

$p : \mathbb{E}_2 \to \mathbb{E}_2$ onto $\operatorname{aff} K$ (which is ordered by $K$) is a linear self-adjoint operator. We define $G : \mathbb{E}_1 \to (\operatorname{aff} K)^\bullet$ by

$$G(x) := \begin{cases} p(F(x)), & x \in \operatorname{dom} F, \\ +\infty_\bullet, & \text{else.} \end{cases}$$

Note that for all $\alpha \in (0,1)$ and $x, y \in \operatorname{dom} F$, we have $\alpha F(x) + (1-\alpha)F(y) - F(\alpha x + (1-\alpha)y) \in K$. Hence, as $K \subset \operatorname{aff} K$ and by linearity of $p$, for all $\alpha \in (0,1)$ and $x, y \in \operatorname{dom} F = \operatorname{dom} G$, we have

$$\alpha F(x) + (1-\alpha)F(y) - F(\alpha x + (1-\alpha)y) = \alpha G(x) + (1-\alpha)G(y) - G(\alpha x + (1-\alpha)y).$$

Therefore, $G$ is $K$-convex. Moreover, if we denote $D := \operatorname{dom} G$ and $H := F - G : D \to \operatorname{aff}(K)$, then

$$\alpha H(x) + (1-\alpha)H(y) - H(\alpha x + (1-\alpha)y) = 0 \quad \forall x, y \in D,\ \alpha \in (0,1).$$

Hence, $H$ is $\{0\}$-convex, and by Proposition 18, there exists an affine function $\hat{H} : \mathbb{E}_1 \to \operatorname{aff} K$ such that $\hat{H}_{|D} = H$.

Now, let $\{(x_k, z_k) \in K\text{-epi}\, G]\} \to (x, z) \in \mathbb{E}_1 \times \operatorname{aff} K$. Then, for all $k \in \mathbb{N}$, $x_k \in \operatorname{dom} G$, and there exists $v_k \in K$ such that

$$z_k = G(x_k) + v_k = F(x_k) - H(x_k) + v_n = F(x_k) - \hat{H}(x_k) + v_k.$$

As $\hat{H} : \mathbb{E}_1 \to \operatorname{aff} K$ is affine, it is continuous, so $\hat{H}(x_k) \to \hat{H}(x)$, and thus $F(x_k) + v_k \to z + \hat{H}(x)$. Therefore $\{(x_k, F(x_k) + v_k) \in K\text{-epi}\, F\} \to (x, z + \hat{H}(x))$. As $F$ is $K$-closed, we have $(x, z + \hat{H}(x)) \in K\text{-epi}\, F$, thus $x \in \operatorname{dom} F = \operatorname{dom} G$, $\hat{H}(x) = H(x)$, and $z + H(x) - F(x) \in K$, so $z - G(x) \in K$ and $(x, z) \in K\text{-epi}\, G$. This proves that $G$ is $K$-closed.

Let $K'$ be the dual cone of $K$ in $\operatorname{aff} K$. As $K \subset -K^\circ$ by assumption, we consequently obtain $K \subset -K^\circ \cap \operatorname{aff} K = K' \subset \operatorname{aff} K$. Hence, $\operatorname{ri} K \subset \operatorname{ri} K'$. Moreover, as $G : \mathbb{E}_1 \to (\operatorname{aff} K)^\bullet$ is proper, $K$-closed and $K$-convex, by [17, Corollary 7.4(ii)], we have $\langle u, G\rangle \in \Gamma_0(\mathbb{E}_1)$ for all $u \in \operatorname{ri} K'$. But for any $u \in \operatorname{ri} K \subset \operatorname{aff} K$, as $p$ is self-adjoint, we have $\langle u, G\rangle = \langle u, p(F)\rangle = \langle u, F\rangle$. Thus, for any $u \in \operatorname{ri} K$ we have $\langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)$. □

**Proposition 48.** *Let $K \subset \mathbb{E}_2$ be a proper, closed, convex cone such that $K \subset -K^\circ$ and let $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be proper, $K$-convex and $K$-closed with $\operatorname{ri} K \subset \operatorname{rge} F$. Moreover, assume that, for any $u \in \operatorname{ri} K$, there exists a nonempty convex set $C^u \subset F^{-1}(\mathbb{R}_+ u)$ such that $f_u := \langle u, F\rangle + \delta_{C^u}$ does not have any affine majorant on its domain. Then, $K\text{-epi}\, F = \overline{\operatorname{conv}}\,(\operatorname{gph} F)$.*

*Proof.* By Lemma 47, for all $u \in \operatorname{ri} K$, we have $\langle u, F\rangle \in \Gamma_0(\mathbb{E}_1)$, hence $f_u \in \Gamma_0(C^u)$. Applying Lemma 45 with $L = \operatorname{ri} K$ yields the desired result. □

### 4.4 Examples

In this section we put our findings from the previous sections to the test on various examples of $K$-convex functions. Throughout, we equip the matrix space $\mathbb{R}^{n\times m}$ with the *Frobenius* inner product $\langle\cdot,\cdot\rangle : \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \to \mathbb{R}$, $\langle X, Y\rangle = \operatorname{tr}(X^T Y)$. In particular, on the space of symmetric matrices $\mathbb{S}^n$, the transposition is superfluous.

#### 4.4.1 $F : X \mapsto \frac{1}{2} X X^T$

We consider the function

$$F : \mathbb{R}^{n\times m} \to \mathbb{S}^n, \quad F(X) = \frac{1}{2} X X^T. \tag{14}$$

It plays a central role in study of the *matrix-fractional* [7–9] and *variational Gram functions* [10,13].

**Proposition 49.** *Let $F$ be given by* (14). *Then the following hold:*

*a)* $\hat{K}_F = K_F = \mathbb{S}^n_+ = \operatorname{conv}(\operatorname{rge} F)$. *In case where $m = n$, the convex hull is superfluous.*

*b)* *$F$ is $\mathbb{S}^n_+$-closed and -convex.*

*c)* $\overline{\operatorname{conv}}\,(\operatorname{gph} F) = \mathbb{S}^n_+\text{-epi}\, F$.

*Proof.* a) We know from [10, Lemma 8] that $K_F = \mathbb{S}^n_+$. But as $F$ is continuous and $K_F$ is closed, we have that $F$ is $K_F$-closed, and hence $K_F = \hat{K}_F$, which shows the first identity. For the third, observe that, clearly $\mathbb{S}^n_+ \supset \operatorname{conv}(\operatorname{rge} F)$. On the other hand for $V \in \mathbb{S}^n_+$, there exists $L \in \mathbb{R}^{n\times n}$ such that $\frac{1}{2}LL^T = V$. This already shows that $\mathbb{S}^n_+ = \operatorname{rge} F$ if $m = n$. If not, we denote the columns of $L$ by $\ell_1, \dots, \ell_n$ and set $x_i := \sqrt{n}\ell_i$ for all $i = 1, \dots, n$, and $X_i := [x_i, 0, \dots, 0] \in \mathbb{R}^{n\times m}$. Then

$$V = \frac{1}{2}\sum_{i=1}^n \ell_i\ell_i^T = \frac{1}{2}\sum_{i=1}^n \left(\frac{x_i}{\sqrt{n}}\right)\left(\frac{x_i}{\sqrt{n}}\right)^T = \sum_{i=1}^n \frac{1}{n}F(X_i) \in \operatorname{conv}(\operatorname{rge} F),$$

which gives the desired inclusion.

b) Follows from a).

c) We prove that $\{0\} \times \mathbb{S}^n_+ \subset \operatorname{conv}(\operatorname{gph} F)$ which then gives the desired result via b) and Corollary 34. To this end, let $(0, V) \in \{0\} \times \mathbb{S}^n_+$. Hence, by a), there exist $\alpha_1, \dots, \alpha_r \geqslant 0$ and $X_1, \dots X_r \in \mathbb{R}^{n\times m}$ such that $\sum_{i=1}^r \alpha_i = 1$ and $V = \sum_{i=1}^r \alpha_i F(X_i)$. However, $F(-X_i) = F(X_i)$. Hence, we also have $V = \sum_{i=1}^r \frac{\alpha_i}{2}F(X_i) + \frac{\alpha_i}{2}F(-X_i)$. As $\sum_{i=1}^r \frac{\alpha_i}{2}X_i + \frac{\alpha_i}{2}(-X_i) = 0$, we then have $(0, V) \in \operatorname{conv}(\operatorname{gph} F)$. □

### 4.4.2 The squared matrix mapping

We consider the map

$$F : \mathbb{S}^n \to \mathbb{S}^n, \quad F(X) = X^2. \tag{15}$$

**Proposition 50.** *Let $F$ be given by* (15). *Then the following hold:*

*a)* $\hat{K}_F = K_F = \mathbb{S}^n_+ = \operatorname{rge} F$.

*b)* $F$ *is* $\mathbb{S}^n_+$*-closed and -convex.*

*c)* $\overline{\operatorname{conv}}(\operatorname{gph} F) = \mathbb{S}^n_+\text{-epi}\, F$.

*Proof.* a) Using Proposition 24 b) we know that $-K_F^\circ = \{V \in \mathbb{S}^n \mid \langle V, F\rangle \text{ convex}\}$. Now for any $V \in \mathbb{S}^n$, we have $\nabla\langle V, F\rangle(X) = VX + XV$. Therefore, for $X, Y \in \mathbb{S}^n$, we find that

$$\langle V, F\rangle(X) - \langle V, F\rangle(Y) + \langle\nabla\langle V, F\rangle(X), Y - X\rangle = -\operatorname{tr}((X-Y)V(X-Y)).$$

For $\langle V, F\rangle$ to be convex, by the gradient inequality, it is therefore necessary and sufficient that

$$\operatorname{tr}((X-Y)V(X-Y)) \geq 0 \quad \forall X, Y \in \mathbb{S}^n,$$

which is equivalent to saying that $V \succeq 0$. Therefore $-K_F^\circ = \mathbb{S}^n_+$, and by bipolarity, we obtain $K_F = \mathbb{S}^n_+$. Since $F$ is continuous, we have $K_F = \hat{K}_F$, and the fact that $\operatorname{rge} F = \mathbb{S}^n_+$ is obvious.

b) Follows from a).

c) Use the same reasoning as in the proof of Proposition 49 c). □

### 4.4.3 The inverse matrix mapping

We consider the map

$$F : \mathbb{S}^n_{++} \to \mathbb{S}^n, \quad F(X) = X^{-1}. \tag{16}$$

**Proposition 51.** *Let $F$ be given by* (16). *Then the following hold:*

*a)* $\hat{K}_F = K_F = \mathbb{S}^n_+$.

*b)* $F$ *is* $\mathbb{S}^n_+$*-convex and -closed.*

*c)* $\overline{\operatorname{conv}}(\operatorname{gph} F) = \mathbb{S}^n_+\text{-epi}\, F$.

*Proof.* a) By Proposition 24 b), we know that $-K_F^\circ = \{V \in \mathbb{S}^n \mid \langle V, F\rangle \text{ convex}\}$. Now let $V \in \mathbb{S}^n$ and observe that $\nabla\langle V, F\rangle(X) = -X^{-1}VX^{-1}$ for all $X \succ 0$. Therefore, for all $X, Y \succ 0$

$$\begin{aligned} &\langle V, F\rangle(X) - \langle V, F\rangle(Y) + \langle\nabla\langle V, F\rangle(X), Y - X\rangle \\ &= \operatorname{tr}(VX^{-1}) - \operatorname{tr}(VY^{-1}) - \operatorname{tr}(X^{-1}VX^{-1}(Y-X)) \\ &= -\operatorname{tr}(V(Y^{-1} + X^{-1}YX^{-1})) \end{aligned}$$

For $\langle V, F\rangle$ to be convex, by the gradient inequality, it is therefore necessary and sufficient that

$$\operatorname{tr}(V(Y^{-1} + X^{-1}YX^{-1}) \geq 0 \quad \forall X, Y \succ 0.$$

Since the trace of a product of two positive semidefinite matrices is nonnegative, it follows that $V \succeq 0$ will be sufficient. On the other hand, let $V \not\succeq 0$ and let $V = \sum_{i=1}^n \lambda_i q_i q_i^T$ be the spectral decomposition with $q_1, \dots, q_n$ an orthonormal basis of $\mathbb{R}^n$ and $\lambda_n < 0$. Now define

$$X_t := \left( \sum_{i=1}^{n-1} q_i q_i^T + t q_n q_n^T + \frac{1}{t} I \right)^{-1} \succ 0 \quad \forall t > 0.$$

Put $Y_t := X_t \succ 0$. Then, since $q_1, \dots, q_n$ are orthonormal, we have

$$\begin{aligned} \operatorname{tr}\left(V(Y_t^{-1} + X_t^{-1} Y_t X_t^{-1}\right) &= 2\operatorname{tr}\left( V \left( \sum_{i=1}^{n-1} q_i q_i^T + t q_n q_n^T + \frac{1}{t} I \right) \right) \\ &= 2 \left[ \operatorname{tr}\left( \sum_{i=1}^{n-1} \lambda_i q_i q_i^T \right) + \operatorname{tr}\left(t \lambda_n q_n q_n^T\right) + \operatorname{tr}\left( \frac{1}{t} V \right) \right] \\ &= 2 \left[ \sum_{i=1}^{n-1} \lambda_i + t\lambda_n + \frac{\operatorname{tr}(V)}{t} \right] \\ &\overset{t\to\infty}{\to} -\infty. \end{aligned}$$

Therefore $-K_F^\circ = \mathbb{S}^n_+$, and by bipolarity, we obtain $K_F = \mathbb{S}^n_+$. In view of Corollary 30 it hence suffices to show that $F$ is $\mathbb{S}^n_+$-closed: To this end, let $\{(X_k, Y_k) \in \mathbb{S}^n_+\text{-epi}\, F\} \to (X, Y)$. In particular, $Y_k = X_k^{-1} + S_k$ for $S_k \succeq 0$. It hence is enough to show that $X \succ 0$. If this were not the case, then the smallest eigenvalue $\lambda_k$ of $X_k$ goes to zero, and hence $1/\lambda_k$, the largest eigenvalue of $X_k^{-1}$, goes to $+\infty$. This cannot be compensated for by $S_k \succeq 0$, which then contradicts that $\{X_k^{-1} + S_k\}$ is convergent, hence bounded.

b) Follows from a).

c) Firs note that $\operatorname{rge} F = \mathbb{S}^n_{++} = \operatorname{ri} \mathbb{S}^n_+$ and that $\mathbb{S}^n_+$ is self-dual, i.e. $\mathbb{S}^n_+ = -(\mathbb{S}^n_+)^\circ$. Moreover, for every $U \in \operatorname{ri} \mathbb{S}^n_+$, we have that $C^U = F^{-1}(\mathbb{R}_+ U) = \mathbb{R}_{++} U^{-1}$ is convex and nonempty. The desired statement will follow from Proposition 48 once we show that $\langle U, F \rangle$ has no affine majorant on $C^U$. To this end, let $V_t = tU^{-1} \in C^U$ for $t > 0$. Then $\langle U, F \rangle (V_t) = \frac{\|U\|^2}{t}$. Since $0 < t \mapsto 1/t$ has no affine majorant on $\mathbb{R}_{++}$, then $\langle U, F \rangle$ cannot have an affine majorant on $C^U$. □

### 4.4.4 Entry-wise convex functions

It is well known [10] that a component-wise convex function $\mathbb{E}_1 \to \mathbb{R}^n$ is $\mathbb{R}_+$-convex. This can be slightly generalized.

**Proposition 52.** *Let $\{b_1, \dots, b_n\} \subset \mathbb{E}_2$ and let $f_i : \mathbb{E}_1 \to \mathbb{R}$ be convex for all $i = 1, \dots, n$. Define $F : \mathbb{E}_1 \to \mathbb{E}_2$ by $F(x) = \sum_{i=1}^n f_i(x) b_i$ and let $K := \operatorname{cone}\{b_1, \dots, b_n\}$. Then the following hold:*

*a) $F$ is $K$-convex and $K$-closed.*

*b) We have $K \subset -K^\circ$ if and only if $\langle b_i, b_j \rangle \geq 0$ for all $i, j = 1, \dots, n$.*

*c) Suppose that, for all $i = 1, \dots, n$, we have $C_i := \bigcap_{i \neq j} \operatorname{argmin} f_j \neq \emptyset$ and that $f_i$ has no affine majorant on $C_i$. Then $K\text{-epi}\, F = \overline{\operatorname{conv}}\,(\operatorname{gph} F)$.*

*Proof.* a) Observe that $-K^\circ = \{ y \mid \langle b_i, y \rangle \geq 0 \ (i = 1, \dots, n) \}$. Therefore for all $z \in -K^\circ$ we have $\langle z, F \rangle = \sum_{i=1}^n \langle b_i, z \rangle f_i \in \Gamma_0(\mathbb{E}_1)$. Hence Propostion 14 yields the desired statement.

b) Clear as $K^{\circ\circ} = K$.

c) By assumption, $\langle F, b_i \rangle + \delta_{C^i} \in \Gamma_0(C^i)$ and $C^i \subset F^{-1}(\mathbb{R}_+\{b_i\})$. The claim hence follows from Proposition 46. □

### 4.4.5 The spectral function

The *spectral function* [10,15,17] is the map $\lambda \colon \mathbb{S}^n \to \mathbb{R}^n$, $\lambda(A) = [\lambda_1(A), \dots \lambda_n(A)]^T$ where $\lambda_1(A) \geq \dots \geq \lambda_n(A)$ are the ordered eigenvalues of $A$ (with multiplicity). Define the cone

$$K_n = \left\{ v \in \mathbb{R}^n \mid \sum_{i=1}^k v_i \geq 0, k = 1, \dots, n-1, \sum_{i=1}^n v_i = 0 \right\}. \tag{17}$$

The following result clarifies the convexity properties of $\lambda$ w.r.t $K_n$ and shows, based on Proposition 46 and Corollary 37, respectively, that the question whether $K_n\text{-epi}\, \lambda = \overline{\operatorname{conv}}\,(\operatorname{gph} \lambda)$ depends on $n$.

**Proposition 53** (Spectral map)**.** *Let $K_n$ be given by* (17)*. Then the following hold:*

*a)* $K_n$ *is closed, convex and pointed with* $K_n^\circ = \{w \in \mathbb{R}^n \mid w_1 \geq \cdots \geq w_n\}$.

*b)* $\lambda$ *is* $K_n$*-convex and* $K_n$*-closed.*

*c) The following are equivalent:*

*i)* $K_n \subset -K_n^\circ$;

*ii)* $n \leq 2$;

*iii)* $K_n\text{-epi}\,\lambda = \overline{\text{conv}}\,(\text{gph}\,\lambda)$.

*Proof.* a) The properties of $K$ are straightforward. The formula for $K_n^\circ$ can be found in e.g. [10, 15, 17].

b) See e.g. [10, 15, 17] for $K_n$-convexity. The $K_n$-closedness follows because $\lambda$ is continuous on $\mathbb{S}^n$ and $K_n$ is closed.

c) Consider the following implications:

'i)⇒ii)': For $n > 2$ we have $[0, \ldots, 0, 1, -1]^T \in K_n \setminus (-K_n^\circ)$, see a).

'ii)⇒iii)': For $n = 1$ there's nothing to prove. For $n = 2$ set $b_1 := [1; -1]^T$ so that $K_n = \text{cone}\,\{b_1\}$ and define $C^{b_1} := \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} \mid \alpha \in \mathbb{R} \right\}$ which is a subspace, hence nonempty and closed, and convex. Then $C^{b_1} \subset \lambda^{-1}(\mathbb{R}_+ b_1)$ and we have $\langle b_1, \lambda \rangle \left( \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} \right) = 2|\alpha|$ for all $\alpha \in \mathbb{R}$. Therefore $\langle b_1, \lambda \rangle + \delta_{C^{b_1}} \in \Gamma_0(\mathbb{S}^n)$ and has no affine majorant on its domain $C^{b_1}$. Therefore Proposition 46 yields the desired implication.

'iii)⇒i)': Corollary 37. □

# 5 Convex-convex composite functions

We start this section with the definition of $K$-increasing functions.

**Definition 54** ($K$-increasing functions)**.** *Let $K \subset \mathbb{E}$ be a cone. The function $g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is said to be* $K$-increasing *if*

$$y \geq_K x \quad \Longrightarrow \quad g(y) \geq g(x) \quad \forall x, y \in \mathbb{E}.$$

It is well known and explored extensively in the literature [4–6,10] that, given $K \subset \mathbb{E}$, a $K$-increasing function $g \in \Gamma(\mathbb{E}_2)$ and a $K$-convex function $F : D \subset \mathbb{E}_1 \to \mathbb{E}_2$, the composition

$$g \circ F : \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}, \quad (g \circ F)(x) := \begin{cases} g(F(x)), & x \in D, \\ +\infty, & \text{else} \end{cases} \tag{18}$$

is convex (and proper if and only if $F(\text{dom}\,F) \cap \text{dom}\,g \neq \emptyset$). One of the questions we address in this section is the following: given $g \in \Gamma(\mathbb{E}_2)$ and $F : D \subset \mathbb{E}_1 \to \mathbb{E}_2$ such that $g \circ F$ is convex, under which conditions does there exist a (closed) cone $K$ such that $F$ is $K$-convex and $g$ is $K$-increasing?

## 5.1 The horizon cone of a closed, proper, convex function

For a proper function $f : \mathbb{E} \to \overline{\mathbb{R}}$, its *horizon function* $f^\infty : \mathbb{E} \to \overline{\mathbb{R}}$ is defined via $\text{epi}\,f^\infty = (\text{epi}\,f)^\infty$. The *horizon cone* $\text{hzn}\,f$ of $f$ is the level set $\text{hzn}\,f := \{x \in \mathbb{E} \mid f^\infty(x) \leq 0\}$. For $f \in \Gamma_0(\mathbb{E})$ the horizon function and horizon cone of $f$ coincide with the respective *recession* objects [18, Chapter 8]. We summarize some fundamental properties of the horizon cone of a closed, proper, convex function.

**Proposition 55.** *Let $g \in \Gamma_0(\mathbb{E})$. Then the following hold:*

*a)* $g^\infty$ *is closed, proper, convex and positively homogenous.*

*b) We have*

$$g^\infty(u) = \sup_{t > 0} \frac{g(x + tu) - g(x)}{t} \quad \forall x \in \text{dom}\,g.$$

*In particular, we have*

$$u \in \text{hzn}\,g \iff g(x + tu) \leqslant g(x) \quad \forall x \in \text{dom}\,g, t > 0.$$

*c)* $\operatorname{hzn} g$ *is a closed convex cone.*

*d)* $g$ *is* $(-\operatorname{hzn} f)$*-increasing.*

*e)* $K$ *is a cone with respect to which* $f$ *is increasing if and only if* $K \subset -\operatorname{hzn} f$*.*

*Proof.* a),b) See [19, Theorem 3.12].
c) From a) and the definition of $\operatorname{hzn} f$.
d) See [10, Lemma 7] or [16, Corollary 3.1].
e) See [16, Proposition 3.2]. □

The next example shows that the convexity in part b) is essential, which also shows that the horizon function is not the recession function (see [16]) without convexity.

**Example 56.** *Consider* $f : \mathbb{R} \to \mathbb{R}$ *given by*

$$f(x) = \begin{cases} 1+x, & x < -1, \\ 0, & x \in [-1,1], \\ 1-x, & x > 1. \end{cases}$$

*Then* $f$ *is continuous (hence proper and lsc), but not convex, and it holds that* $f^\infty(u) = -|u|$*. Moreover, for any* $u \in \mathbb{R}$*,* $\sup_{x\in\mathbb{R},t>0} \frac{f(x+tu)-f(x)}{t} = |u|$*. Thus* $f^\infty(u) \neq \sup_{x\in\mathbb{R},t>0} \frac{f(x+tu)-f(x)}{t}$*.*

## 5.2 The $K$-increasing case

The next proposition characterizes the situation where there exists a cone with respect which $F : D \to \mathbb{E}_2$ is convex and $g \in \Gamma_0(\mathbb{E}_2)$ is increasing. At this, the cone $K_F$, the smallest closed cone with respect to which $F$ is convex, comes in to play, which ties our study here to our results from Section 3.

**Proposition 57.** *Let* $g \in \Gamma_0(\mathbb{E}_2)$ *and* $F : D \to \mathbb{E}_2^\bullet$ *with* $D \subset \mathbb{E}_1$ *convex such that* $g \circ F$ *is proper. Then the following statements are equivalent.*

*i) There exists a cone* $K$ *such that* $g$ *is* $K$*-increasing and* $F$ *is* $K$*-convex;*

*ii)* $g$ *is* $K_F$*-increasing;*

*iii)* $K_F \subset -\operatorname{hzn} g$*;*

*iv)* $(\operatorname{hzn} g)^\circ \subset -K_F^\circ$*.*

*Proof.* We only (need to) show that i), ii) and iii) are equivalent. The equivalence of iii) and iv) follows from (bi)polarization and the fact that both cones in play are closed and convex.
i)⇒ii) : Let $K \subset \mathbb{E}_2$ such that $F$ is $K$-convex and $g$ is $K$-increasing. In particular, by Proposition 55 e), we have $K \subset -\operatorname{hzn} g$, and thus $F$ is $(-\operatorname{hzn} g)$-convex and $g$ is $(-\operatorname{hzn} g)$-increasing, by Proposition 55 d). As $(-\operatorname{hzn} g)$ is closed and convex, see Proposition 55 c), by definition of $K_F$ we have $K_F \subset -\operatorname{hzn} g$. By Proposition 55 e) we find that $g$ is $K_F$-increasing.
ii)⇒iii): From Proposition 55 e).
iii)⇒i): Let $K := K_F$. Clearly, $F$ is $K$-convex and, by Proposition 55 e), $g$ is $K$-increasing. □

Proposition 57 yields the following concrete example.

**Example 58.** *Consider* $g\colon (x,y) \in \mathbb{R}^2 \mapsto |x|$ *and* $F\colon (x,y) \in \mathbb{R}^2 \mapsto (x^2, y)$*. Hence* $g \in \Gamma_0(\mathbb{E}_2)$ *and* $g \circ F\colon (x,y) \mapsto x^2 \in \Gamma_0(\mathbb{E}_1)$*. Using Proposition 24 b), we find that* $K_F = \mathbb{R}_+ \times \{0\}$*. However,* $(-1,0) \leqslant_{K_F} (0,0)$ *and* $g((-1,0)) = 1 > 0 = g((0,0))$*. Thus,* $g$ *is not* $K_F$*-increasing, and consequently, by Proposition 57, there is no closed cone* $K$ *such that* $F$ *is* $K$*-convex and* $g$ *is* $K$*-increasing.*

We close out by remarking that, if $\phi\colon \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}$ is proper convex, there always exists a decomposition $\phi = g \circ F$ with $g \in \Gamma_0(\mathbb{E}_2)$, $F\colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ proper with $g$ $K_F$-increasing: for instance, define $F(x) := (\Phi(x), 0, \dots, 0) \in \mathbb{E}_2$ with $\operatorname{dom} F = \operatorname{dom} \Phi$, and $g(y) = y_1$. Then, $g \in \Gamma_0(\mathbb{E}_2)$, $F$ is $(\mathbb{R}_+ \times \{0\} \times \cdots \times \{0\})$-convex and $g$ is $(\mathbb{R}_+ \times \{0\} \times \cdots \times \{0\})$-increasing.

### 5.3 Beyond $K$-monotonicity

It was already observed by Pennanen [17] and Burke et al. [10] that, in order to obtain convexity of the composition $g \circ F$ in (18), the assumption that $g$ be $K$-increasing can be weakened to

$$g(F(x)) \leq g(y) \quad \forall (x, y) \in K\text{-epi}\, F, \tag{19}$$

in which case we call $g$ *increasing w.r.t* $K$-epi $F$. Concretely, the following result holds.

**Proposition 59** ( [10, Proposition 1])**.** *Let $K \subset \mathbb{E}_2$ be a convex cone such that that $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ is $K$-convex and such that $g \in \Gamma(\mathbb{E}_1)$ is increasing w.r.t $K$-epi $F$ in the sense of (19). Then $g \circ F$ is convex.*

The next proposition gives a characterization of the situation where there exists a closed convex cone $K$ such that $g \in \Gamma_0(\mathbb{E}_1)$ is increasing w.r.t $K$-epi $F$ and $F$ is $K$-convex.

**Proposition 60.** *Let $g \in \Gamma_0(\mathbb{E}_2)$ and $F : D \to \mathbb{E}_2$ for $D \subset \mathbb{E}_1$ (nonempty convex) such that $g \circ F$ is proper. Then there exists a closed (convex) cone $K$ such that $g$ is increasing w.r.t $K$-epi $F$ (in the sense of (19)) and $F$ is $K$-convex if and only if $g$ is increasing w.r.t $K_F$-epi $F$.*

*Proof.* Suppose that $g$ is increasing w.r.t $K_F$-epi $F$ and set $K := K_F$. Then $K$ is closed and convex and $F$ is $K$-convex and $g$ is increasing w.r.t $K$-epi $F$ (by assumption).

Conversely, suppose now that there exists a closed (convex) cone $K$ such that $g$ is increasing w.r.t $K$-epi $F$ and $F$ is $K$-convex. As $K$ is closed and $F$ is $K$-convex, we have$K_F \subset K$, by definition of $K_F$. Therefore $K_F\text{-epi}\, F = \text{gph}\, F + \{0\} \times K_F \subset \text{gph}\, F + \{0\} \times K = K_F\text{-epi}\, F$, and thus $g$ is increasing w.r.t $K_F$-epi $F$.

□

It turns out that, in Proposition 59, the assumption that $g$ be increasing w.r.t. to $K$-epi $F$ can even be further weakened substituting conv (gph $F$) for $K$-epi $F$ by which, again, ties our considerations here to our previous study.

**Proposition 61.** *Let $D \subset \mathbb{E}_1$ be (nonempty) convex, $F : D \to \mathbb{E}_2$, and let $g \in \Gamma(\mathbb{E}_2)$ be increasing w.r.t.* conv (gph $F$)*, i.e.*

$$g(F(x)) \leq g(y) \quad \forall (x, y) \in \text{conv}\, (\text{gph}\, F). \tag{20}$$

*Then $g \circ F$ is convex.*

*Proof.* Let $x, y \in \text{dom}\, (g \circ F)$ and $\alpha \in (0, 1)$. Then $(\alpha x + (1 - \alpha)y, \alpha F(x) + (1 - \alpha)F(y)) \in \text{conv}\, (\text{gph}\, F)$. By (20),and the convexity of $g$ we find

$$g(F(\alpha x + (1 - \alpha)y)) \leq g(\alpha F(x) + (1 - \alpha)F(y)) \leq \alpha g(F(x)) + (1 - \alpha)g(F(y)).$$

Hence $g \circ F$ is convex. □


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# References


[1] H.H. Bauschke and P.L. Combettes: *Convex analysis and Monotone Operator Theory in Hilbert Spaces.* CMS Books in Mathematics, Springer, New York, 2nd Edition, 2017.

[2] S. Boyd and L. Vandenbergh: *Convex Optimization.* Cambridge University Press, 2004.

[3] J. M. Borwein: *Optimization with respect to Partial Orderings.* Ph.D. Thesis, University of Oxford, 1974.

[4] R.I. Boţ: *Conjugate Duality in Convex Optimization. Lecture Notes in Economics and Mathematical Systems, 637.* Springer, Berlin, 2010.

[5] R.I. Boţ, I.B. Hodrea, and G. Wanka: *Farkas-type results for inequality systems with composed convex functions via conjugate duality.* Journal of Mathematical Analysis and Applications 322(1), 2006, pp. 316–328.

[6] R.I. Boţ, S.-M. Grad, and G. Wanka: *New constraint qualification and conjugate duality for composed convex optimization problems.* Journal of Optimization Theory and Applications 135(2), 2007, pp. 241–255.

[7] J. V. Burke and T. Hoheisel: *Matrix support functionals for inverse problems, regularization, and learning.* SIAM Journal on Optimization 25(2), 2015, pp. 1135–1159.

[8] J. V. Burke, Y. Gao and T. Hoheisel: *Convex geometry of the generalized matrix-fractional function.* SIAM Journal on Optimization 28(3), 2018, pp. 2189–2200.

[9] J. V. Burke, Y. Gao, and T. Hoheisel: *Variational properties of matrix functions via the generalized matrix-fractional function.* SIAM Journal on Optimization 29(3), 2019, pp. 1958–1987.

[10] J.V. Burke, T. Hoheisel, and Q. Van Nguyen:*A study of convex convex-composite functions via infimal convolution with applications.* Mathematics of Operations Research, 2021, https://doi.org/10.1287/moor.2020.1099.

[11] J.-B. Hiriart-Urruty: *A Note on the Legendre-Fenchel Transform of Convex Composite Functions.* In *Nonsmooth Mechanics and Analysis.* Eds. P. Alart, O. Maisonneuve, and R. T. Rockafellar, Springer, 2006, pp. 35–46.

[12] R.A. Horn and C.R. Johnson: *Topics in Matrix Analysis.* Cambridge University Press, New York, 1991.

[13] A. Jalali, M. Fazel, and L. Xiao: *Variational Gram functions: convex analysis and optimization.* SIAM Journal on Optimization 27(4), 2017, pp. 2634–2661.

[14] A.G. Kusraev and S.S. Kutateladze: *Subdifferentials: theory and applications.* Mathematics and its Applications, 323. Kluwer Academic Publishers Group, Dordrecht, 1995.

[15] A.S. Lewis:*Convex analysis on the Hermitian matrices.* SIAM Journal on Optimization 6(1), 1996, pp. 164–177.

[16] G. Li , S. Li, and M. You:*Recession function and its applications in optimization.* Optimization, 2020, https://doi.org/10.1080/02331934.2020.1786569.

[17] T. Pennanen:*Graph-convex mappings and $K$-convex functions.* Journal of Convex Analysis 6(2), 1999, pp. 235–266.

[18] R.T. Rockafellar: *Convex Analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J. 1970.

[19] R.T. Rockafellar and R.J.-B. Wets: *Variational Analysis.* Grundlehren der Mathematischen Wissenschaften, Vol. 317, Springer-Verlag, Berlin, 1998.