Stability of nonsmooth optimization problems

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Motivation

Consider the optimization problem

$$\min_{x \in \mathbb{R}^n} h(p, x) + \varphi(x) \tag{1}$$

where

- $h : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}$ (locally) smooth and convex in x;
- $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ closed, proper, convex.

$$S(p) := \operatorname*{argmin}_{x \in \mathbb{R}^n} \{h(p, x) + \varphi(x)\} \quad \text{(solution map).}$$

References: Bonnans/Shapiro (general NLP), Bolte et al. (monotone operators), Vaiter et al. (regularized LLS).

Examples

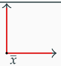
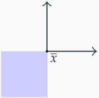
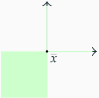
- (prox operator) $p := (\bar{x}, \lambda),\ h(p, x) := \frac{1}{2\lambda}\|x - \bar{x}\|^2$: $S(\bar{x}, \lambda) = P_\lambda \varphi(\bar{x})$.
- (unconstrained LASSO) $p := (A, b, \lambda),\ h(p, x) = \frac{1}{2\lambda}\|Ax - b\|^2,\ \varphi = \|\cdot\|_1$.

By convexity

$$S(p) = \{x \in \mathbb{R}^n \mid 0 \in \nabla_x h(x, p) + \partial\varphi(x)\}.$$

Tailor-made for the implicit function theorems of variational analysis based on graphical differentiation.

Variational analysis: normal cones and graphical differentiation

Name	Definition	Properties	Example
tangent cone	$T_A(\bar{x}) := \operatorname{Lim\,sup}_{t \downarrow 0} \frac{A-\bar{x}}{t}$	closed	
regular normal cone	$\hat{N}_A(\bar{x}) := T_A(\bar{x})^{\circ}$	closed, convex	
limiting normal cone	$N_A(\bar{x}) := \operatorname{Lim\,sup}_{x \to \bar{x}} \hat{N}_A(x)$	closed	

$S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $(\bar{x}, \bar{y}) \in \operatorname{gph} S := \{(x, y) \mid y \in S(x)\}$.

- Graphical derivative (Aubin '81, Benko '21): $DS(\bar{x}|\bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ via

$$v \in DS(\bar{x}|\bar{y})(u) \quad :\Longleftrightarrow \quad (u, v) \in T_{\operatorname{gph} S}(\bar{x}, \bar{y}).$$

- Coderivative (Mordukhovich '80, Ioffe '84): $D^*S(\bar{x}|\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ via

$$v \in D^*S(\bar{x}|\bar{y})(u) \quad :\Longleftrightarrow \quad (v, -u) \in N_{\operatorname{gph} S}(\bar{x}, \bar{y}).$$

Variational analysis: proto-differentiability

Observe that graphical derivative of $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x}, \bar{u}) \in \operatorname{gph} S$ is (by definition)

$$DS(\bar{x} \mid \bar{u})(\bar{w}) = \operatorname*{Lim\,sup}_{t\downarrow 0,\, w\to\bar{w}} \frac{S(\bar{x} + tw) - \bar{u}}{t} \quad \forall\, \bar{w} \in \mathbb{R}^n. \tag{2}$$

Definition (Proto-differentiability (Rockafellar '89))

We call S is *proto-differentiable* at $(\bar{x}, \bar{u}) \in \operatorname{gph} S$ if the following hold:

$$\forall \bar{z} \in DS(\bar{x} \mid \bar{u})(\bar{w}),\ \{t_k\} \downarrow 0\ \exists \{w_k\} \to \bar{w},\ \{z_k\} \to \bar{z} :\ z_k \in \frac{S(\bar{x} + t_k w_k) - \bar{u}}{t_k}\ \forall k \in \mathbb{N}.$$

- Relates to semidifferentiability (Penot) which will yield directional differentiability for our purposes.
- Graphical regularity implies proto-differentiability.
- ∂f is proto-differentiable at $(\bar{x}, \bar{u})$, e.g., if $f = g \circ F$ is *fully amenable*, i.e., g PLQ and $F \in C^2$ such that

$$\ker F'(\bar{x})^* \cap N_{\operatorname{dom} g}(F(\bar{x})) = \{0\} \quad \text{(basic constraint qualification)}$$

- For more (subtle) conditions implying proto-differentiability, see, e.g., Hang and Sarabi (SIOPT 2024).

Variational analysis: directional normal cone and semismoothness*

Directional normal cone of A at $\bar{x}$ in direction $\bar{u}$:

$$N_A(\bar{x}; \bar{u}) := \operatorname*{Lim\,sup}_{u \to \bar{u},\ t \downarrow 0} \hat{N}_A(\bar{x} + tu).$$

- $N(\bar{x}; \bar{u}) = \emptyset$ if $\bar{u} \notin T_A(\bar{x})$;
- $N(\bar{x}; \bar{u}) \subset N_A(\bar{x})$ for all $u \in \mathbb{R}^n$.

Semismoothness* (Gfrerer et al.):

i) $A \subset \mathbb{R}^n$ *semismooth** at $\bar{x} \in A$ $\quad :\Longleftrightarrow \quad \langle x^*, u \rangle = 0 \quad \forall u \in \mathbb{R}^n,\ x^* \in N_A(\bar{x}; u)$.

ii) $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *semismooth** at $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ $\quad :\Longleftrightarrow \quad \operatorname{gph} S$ semismooth* at $(\bar{x}, \bar{y})$.

(Gfrerer and Outrata '19): For $F : D \subset \mathbb{R}^n \to \mathbb{R}^m$ locally Lipschitz at $\bar{x} \in \operatorname{int} D$, the following are equivalent:

- F semismooth (in the sense of Qi and Sun) at $\bar{x}$.
- F semismooth* and directionally differentiable at $\bar{x}$.

The workhorse (Dontchev/Rockafellar, Berk/Brugiapaglia/H.)

Let $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable at $(\bar{p}, \bar{x})$ such that $f(p, \cdot)$ is monotone near $\bar{p}$, let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be maximally monotone at. Define $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ by

$$S(p) = \{x \in \mathbb{R}^n \mid 0 \in f(p, x) + F(x)\}, \quad \forall p \in \mathbb{R}^d.$$

The following hold if $(\bar{p}, \bar{x}) \in \operatorname{gph} S$ is such that

$$\ker \left(D_x f(\bar{p}, \bar{x})^* + D^* F(\bar{x}| - f(\bar{p}, \bar{x})\right) = \{0\} \quad \text{(Mordukhovich criterion).}$$

(a) S is locally Lipschitz at $\bar{p}$ with modulus

$$L \leq \limsup_{p \to \bar{p}} \max_{\|q\| \leq 1} \inf_{w \in DS(p)(q)} \|w\|.$$

(b) If F is *proto-differentiable* at $(\bar{x}, -f(\bar{p}, \bar{x}))$, S is directionally differentiable at $\bar{p}$ with locally Lipschitz directional derivative (for $G(p, x) := f(p, x) + F(x)$) given by

$$S'(\bar{p}; q) = \{w \in \mathbb{R}^n \mid 0 \in DG(\bar{p}, \bar{x}|0)(q, w)\} \quad \forall q \in \mathbb{R}^d.$$

(c) If F is semismooth* and the following implication is satisfied:

$$\left.\begin{array}{rcl} -(v, w) & \in & N_{\operatorname{gph} F}(\bar{x}, -f(\bar{p}, \bar{x})), \\ 0 & = & D_p f(\bar{p}, \bar{x})^* w, \\ v & = & D_x f(\bar{p}, \bar{x})^* w \end{array}\right\} \implies (v, w) = (0, 0),$$

then S is semismooth at $\bar{p}$.

(d) If $S'(\bar{p}; \cdot)$ is linear, then S is differentiable at $\bar{p}$.

The Mordukhovich criterion for regularized linear least-squares

Consider the regularized least-squares problem

$$\min_x \frac{1}{2}\|Ax-b\|^2 + \lambda g(x) \tag{3}$$

for $\lambda > 0$ and g closed, proper, convex.

Let $\bar{x}$ solve (3), i.e. $\bar{u} := \frac{1}{\lambda}A^T(b - A\bar{x}) \in \partial g(\bar{x})$, i.e.

$$0 \in \underbrace{\frac{1}{\lambda}A^*(A\bar{x}-b)}_{=f(A,b,\lambda,\cdot)(\bar{x})} + \underbrace{\partial g}_{F}(\bar{x}).$$

Let $0 \in D_x f(A,b,\lambda,\bar{x})^* w + D^* F(\bar{x}|\bar{u})(w) = \frac{1}{\lambda}A^*Aw + D^*(\partial g)(\bar{x}|\bar{u})(w)$, i.e.

$$-\frac{1}{\lambda}A^*Aw \in D^*(\partial g)(\bar{x}|\bar{u})(w). \tag{4}$$

By ‘positive semidefiniteness’ of $D^*(\partial g)(\bar{x}|\bar{u})$ we have

$$0 \le \langle w, -A^*Aw \rangle = -\|Aw\|^2 \quad \Longleftrightarrow \quad w \in \ker A$$

Inserting into (4) yields

$$0 \in D^*(\partial g)(\bar{x}|\bar{u})(w) \quad \overset{(\partial g)^{-1} = \partial g^*}{\Longleftrightarrow} \quad -w \in D^*(\partial g^*)(\bar{u}|\bar{x})(0).$$

Hence

$$\ker A \bigcap D^*(\partial g^*)(\bar{u}|\bar{x})(0) = \{0\} \quad \Longleftrightarrow \quad \text{Mordukhovich criterion holds} \tag{5}$$

Tangible conditions for the Mordukhovich criterion

Example Let $\bar{x}$ be a solution of the regularized linear least-squares problem

$$\min_x \frac{1}{2}\|Ax - b\|^2 + \lambda g(x), \tag{6}$$

i.e., $\bar{u} := \frac{1}{\lambda}A^*(b - A\bar{x}) \in \partial g(\bar{x})$.

- ($g^* \in C^{1,1}$) *If g^* has locally Lipschitz gradient*[1] *at $\bar{u}$, then*

$$D^*(\partial g^*)(\bar{u}|\bar{x})(0) \subset \partial^C(\nabla g^*)(\bar{u})^* 0 = \{0\}.$$

- (Polyhedral support) *Let $\mathcal{P} = \{x \mid \langle p_i, x\rangle \le \beta_i \ \forall i = 1, \dots, l\}$, and let $g = \sigma_{\mathcal{P}}$ be its support function. Then*

$$D^*(\partial g^*)(\bar{u}|\bar{x})(0) = D^* N_{\mathcal{P}}(\bar{u}|\bar{x})(0) = \operatorname{span}\{p_i \mid i : \langle p_i, \bar{u}\rangle = \beta_i\} = \operatorname{par}\partial g^*(\bar{u}).$$

We define the qualification condition

$$\operatorname{par}\partial g^*(\bar{u}) \cap \ker A = \{0\} \quad (\mathbf{R}).$$

Note: The condition **(R)** is (equivalent to) *generalized LICQ*[2] for the dual problem of (6)

$$\min_{y,t} \frac{\lambda}{2}\|y\|^2 - \langle b, y\rangle + t \quad \text{s.t.} \quad (A^*y, t) \in \operatorname{epi} g^*.$$

[1] See Goebel and Rockafellar (Journal of Convex Analysis, 2008) for a primal characterization.

[2] Or *partial constraint nondegeneracy*

Towards more general results

Proposition (Tran, H./Sarabi, H. '24) Let $\bar{x}$ be a solution of the regularized linear least-squares problem

$$\min_x \frac{1}{2}\|Ax - b\|^2 + \lambda g(x), \quad \lambda > 0 \tag{7}$$

with $\bar{u} = \frac{1}{\lambda}A^*(b - A\bar{x})$. Assume that g is in either of the following classes:

(i) (C^2-cone reducible conjugate) $\operatorname{epi} g^*$ is C^2*-cone reducible*[3]

(ii) (PLQ penalty) $g = \theta_{\mathcal{P},B}$ with

$$\theta_{\mathcal{P},B}(y) = \sup_{z\in\mathcal{P}} \left\{ \langle y, z\rangle - \frac{1}{2}\langle Bz, z\rangle \right\}, \quad B \succeq 0,\ \mathcal{P} \text{ polyhedron.}$$

Let $\bar{x}$ be a solution of (7) such that **(R)** holds. Then the solution map

$$(\hat{A}, \hat{b}, \hat{\lambda}) \mapsto \operatorname*{argmin}_x \frac{1}{2}\|\hat{A}x - \hat{b}\|^2 + \hat{\lambda} g(x)$$

is locally Lipschitz around (A, b, λ).

[3] See Bonnans/Shapiro (2000)

Application: unconstrained LASSO (constraint qualifications)

The unconstrained LASSO[4] for $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, \lambda > 0$ reads

$$\min_x \frac{1}{2}\|Ax - b\|^2 + \lambda\|x\|_1. \tag{8}$$

For a solution $\bar{x}$ of (8) define:

- $I := \{i \in \{1,\dots,n\} \mid \bar{x}_i \neq 0\}$ (support);
- $J := \left\{i \in \{1,\dots,n\} \;\middle|\; |A_i^T(b - A\bar{x})| = \lambda\right\}$ (equicorrelation set).

Note: $\boxed{I \subset J.}$

Qualification conditions

- (Intermediate) $\ker A_J = \{0\}$ ($\Leftrightarrow$ **(R)**) ;
- (Strong) $I = J$ and $\ker A_I = \{0\}$.

(Strong) $\implies$ (Intermediate) $\implies$ $\bar{x}$ is unique solution of (8)

[4] Santosa and Symes (1986), Tibshirani (1996)

Application: unconstrained LASSO (stability)

Apply the main theorem with $f(b,\lambda,x) := \frac{1}{\lambda}A^T(Ax-b), \quad F := \partial\|\cdot\|_1$ such that

$$S(b,\lambda) = \{x \mid 0 \in f(b,\lambda,x) + F(x)\} = \operatorname*{argmin}_{x\in\mathbb{R}^n}\left\{\frac{1}{2}\|Ax-b\|^2 + \lambda\|x\|_1\right\} \quad (\lambda > 0).$$

For $(\bar{b},\bar{\lambda}) \in \mathbb{R}^n \times \mathbb{R}_{++}$ let $\bar{x} \in S(\bar{b},\bar{\lambda})$. Then:

(a) If the intermediate condition holds, S is semismooth at $(\bar{b},\bar{\lambda})$ with Lipschitz modulus

$$L \leq \frac{1}{\sigma_{\min}(A_J)^2}\left(\sigma_{\max}(A_J) + \left\|\frac{A_J^T(A\bar{x}-\bar{b})}{\bar{\lambda}}\right\|\right).$$

Moreover, the directional derivative $S'((\bar{b},\bar{\lambda});(\cdot,\cdot)) : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz and given as follows: for $(q,\alpha) \in \mathbb{R}^m \times \mathbb{R}$ there exists an index set $K = K(q,\alpha)$ with $I \subseteq K \subseteq J$ such that

$$S'((\bar{b},\bar{\lambda});(q,\alpha)) = L_K\left((A_K^TA_K)^{-1}A_K^T\left(q + \frac{\alpha}{\bar{\lambda}}(A\bar{x}-\bar{b})\right), 0\right).$$

(b) If the strong assumptions holds, S is continuously differentiable at $(\bar{b},\bar{\lambda})$ with

$$DS(\bar{b},\bar{\lambda})(q,\alpha) = L_I\left((A_I^TA_I)^{-1}A_I^T\left(q + \frac{\alpha}{\bar{\lambda}}(A\bar{x}-\bar{b})\right), 0\right), \quad \forall (q,\alpha) \in \mathbb{R}^m \times \mathbb{R}.$$

In particular, S is locally Lipschitz with modulus given above with $I = J$.

Application: unconstrained LASSO (tuning parameter sensitivity)

Suppose

$$b = Ax_0 + e:$$

- $n = 200$,
- $A_{ij} \overset{\text{iid}}{\sim} \mathcal{N}(0, 1/m)$,
- $e_i \overset{\text{iid}}{\sim} \mathcal{N}(0, 0.01)$ and
- x_0 s-sparse: $(x_0)_j \overset{\text{iid}}{\sim} \mathcal{N}(m, m)$ $(j \in I)$.

- $x(\lambda) := \operatorname{argmin}_x \left\{ \frac{\|Ax - b\|^2}{2} + \lambda\|x\|_1 \right\}$,
- $\lambda^* := \inf \operatorname{argmin}_{\lambda > 0} \|x(\lambda) - x_0\|$,
- $\bar{x} := x(\lambda^*)$.

Under the strong assumption at $\bar{x}$, $x(\cdot)$ is locally Lipschitz with $L := \frac{\sqrt{|I|}}{\sigma_{\min}(A_I)^2}$.

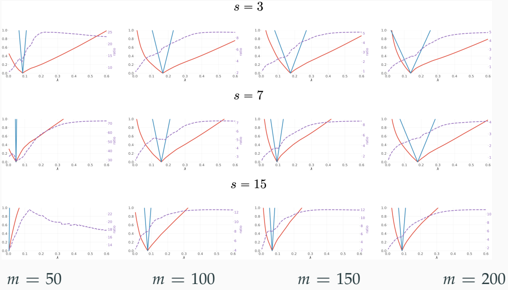


Figure 1: $\|x(\lambda) - \bar{x}\|$, $L|\lambda - \lambda^*|$, $\frac{L|\lambda - \lambda^*|}{\|x(\lambda) - \bar{x}\|}$.

References and Future directions

References

- T. Hoheisel and E. Sarabi: *Stability of regularized least-squares with PLQ regularizers.* Working paper, 2024.
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Future directions

- Expand *qualitative* analysis.
- Clarify the relation between proto-differentiability and semismoothness*.
- Explore implications in bilevel optimization.

Thanks for coming!