Some applications of implicit function theorems from variational analysis

Tim Hoheisel (McGill)

Joint work with Aaron Berk (McGill), Simone Brugiapaglia (Concordia), Michael Friedlander (UBC), Ariel Goodwin (Cornell)

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Motivation

Consider the optimization problem

$$\min_{x \in \mathbb{R}^n} h(p, x) + \varphi(x) \tag{1}$$

where

- $h : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}$ (locally) smooth and convex in x;
- $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ closed, proper, convex.

$$S(p) := \operatorname*{argmin}_{x \in \mathbb{R}^n} \{h(p, x) + \varphi(x)\} \quad \text{(solution map)}.$$

References: Bonnans/Shapiro (general NLP), Bolte et al. (monotone operators), Vaiter et al. (regularized LLS).

Examples

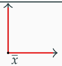
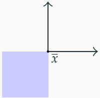
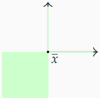
- (prox operator) $p := (\bar{x}, \lambda),\ h(p, x) := \frac{1}{2\lambda}\|x - \bar{x}\|^2$: $S(\bar{x}, \lambda) = P_\lambda \varphi(\bar{x})$.
- (unconstrained LASSO) $p := (A, b, \lambda),\ h(p, x) = \frac{1}{2\lambda}\|Ax - b\|^2,\ \varphi = \|\cdot\|_1$.

By convexity

$$S(p) = \{x \in \mathbb{R}^n \mid 0 \in \nabla_x h(x, p) + \partial\varphi(x)\}.$$

Tailor-made for the implicit function theorems of variational analysis based on graphical differentiation.

Variational analysis: normal cones and graphical differentiation

Name	Definition	Properties	Example
tangent cone	$T_A(\bar{x}) := \operatorname{Lim\,sup}_{t \downarrow 0} \frac{A - \bar{x}}{t}$	closed	
regular normal cone	$\hat{N}_A(\bar{x}) := T_A(\bar{x})^\circ$	closed, convex	
limiting normal cone	$N_A(\bar{x}) := \operatorname{Lim\,sup}_{x \to \bar{x}} \hat{N}_A(x)$	closed	

$S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $(\bar{x}, \bar{y}) \in \operatorname{gph} S := \{(x, y) \mid y \in S(x)\}$.

- Graphical derivative (Aubin '81, Benko '21): $DS(\bar{x}|\bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ via

$$v \in DS(\bar{x}|\bar{y})(u) \;\; :\Longleftrightarrow \;\; (u, v) \in T_{\operatorname{gph} S}(\bar{x}, \bar{y}).$$

- Coderivative (Mordukhovich '80, Ioffe '84): $D^*S(\bar{x}|\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ via

$$v \in D^*S(\bar{x}|\bar{y})(u) \;\; :\Longleftrightarrow \;\; (v, -u) \in N_{\operatorname{gph} S}(\bar{x}, \bar{y}).$$

Variational analysis: proto-differentiability

Observe that graphical derivative of $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x}, \bar{u}) \in \operatorname{gph} S$ is (by definition)

$$DS(\bar{x} \mid \bar{u})(\bar{w}) = \operatorname*{Lim\,sup}_{t \downarrow 0,\, w \to \bar{w}} \frac{S(\bar{x} + tw) - \bar{u}}{t} \quad \forall\, \bar{w} \in \mathbb{R}^n. \tag{2}$$

Definition (Proto-differentiability (Rockafellar '89))

We call S is *proto-differentiable* at $(\bar{x}, \bar{u}) \in \operatorname{gph} S$ if the following hold:

$$\forall \bar{z} \in DS(\bar{x} \mid \bar{u})(\bar{w}),\ \{t_k\} \downarrow 0\ \exists \{w_k\} \to \bar{w},\ \{z_k\} \to \bar{z} :\ z_k \in \frac{S(\bar{x} + t_k w_k) - \bar{u}}{t_k}\ \forall k \in \mathbb{N}.$$

- Relates to semidifferentiability (Penot) which will yield directional differentiability for our purposes.
- Graphically regularity implies proto-differentiability.
- ∂f is proto-differentiable at $(\bar{x}, \bar{u})$, e.g., if $f = g \circ F$ is *fully amenable*, i.e., g PLQ and $F \in C^2$ such that

$$\ker F'(\bar{x})^* \cap N_{\operatorname{dom} g}(F(\bar{x})) = \{0\} \quad \text{(basic constraint qualification).}$$

Variational analysis: directional normal cone and semismoothness*

Directional normal cone of A at $\bar{x}$ in direction $\bar{u}$:

$$N_A(\bar{x};\bar{u}) := \operatorname*{Lim\,sup}_{u\to\bar{u},\ t\downarrow 0} \hat{N}_A(\bar{x}+tu).$$

- $N(\bar{x};\bar{u}) = \emptyset$ if $\bar{u} \notin T_A(\bar{x})$;
- $N(\bar{x};\bar{u}) \subset N_A(\bar{x})$ for all $u \in \mathbb{R}^n$.

Semismoothness* (Gfrerer et al.):

i) $A \subset \mathbb{R}^n$ *semismooth** at $\bar{x} \in A$ $\quad :\Longleftrightarrow \quad \langle x^*, u\rangle = 0 \quad \forall u \in \mathbb{R}^n,\ x^* \in N_A(\bar{x};u)$.

ii) $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *semismooth** at $(\bar{x},\bar{y}) \in \operatorname{gph} S$ $\quad :\Longleftrightarrow \quad$ $\operatorname{gph} S$ semismooth* at $(\bar{x},\bar{y})$.

(Gfrerer and Outrata '19): For $F : D \subset \mathbb{R}^n \to \mathbb{R}^m$ locally Lipschitz at $\bar{x} \in \operatorname{int} D$, the following are equivalent:

- F semismooth (in the sense of Qi and Sun) at $\bar{x}$.
- F semismooth* and directionally differentiable at $\bar{x}$.

Let $f:\mathbb{R}^d\times\mathbb{R}^n\to\mathbb{R}^n$ be continuously differentiable at $(\bar p,\bar x)$ such that $f(\bar p,\cdot)$ is monotone, let $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ be monotone at $(\bar x,-f(\bar p,\bar x))$. Define $S:\mathbb{R}^d\rightrightarrows\mathbb{R}^n$ by

$$S(p)=\{x\in\mathbb{R}^n \mid 0\in f(p,x)+F(x)\},\quad \forall p\in\mathbb{R}^d.$$

The following hold if $(\bar p,\bar x)\in\operatorname{gph} S$ is such that

$$\ker\left(D_xf(\bar p,\bar x)^*+D^*F(\bar x|-f(\bar p,\bar x)\right)=\{0\}\quad \text{(Mordukhovich criterion).}$$

(a) S is locally Lipschitz at $\bar p$ with modulus

$$L\le\limsup_{p\to\bar p}\max_{\|q\|\le 1}\inf_{w\in DS(p)(q)}\|w\|.$$

(b) If F is *proto-differentiable* at $(\bar x,-f(\bar p,\bar x))$, S is directionally differentiable at $\bar p$ with locally Lipschitz directional derivative (for $G(p,x):=f(p,x)+F(x)$) given by

$$S'(\bar p;q)=\{w\in\mathbb{R}^n \mid 0\in DG(\bar p,\bar x|0)(q,w)\}\quad\forall q\in\mathbb{R}^d.$$

(b) If F is semismooth* and the following implication is satisfied:

$$\left.\begin{array}{rcl}-(v,w)&\in&N_{\operatorname{gph}F}(\bar x,-f(\bar p,\bar x)),\\ 0&=&D_pf(\bar p,\bar x)^*w,\\ v&=&D_xf(\bar p,\bar x)^*w\end{array}\right\}\implies (v,w)=(0,0),$$

then S is semismooth at $\bar p$.

(c) If $S'(\bar p;\cdot)$ is linear, then S is differentiable at $\bar p$.

Application: unconstrained LASSO (constraint qualifications)

The unconstrained LASSO[1] for $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, \lambda > 0$ reads

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - b\|^2 + \lambda \|x\|_1. \tag{3}$$

For a solution $\bar{x}$ of (**??**) define:

- $I := \{i \in \{1, \ldots, n\} \mid \bar{x}_i \neq 0\}$ (support);
- $J := \left\{i \in \{1, \ldots, n\} \;\middle|\; |A_i^T(b - A\bar{x})| = \lambda\right\}$ (equicorrelation set).

Note: $\boxed{I \subset J.}$

<u>Qualification conditions</u>

- (Intermediate) $\ker A_J = \{0\}$;
- (Strong) $I = J$ and $\ker A_I = \{0\}$.

(Strong) $\implies$ (Intermediate) $\implies$ $\bar{x}$ is unique solution of (**??**)

[1] Santosa and Symes (1986), Tibshirani (1996)

Application: unconstrained LASSO (stability)

Apply the main theorem with $f(b,\lambda,x) := \frac{1}{\lambda}A^T(Ax-b), \quad F := \partial\|\cdot\|_1$ such that

$$S(b,\lambda) = \{x \mid 0 \in f(b,\lambda,x) + F(x)\} = \operatorname*{argmin}_{x\in\mathbb{R}^n}\left\{\frac{1}{2}\|Ax-b\|^2 + \lambda\|x\|_1\right\} \quad (\lambda > 0).$$

For $(\bar{b},\bar{\lambda}) \in \mathbb{R}^n \times \mathbb{R}_{++}$ let $\bar{x} \in S(\bar{b},\bar{\lambda})$. Then:

(a) If the intermediate condition holds, S is semismooth at $(\bar{b},\bar{\lambda})$ with Lipschitz modulus

$$L \le \frac{1}{\sigma_{\min}(A_J)^2}\left(\sigma_{\max}(A_J) + \left\|\frac{A_J^T(A\bar{x}-\bar{b})}{\bar{\lambda}}\right\|\right).$$

Moreover, the directional derivative $S'((\bar{b},\bar{\lambda});(\cdot,\cdot)) : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz and given as follows: for $(q,\alpha) \in \mathbb{R}^m \times \mathbb{R}$ there exists an index set $K = K(q,\alpha)$ with $I \subseteq K \subseteq J$ such that

$$S'((\bar{b},\bar{\lambda});(q,\alpha)) = L_K\left((A_K^TA_K)^{-1}A_K^T\left(q + \frac{\alpha}{\bar{\lambda}}(A\bar{x}-\bar{b})\right), 0\right).$$

(b) If the strong assumptions holds, S is continuously differentiable at $(\bar{b},\bar{\lambda})$ with

$$DS(\bar{b},\bar{\lambda})(q,\alpha) = L_I\left((A_I^TA_I)^{-1}A_I^T\left(q + \frac{\alpha}{\bar{\lambda}}(A\bar{x}-\bar{b})\right), 0\right), \quad \forall (q,\alpha) \in \mathbb{R}^m \times \mathbb{R}.$$

In particular, S is locally Lipschitz with modulus given above with $I = J$.

Application: unconstrained LASSO (Mordukhovich criterion verified)

Let $\bar{x}$ solve the unconstrained LASSO, i.e.

$$0 \in \underbrace{\frac{1}{\lambda} A^T(A\bar{x} - b)}_{=f(b,\lambda,\bar{x})} + \underbrace{\partial\|\cdot\|_1(\bar{x})}_{F(\bar{x})}.$$

Assume that the intermediate assumption holds, i.e. (with $\bar{u} := \frac{1}{\lambda} A^T(b - A\bar{x}) \in \partial\|\cdot\|_1(\bar{x})$)

$$\ker A_J = \{0\} \quad \text{for} \quad J := \{\, i \in [1:n] \mid |\bar{u}_i| = 1 \,\}. \tag{4}$$

Let $0 \in D_x f(b, \lambda, \bar{x})^* w + D^* F(\bar{x}|\bar{u})(w) = \frac{1}{\lambda} A^T A w + D^*(\partial\|\cdot\|_1)(\bar{x}|\bar{u})(w)$, i.e.

$$-\frac{1}{\lambda} A^T A w \in D^*(\partial\|\cdot\|_1)(\bar{x}|\bar{u})(w). \tag{5}$$

By ‘positive semidefiniteness’ of $D^*(\partial\|\cdot\|_1)(\bar{x}|\bar{u})$ it follows that

$$w \in \ker A. \tag{6}$$

Therefore (**??**) implies

$$\begin{aligned} 0 \in D^*(\overbrace{\partial\|\cdot\|_1}^{=N_{\mathbb{B}_\infty}^{-1}})(\bar{x}|\bar{u})(w) \quad &\iff \quad w \in D^* N_{\mathbb{B}_\infty}(\bar{u}|\bar{x})(0) = \operatorname{span}\{e_i \mid i \in J\} \\ &\implies \quad w_{J^C} = 0 \\ &\overset{(\mathbf{??})}{\implies} \quad w \in \ker A_J \\ &\overset{(\mathbf{??})}{\implies} \quad w = 0. \end{aligned}$$

Application: unconstrained LASSO (tuning parameter sensitivity)

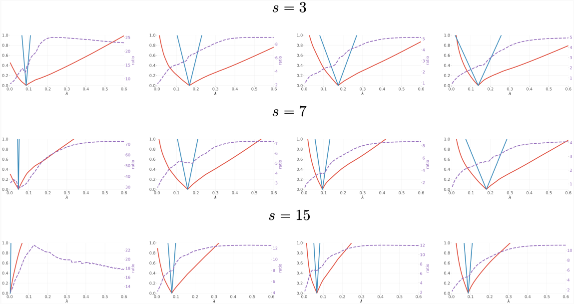
Suppose

$$b = Ax_0 + e :$$

- $n = 200$,
- $A_{ij} \overset{\text{iid}}{\sim} \mathcal{N}(0, 1/m)$,
- $e_i \overset{\text{iid}}{\sim} \mathcal{N}(0, 0.01)$ and
- x_0 s-sparse: $(x_0)_j \overset{\text{iid}}{\sim} \mathcal{N}(m, m)$ $(j \in I)$.

- $x(\lambda) := \underset{x}{\operatorname{argmin}} \left\{ \frac{\|Ax - b\|^2}{2} + \lambda \|x\|_1 \right\}$,
- $\lambda^* := \inf \underset{\lambda > 0}{\operatorname{argmin}} \|x(\lambda) - x_0\|$,
- $\bar{x} := x(\lambda^*)$.

Under the strong assumption at $\bar{x}$, $x(\cdot)$ is locally Lipschitz with $L := \frac{\sqrt{|I|}}{\sigma_{\min}(A_I)^2}$.



$m = 50$ $m = 100$

Figure 1: $\|x(\lambda) - \bar{x}\|$, $L|\lambda - \lambda^*|$, $L|\lambda - \lambda^*|$ ...

Application: the proximal operator

For $f \in \Gamma_0 := \{f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \mid f$ lsc, proper, convex $\}$, the proximal operator is

$$P_\lambda f(x) = \underset{u \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ \varphi(x) + \frac{1}{2\lambda} \|x - u\|^2 \right\} \quad \forall x \in \mathbb{R}^n,\ \lambda > 0.$$

Theorem

The following hold for $P_f : (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}_{++} \mapsto P_\lambda f(x)$:

(a) P_f *is locally Lipschitz at* $(\bar{x}, \bar{\lambda}) \in \mathbb{R}^n \times \mathbb{R}_{++}$.

(b) *If* ∂f *is proto-differentiable at* $\left(P_f(\bar{x}, \bar{\lambda}), \frac{\bar{x} - P_f(\bar{x}, \bar{\lambda})}{\bar{\lambda}}\right)$, *then* P_f *is directionally differentiable at* $(\bar{x}, \bar{\lambda})$ *with*

$$P_f'((\bar{x}, \bar{\lambda}); (d, \Delta)) = \left[\bar{\lambda} D(\partial f)\left(P_f(\bar{x}, \bar{\lambda}) \middle| \frac{\bar{x} - P_f(\bar{x}, \bar{\lambda})}{\bar{\lambda}}\right) + I\right]^{-1} \left(d - \frac{\Delta}{\bar{\lambda}}(\bar{x} - P_f(\bar{x}, \bar{\lambda}))\right).$$

(c) *If* ∂f *is proto-differentiable and semismooth* at* $\left(P_f(\bar{x}, \bar{\lambda}), \frac{\bar{x} - P_f(\bar{x}, \bar{\lambda})}{\bar{\lambda}}\right)$ *then* P_f *is semismooth at* $(\bar{x}, \bar{\lambda})$.

Note: $f \in C^2$ or f PLQ $\Longrightarrow \partial f$ proto-differentiable and semismooth*.

References

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Future directions

- Explore new techniques by Gfrerer and Outrata (*subspace containing derivatives*) for establishing strong metric regularity of hypomonotone operators (e.g., subdifferentials of weakly convex functions).
- Clarify the relation between proto-differentiability and semismoothness*.
- Apply the graphical derivative-based implicit function framework to, e.g.,:
  - regularized (linear) least-squares with PLQ regularizers;
  - nuclear norm regularized minimization.
- Explore implications in bilevel optimization.

Thanks for coming!