Topics in Geometric Fully Nonlinear Equations

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Preface

There have been some renewed interests on fully nonlinear equations on manifolds recently, in particular in connection to some problems in classical Euclidean geometry and conformal geometry. These equations are of some kind of interpolation of the Monge-Ampère equation and Laplace equation with respect to certain quadratic form involving second order covariant derivatives. Due to the fundamental work of Krylov [**87**] and Evans [**42**], there is a general theorem on $C^{2,\alpha}$-regularity of the solutions of these equations once C^2 a priori estimates are established. The basic structure of these equations in Euclidean domains have been studied thoroughly in [**26**]. When dealing with equations arising from geometry, the treatments may vary according to the underline geometric situation.

In this lecture notes, we will restrict ourselves on fully nonlinear elliptic and parabolic equations related to classical Euclidean geometry and conformal geometry. Some algebraic and analytic properties of concave symmetric functions and Garding's theory of hyperbolic polynomials are collected in the appendix. The choice of the topics is solely based on author's personal taste and the material familiar to him.

This an expanded and updated version of the notes delivered in a series of lectures in the workshop of Monge-Ampère equations and summer school of mathematics in Zhejiang University, Hangzhou, 2002 and 2004 respectively. These notes are compiled from some of joint works with B. Guan, C.S. Lin, X. Ma and G. Wang in recent years. This is a record of their contributions to the subject. Of course, any errors, mistakes and omissions in the notes lies completely on the author. I have learned a great deal from them during the pleasant period of collaborations. I would like to take this opportunity to thank them for their friendship and impact on me.

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Part 1

Curvature equations of hypersurfaces in $\mathbb{R}^n$

CHAPTER 1

Theory of convex bodies: Alexandrov-Fenchel inequality

Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain with reasonable smooth boundary, let's denote $V(\Omega)$ and $A(\partial\Omega)$ the volume of Ω and surface area of $\partial\Omega$ respectively. The isopermetric inequality says that

$$V(\Omega)^{\frac{1}{n+1}} \leq c_n A(\partial\Omega)^{\frac{1}{n}},$$

where c_n is a dimensional constant, with the equality holds if and only if Ω is a ball.

If Ω is convex, there is a sequence of geometric quantities called quermassintegrals $\mathcal{W}_k(\Omega)$ for $k = 1, 2, ..., n+1$ with $\mathcal{W}_{n+1}(\Omega) = V(\Omega)$ and $\mathcal{W}_n(\Omega) = A(\partial\Omega)$, They defined as

$$\mathcal{W}_k(\Omega) = \int_{\pi \in G(k,n+1)} vol(\Omega|\pi) d\pi,$$

where π is any k-dim hyperplane (as a point in Grassmannian manifold $G(k, n+1)$) and $vol(\Omega|\pi)$ is the volume of the projection of Ω to π. The Cauchy-Crofton formula states that

$$\mathcal{W}_k(\Omega) = c_{n,k} \int_{\partial\Omega} \sigma_{n-k}(\kappa_1, ..., \kappa_n),$$

where $c_{n,k}$ is a positive constant depending only on n, k, σ_k is the k-th elementary symmetric function and $\kappa_1, ..., \kappa_n$ the principal curvature functions on $\partial\Omega$. We note that if $\partial\Omega$ is smooth, $\sigma_{n-k}(\kappa_1, ..., \kappa_n)$ is a smooth function, while $\kappa_1, ..., \kappa_n$ may not necessary smooth (but they are continuous).

Support function: we define

$$u(x) = \max_{y \in \Omega} < x, y >, \quad x \in \mathbb{S}^n.$$

If $\partial\Omega$ is strictly convex, one may check that $u(x) =< x, \mathbf{n}^{-1}(x) >$ for $x \in \mathbb{S}^n$.

The support function carries all the information of $\partial\Omega$. There is one-to-one correspondence of support function and convex body. For any function on $\mathbb{S}^n$, we may extend it as a homogeneous function of degree one in $\mathbb{R}^{n+1}$. A function u on $\mathbb{S}^n$ is a support function of some convex body if and only if it is a convex function in $\mathbb{R}^{n+1}$ after this extension.

For convex bodies, one can define Minkowski summation. Together with the concept of support functions, they play fundamental roles, in the theory of convex bodies.

Minkowski summation: For two convex bodies Ω_1 and Ω_2, define

$$\Omega_1 + \Omega_2 := \{x + y | x \in \Omega_1, y \in \Omega_2\}.$$

and for $\lambda > 0$, define

$$\lambda\Omega := \{\lambda x | x \in \Omega\}.$$

So, for any positive numbers $t_1, ..., t_m$ and convex bodies $\Omega_1, ..., \Omega_m$ with support function $u_1, ..., u_m$ respectively. We can define $t_1\Omega_1 + ... + t_m\Omega_m$, it is still convex. It is easy to check that the corresponding support function is $t_1u_1 + ... + t_mu_m$.

Minkowski proved that the volume of $t_1\Omega_1 + ... + t_{n+1}\Omega_{n+1}$ is a homogeneous polynomial in $t_1, ..., t_{n+1}$. The coefficient in front of the minomial $t_1 \times ... \times t_{n+1}$ is called the mixed volume, often write as $V(\Omega_1, ..., \Omega_{n+1})$.

From now on, we assume $\partial\Omega$ is C^2 and strictly convex. By the Hadamard's theorem, it is equivalent to $\kappa_1, ..., \kappa_n$ are positive functions on $\partial\Omega$. If we view $\partial\Omega$ as a Riemannian manifold embedded in $\mathbb{R}^{n+1}$ as a compact hypersurface, let X be its position vector and $\mathbf{n}$ be its outer normal (Gauss map), the first and second fundamental forms are given by $I = dXdX$ and $II = d\mathbf{n}dX$ respectively. $\kappa_1, ..., \kappa_n$ are the eigenvalues of II with respect to the first fundamental form I. When $\partial\Omega$ is strictly convex, the Gauss map $\mathbf{n}$ is a diffeomorphism from $\partial\Omega$ onto $\mathbb{S}^n$. We may view the inverse Gauss map as a natural parametrization of $\partial\Omega$. This is a starting point for the theory of convex bodies.

There is a magic connection of the support function and curvature functions of $\partial\Omega$. Write

$$W = (u_{ij} + \delta_{ij}u),$$

where u_{ij} indicates the second order covariant derivatives of u with respect to any orthonormal frame on S^n. The eigenvalues of W are the principal radii of $\partial\Omega$. By divergence theorem, $V(\Omega) = c_n \int_{\partial\Omega} u$. For the rest, we will assume $\partial\Omega$ is strictly convex and C^2. So, we have the formula

$$\mathcal{W}_k(\Omega) = c_{n,k} \int_{\mathbb{S}^n} \sigma_k(W), \quad 1 \leq n. \tag{1.1}$$

The volume $V(\Omega)$, in general $\mathcal{W}_{k+1}(\Omega)$ can also be expressed as (via Minkowski formula, see (1.7))

$$\mathcal{W}_{k+1}(\Omega) = \tilde{c}_{n,k} \int_{\mathbb{S}^n} u\sigma_k(W), \quad 1 \leq n.$$

For the above, it's easy to see that $V(t_1\Omega_1 + ... + t_{n+1}\Omega_{n+1})$ is a homogeneous polynomial of degree $n + 1$. Though the above expression only proved for strictly smooth convex bodies, the general case can be valid by approximation.

$\sigma_k(W)$ is called the k-th area function of Ω. The problem of prescribing k-th area function on $\mathbb{S}^n$ is called The Christoffel-Minkowski problem. The main subject of the theory of convex bodies is to study the mixed volumes and their local versions: area measures and curvature measures.

We now start differential calculations with respect to support functions. Let $e_1, ..., e_n$ is an orthonormal frame on $\mathbb{S}^n$, let $\omega_1, ..., \omega_n$ be the corresponding dual 1-forms. For each function $u \in C^2(\mathbb{S}^n)$, let u_i be the covariant derivative of u with respect to e_i. We define a vector valued function

$$Z = \sum_{i=1}^{n} u_ie_i + ue_{n+1}.$$

where e_{n+1} is the position vector on $\mathbb{S}^n$, that is, the outer normal vector field of $\mathbb{S}^n$. We note that Z is globally defined on $\mathbb{S}^n$. We write the hessian matrix of u with respect to the frame as

$$W = \{u_{ij} + u\delta_{ij}\}.$$

We calculate that,

$$\begin{aligned} u &= Z \cdot e_{n+1}, \\ dZ &= \sum_{i=1}^{n}(du_i e_i + u_i de_i) + due_{n+1} + ude_{n+1} \\ &= \sum_{i=1}^{n}(\sum_{j=1}^{n} u_{ij}\omega^j - \sum_{j=1}^{n} u_j\omega_j^i)e_i + \sum_{i=1}^{n}(\sum_{\alpha=1}^{n+1} u_i\omega_i^\alpha e_\alpha) \\ &\quad + \sum_{i=1}^{n}(u_i\omega^i)e_{n+1} + u\sum_{i=1}^{n}\omega^i e_i \\ &= \sum_{j=1}^{n}(\sum_{i=1}^{n}(u_{ij} + \delta_{ij}u)e_i)\omega^j. \end{aligned}$$

Let $u^1, ..., u^{n+1} \in C^2(\mathbb{S}^n)$, we define $\forall l = 1, ..., n+1$,

$$Z^l = \sum_{i=1}^{n} u_i^l e_i + u^l e_{n+1},$$

and

$$W^l = \{u_{ij}^l + u^l\delta_{ij}\}$$

Set,

$$\Omega(u^1, ..., u^{n+1}) = (Z^1, dZ^2, dZ^3, ..., dZ^{n+1}). \tag{1.2}$$

and

$$V(u^1, u^2..., u^{n+1}) = \int_{\mathbb{S}^n} \Omega(u^1, ..., u^{n+1}). \tag{1.3}$$

We note that

$$\Omega(u^1, ..., u^{n+1}) = u^1\sigma_n(W^2, ..., W^{n+1})ds \tag{1.4}$$

where $\sigma_n(W^2, ..., W^{n+1})$ is the mixed determinant and ds is the standard area form on $\mathbb{S}^n$. In particular, $\forall 1 \leq k \leq n$, if we set $u^{k+2} = ... = u^{n+1} = 1$, we obtain

$$\Omega(u^1, ..., u^{n+1}) = \binom{n}{k}^{-1} u^1\sigma_k(W^2, ..., W^{k+1})ds \tag{1.5}$$

where $\sigma_k(W^2, ..., W^{k+1})$ is the complete polarization of the symmetric function σ_k defined for symmetric matrices.

Lemma 1.1. *V is a symmetric multilinear form on $(C^2(\mathbb{S}^n))^{n+1}$.*

Proof. The multilinearility follows directly from the definition. Also, by the definition, for any permutation σ of $\{2, ..., n+1\}$,

$$\Omega(u^1, u^2..., u^{n+1}) = \Omega(u^1, u^{\sigma(2)}..., u^{\sigma(n+1)}),$$

so $V(u^1, u^2..., u^{n+1}) = V(u^1, u^{\sigma(2)}..., u^{\sigma(n+1)})$. To see V is a symmetric form, we only need to show

$$V(u^1, u^2, u^3..., u^{n+1}) = V(u^2, u^1, u^3..., u^{n+1}). \tag{1.6}$$

We first assume $u^i \in C^3(\mathbb{S}^n), \forall i$. Let,

$$\omega(u^1, ..., u^{n+1}) = (Z^1, Z^2, dZ^3, ..., dZ^{n+1}),$$

we have

$$d\omega(u^1, ..., u^{n+1}) = -\Omega(u^2, u^1, u^3..., u^{n+1}) + \Omega(u^1, u^2, u^3..., u^{n+1}),$$

Now, (1.6) follows from Stokes theorem. The identity (1.6) is valid for C^2 function by approximation. ■

Remark: If $u^1, ..., u^{n+1}$ are the support functions of convex bodies $K_1, ..., K_{n+1}$ respectively, then $V(u^1, u^2..., u^{n+1})$ is the Minkowski mixed volume $V(K_1, ..., K_{n+1})$.

The following is a direct corollary of the lemma. If u is a support function of a convex body, it is well known as Minkowski type integral.

Corollary 1.1. *For any function $u \in C^2(\mathbb{S}^n)$, $W = \{u_{ij} + \delta_{ij}u\}$. For any $1 \le k < n$, we have the Minkowski type integral formulas.*

$$(n-k)\int_{\mathbb{S}^n} u\sigma_k(W)\, ds = (k+1)\int_{\mathbb{S}^n} \sigma_{k+1}(W)\, ds, \tag{1.7}$$

where ds is the standard area element on $\mathbb{S}^n$.

For any $n \times n$ symmetric matrices $W_1, ..., W_k$, let $\sigma_k(W_1, ..., W_k)$ be the complete polarization of σ_k. Let u and $\tilde{u}$ are two C^2 functions on $\mathbb{S}^n$. Let W and $\tilde{W}$ are the corresponding Hessian matrices of u and $\tilde{u}$ respectively. Define $P_{rs} = \sigma_{r+s}(W, ..., W, \tilde{W}, ...\tilde{W})$ where W appears r times and $\tilde{W}$ appears s times. So, P_{rs} is a polynomial in W_{ij}, $\tilde{W}_{ij}$, homogeneous of degrees r and s respectively. The following is another corollary of Lemma 1.1.

Corollary 1.2. *Suppose u and $\tilde{u}$ are two C^2 functions on $\mathbb{S}^n$, then the following identities hold.*

$$\int_{\mathbb{S}^n} [uP_{0k} - \tilde{u}P_{1,k-1}]dx = 0, \tag{1.8}$$

$$\int_{\mathbb{S}^n} [uP_{k-1,1} - \tilde{u}P_{k0}]dx = 0, \tag{1.9}$$

and,

$$\begin{aligned} &2\int_{\mathbb{S}^n} u(P_{0k} - P_{k-1,1})dx \\ &= \int_{\mathbb{S}^n} \{\tilde{u}(P_{1,k-1} - P_{k0}) - u(P_{k-1,1} - P_{0k})\}dx. \end{aligned} \tag{1.10}$$

Now, we consider functions satisfying the following equation,

$$\sigma_k(W) = \varphi \quad \text{on} \quad \mathbb{S}^n. \tag{1.11}$$

Definition 1.1. *For $1 \leq k \leq n$, let Γ_k is a convex cone in $\mathbb{R}^n$ determined by*

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \quad \sigma_1(\lambda) > 0, ..., \sigma_k(\lambda) > 0\}$$

Suppose $u \in C^2(\mathbb{S}^n)$, we say u is k-convex, if $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\}$ is in Γ_k for each $x \in \mathbb{S}^n$. u is convex on $\mathbb{S}^n$ if W is semi-positive definite on $\mathbb{S}^n$. Furthermore, u is called an admissible solution of (1.11), if u is k-convex and satisfies (1.11).

The next is a uniqueness theorem.

Theorem 1.1. *Suppose u and $\tilde{u}$ are two C^2 k-convex functions on $\mathbb{S}^n$ satisfying (1.11). If $\sigma_k(W) = \sigma_k(\tilde{W})$, and if one of u and $\tilde{u}$ is nonnegative, then $u - \tilde{u} \in Span\{x_1, ..., x_{n+1}\}$ on $\mathbb{S}^n$.*

Proof of Theorem 1.1. We may assume u is nonnegative. Since $\sigma_k(W)$ is positive, we conclude that u is positive almost everywhere on $\mathbb{S}^n$. As σ_k is complete hyperbolic, and $\forall W^i \in \Gamma_k, i = 1, ..., k$,

$$\sigma_k(W^1, ..., W^k) \geq \sigma_k^{\frac{1}{k}}(W^1) \cdots \sigma_k^{\frac{1}{k}}(W^k), \tag{1.12}$$

with the equality holds if and only if these k matrices are pairwise proportional.

If $W, \tilde{W} \in \Gamma_k$, from (1.12) we have

$$P_{0,k} \leq P_{k-1,1}, \tag{1.13}$$

with the equality holds if and only if W and $\tilde{W}$ are proportional.

Suppose $\sigma_k(W) = \sigma_k(\tilde{W})$ on $\mathbb{S}^n$, where $W = \{u_{ij} + \delta_{ij}u\}$ and $\tilde{W} = \{\tilde{u}_{ij} + \delta_{ij}\tilde{u}\}$. The left hand side of the integral formula (1.10) in Corollary 1.2 is non-positive. The same is therefore true of the right hand side of (1.10). The latter is anti-symmetric on the two function u and $\tilde{u}$, and hence must be zero. It follows that $P_{k-1,1} = P_{0,k}$ by (1.13). Again, the equality gives that W and $\tilde{W}$ are proportional. Since $\sigma_k(W) = \sigma_k(\tilde{W})$, we conclude that $W = \tilde{W}$ at each point of $\mathbb{S}^n$. In particular,

$$L(u - \tilde{u}) = \Delta(u - \tilde{u}) + n(u - \tilde{u}) = 0, \quad \text{on} \quad \mathbb{S}^n.$$

We know that L is a self-adjoint linear elliptic operator on $\mathbb{S}^n$, $Span\{x_1, ..., x_{n+1}\}$ is exactly the kernel of L. This gives $u - \tilde{u} \in Span\{x_1, ..., x_{n+1}\}$. ■

The following is an infinitesimal version of Theorem 1.1.

Proposition 1.1. *$\forall u^2, ..., u^k \in C^2(\mathbb{S}^n)$ fixed, define*

$$L(v) = \Omega(1, v, u^2 ..., u^k, 1, ..., 1), \tag{1.14}$$

then, L is self-adjoint. If in addition, $u^2, ..., u^k$ are k-convex, and at least one of them is non-negative, the kernel of L is $Span\{x_1, ..., x_{n+1}\}$.

Proof of Proposition 1.1. First, L is self-adjoint is self-adjoint by Lemma 1.1. To compute the kernel, we may assume u^2 is nonnegative. Since u^2 is k-convex, it is positive almost everywhere. Suppose v is in kernel of L, i.e.,

$$L(v) = 0. \tag{1.15}$$

Simple calculation shows that

$$\Omega(1, v, v, u^3, ..., u^k, 1, ..., 1) = \binom{n}{k}^{-1} \sigma_k(A, A, W^3, ..., W^k)ds,$$

where $A = \{v_{ij} + \delta_{ij}v\}$ and $W^l = \{u^l_{ij} + \delta_{ij}u^l\}$.

We *claim* that, if (1.15) holds, then

$$\sigma_k(A, A, W^3, ..., W^k) \leq 0, \tag{1.16}$$

with equality if and only if $A = 0$, i.e., $v \in Span\{x_1, ..., x_{n+1}\}$.

We note that,

$$\begin{aligned} 0 &= \int_{\mathbb{S}^n} vL(v) = \int_{\mathbb{S}^n} \Omega(v, v, u^2, u^3, ..., u^k, 1, ..., 1) \\ &= V(v, v, u^2, u^3, ..., u^k, 1, ..., 1) = V(u^2, v, v, u^3, ..., u^k, 1, ..., 1) \\ &= \int_{\mathbb{S}^n} u^2\Omega(1, v, v, u^3, ..., u^k, 1, ..., 1) \\ &= \binom{n}{k}^{-1} \int_{\mathbb{S}^n} u^2\sigma_k(A, A, W^3, ..., W^k)ds. \end{aligned}$$

If the *claim* is true, we will conclude that v is in $Span\{x_1, ..., x_{n+1}\}$ since u^2 is positive almost everywhere.

To prove the claim, we make use of hyperbolicity of σ_k in the cone Γ_k (Corollary 13.1). Since u^l is k-convex, $W^l \in \Gamma_k, \forall 2 \leq l \leq k$. For $W^3, ..., W^k$ fixed, the polarization $\sigma_k(B, B, W^3, ..., W^k)$ is also hyperbolic and complete for $B \in \Gamma_k$. Let $W_t = W^2 + tA$, we have

$$\begin{aligned} &\sigma_k(W_t, W_t, W^3, ...W^k) = \sigma_k(W^2, W^2, W^3, ..., W^k) \\ &+2t\sigma_k(A, W^2, W^3, ..., W^k) + t^2\sigma_k(A, A, W^3, ..., W^k). \end{aligned}$$

Since

$$\sigma_k(W^2, W^2, W^3.., W^k) > 0,$$

and

$$\sigma_k(A, W^2, ..., W^k) = 0.$$

By the hyperbolicity, $\sigma_k(W_t, W_t, W^3, ..., W^k)$ has only real roots in t variable, so (1.16) must be true. If in addition, $\sigma_k(A, A, W, ..., W) = 0$, we would have

$$\sigma_k(W_t, W_t, W, ..., W) = \sigma_k(W, ..., W),$$

for all $t \in \mathbb{R}$. By Lemma 13.2 and the completeness of $\sigma_k(W, W, W^3, ..., W^k)$, $A = 0$. The *claim* is proved. ■

For any $n \geq k \geq 1$ fixed, set $u^{k+2} = ... = u^{n+1} = 1$ we define $\forall u^1, ..., u^{k+1} \in C^2(\mathbb{S}^n)$,

$$V_{k+1}(u^1, u^2, ..., u^{k+1}) = V(u^1, u^2, ..., u^{n+1}). \tag{1.17}$$

Now we state a form of Alexandrov-Fenchel inequality for positive k-convex functions.

THEOREM 1.2. *If $u^1, ..., u^k$ are k-convex, and u_1 positive, and at least one of u^l is nonnegative on $\mathbb{S}^n$ (for $2 \le l \le k$), then $\forall v \in C^2(\mathbb{S}^n)$,*

$$V_{k+1}^2(v, u^1, ..., u^k) \ge V_{k+1}(u^1, u^1, u^2, ..., u^k) V_{k+1}(v, v, u^2, ..., u^k), \tag{1.18}$$

the equality holds if and only if $v = au^1 + \sum_{i=1}^{n+1} a_i x_i$ for some constants $a, a_1, ..., a_{n+1}$.

Proof. The theorem follows directly from the next statement.

Statement: If

$$V_{k+1}(v, u^1, u^2, ..., u^k) = 0, \quad \text{for some} \quad v \in C^2(\mathbb{S}^n), \tag{1.19}$$

then

$$V_{k+1}(v, v, u^2, ..., u^k) \le 0, \tag{1.20}$$

with equality if and only if $v = \sum_{i=1}^{n+1} a_i x_i$.

The proof of the **Statement** will be reduced to an eigenvalue problem for certain elliptic differential operators.

First, for $u^2, ..., u^k \in \Gamma_k$ fixed, we set

$$L(v) = \Omega(1, v, u^2 ..., u^k, 1, ..., 1).$$

By Lemma 13.1, $L(v) > 0$ if v is k-convex. We claim that L is an elliptic differential operator with negative principal symbol. The principal symbol of L at the co-tangent vector $\theta = (\theta_1, ..., \theta_n)$ is obtained when A is replaced by $-\theta \otimes \theta$ in

$$\sigma_k(A, W^2, ..., W^k).$$

So it is equal to

$$-\sigma_k(\theta \otimes \theta, W^2, ..., W^k).$$

Since σ_k is hyperbolic with respect to the positive cone Γ_k, and $\theta \otimes \theta$ is semi-positive definite and is not a 0 matrix if θ not 0. By the complete hyperbolicity,

$$-\sigma_k(\theta \otimes \theta, W^2, ..., W^k) < 0.$$

We now use continuity method to finish the job. For $0 \le t \le 1$, let $u_t^i = t + (1-t)u^i$, and set

$$\rho_t = \frac{\Omega(1, u_t^1, u_t^2 ..., u_t^k, 1, ..., 1)}{u_t^1},$$

We examine the eigenvalue problem:

$$L_t(v) = \lambda \rho_t v. \tag{1.21}$$

If for we set $Q_t(u, v) = \int_{\mathbb{S}^n} u L_t(v)$, the eigenvalue problem (1.21) is corresponding to the quadratic form Q_t with respect to the inner-product $< u, v >_{\rho_t} = \int_{\mathbb{S}^n} uv\rho_t$.

We want to show **Claim:** $\lambda = 1$ is the only positive eigenvalue of multiplicity 1 with eigenfunction u_t^1, and $\lambda = 0$ is the eigenvalue of multiplicity $n+1$ with eigenspace $Span\{x_1, ..., x_{n+1}\}$ for the eigenvalue problem of (1.21).

We note that u_t^1 is an eigenfunction corresponding to the eigenvalue $\lambda = 1$. If the *Claim* is true, (1.19) implies that v is orthogonal to eigenspace corresponding to $\lambda = 1$ with respect to

the inner product $< .,. >_{\rho_1}$. If the claim is true, **Statement** follows from the standard spectral theory of self-adjoint elliptic operators.

We now prove the *Claim*. When $t = 0$, the problem can be reduced to the following simple form by straightforward calculations:

$$\Delta v + nv = n\lambda v.$$

The eigenvectors of Δ are the spherical harmonics of degree $\nu = 0, 1, ...$, with the corresponding eigenvalues $-\nu(\nu + n - 1)$. $\nu = 0$ corresponds to $\lambda = 1$ and $\nu = 1$ corresponds to $\lambda = 0$ in the eigenvalue problem (1.21) respectively in this special case. And $\lambda < 0$ when $\nu > 1$. It is well known that spherical harmonics of degree 0 are constants, and spherical harmonics of degree 1 are linear functions, i.e., $Span\{x_1, ..., x_{n+1}\}$. Therefore, the **Claim** is true for $t = 0$. For arbitrary t, since 1 is an eigenvalue of the problem (1.21) with eigenfunction u_t^1, by the theory of elliptic equations, we only need to prove that 0 is the eigenvalue of multiplicity $n + 1$. It's obvious that $x_1, ..., x_{n+1}$ are the eigenfunctions of L corresponding to the eigenvalue 0. The theorem now follows from Proposition 1.1. ■

Now, we consider a class of domains which will be named k^*-convex. They can be viewed as a generalization of convex bodies via polar dual. Let D be a star-shaped bounded domain in $\mathbb{R}^{n+1}$ with C^2 boundary. The distance function of D is defined as,

$$u(x) = \min\{\lambda | x \in \lambda D\}, \quad \forall x \in \mathbb{S}^n. \tag{1.22}$$

When D is convex, the distance function is also called the gauge function of D.

Definition 1.2. *Let D be a star-shaped bounded domain in $\mathbb{R}^{n+1}$ with C^2 boundary. We say D is k^*-convex if its distance function u is k-convex on $\mathbb{S}^n$. We day D is polar centrized if its distance function u satisfies*

$$\int_{\mathbb{S}^n} x_j u(x) ds = 0, \quad \forall j = 1, 2, ..., n + 1.$$

If $D_1, ..., D_{k+1}$ are k^*-convex bodies, let $u_1, ..., u_{k+1}$ are the corresponding distance functions, and $W_1, ..., W_{k+1}$ be the corresponding hessians of the gauge functions respectively. For $0 \le l \le k$, we define mixed polar surface area functions

$$\sigma_l(D_1, ..., D_l, x) = \sigma_l(W_1, ..., W_l). \tag{1.23}$$

We call $\sigma_l(D, x) = \sigma_l(W, ..., W)$ the lth polar surface are function of D. We also define a mixed polar volume,

$$V^*_{k+1}(D_1, ..., D_{k+1}) = \frac{1}{V_{k+1}(u_1, ..., u_{k+1})} \tag{1.24}$$

where $V_{k+1}(u_1, ..., u_{k+1})$ defined as in (1.17). We also write, $\forall 0 \le l \le k + 1$, $V^*_l(D) = V^*_{k+1}(D, ..., D, B, ..., B)$, where B is the unit ball centered at the origin in $\mathbb{R}^{n+1}$, D appears l times, and B appears $k + 1 - l$ times in the formula.

As an application, we have the following consequences of Theorem 1.1 and Theorem 1.2.

THEOREM 1.3. *Suppose D_1, D_2 are two k^*-convex domains in $\mathbb{R}^{n+1}$. If kth polar surface area functions of D_1 and D_2 are the same, i.e.,*

$$\sigma_l(D_1, x) = \sigma_l(D_2, x), \quad \forall x \in \mathbb{S}^n,$$

then, the distance functions of D_1, D_2 are equal upto a linear function. In particular, if both D_1 and D_2 are polar centrized, then $D_1 = D_2$.

THEOREM 1.4. *Suppose $D_1, ..., D_{k+1}$ are k^*-convex domains in $\mathbb{R}^{n+1}$,then we have the following Alexandrov-Fenchel inequality for the mixed polar volumes:*

$$(V^*_{k+1}(D_1, ..., D_{k+1}))^2 \leq V^*_{k+1}(D_1, D_1, D_3..., D_{k+1})V^*_{k+1}(D_2, D_2, D_3..., D_{k+1}),$$

with the equality if and only if the distance functions of D_1 and D_2 are equal upto a linear function. In particular, if both D_1, D_2 are polar centerized, then $D_1 = \lambda D_2$ for some $\lambda > 0$.

The above theorem indicates that the reciprocal of the mixed polar volume is log-concave. Therefore, one may deduce a sequence of inequalities for k^*-convex domains from Theorem 1.2. In particular, one can obtain the corresponding Brunn-Minkowski inequality and quermassintegral inequalities for V^*.

COROLLARY 1.3. *Suppose D_1, D_2 are k^*-convex, then for $0 \leq t \leq 1$,*

$$V^*_{k+1}((1-t)D_1 + tD_2)^{\frac{-1}{k+1}} \geq (1-t)V^*_{k+1}(D_1)^{\frac{-1}{k+1}} + tV^*_{k+1}(D_2)^{\frac{-1}{k+1}},$$

if D_1, D_2 are polar centralized, the equality for some $0 < t < 1$ holds if and only if $D_1 = \lambda D_2$ for some $\lambda > 0$. If D is k^*-convex, then for $0 \leq i < j < l \leq k+1$,

$$(V^*_j(D))^{l-i} \leq (V^*_i(D))^{l-j}(V^*_l(D))^{j-i}.$$

if D is polar centralized, the equality holds if and only if D is a ball centered at the origin. In particular, if we let σ_n be the volume of the unit ball B in $\mathbb{R}^{n+1}$,

$$\sigma_n^{i-j}(V^*_j(D))^{k-i} \leq (V^*_i(D))^{k-j},$$

if D is polar centralized, the equality holds if and only if D is a ball centered at the origin.

At the end of this chapter, we discuss the geometric obstructions and uniqueness problem for equation

$$\frac{\sigma_n}{\sigma_{n-k}}(u_{ij} + \delta_{ij}u) = f \quad \text{on } \mathbb{S}^n. \tag{1.25}$$

This equation arises from the problem of prescribing Weingarten curvatures on outer normals (see [**4**], [**36**]). It was discovered in [**57**] that the necessary conditions for the Minkowski problem are not valid for equation (1.25) if $k \neq n$.

We start with some calculation. Let $v \in C^\infty(\mathbb{S}^n)$ and consider $u_t = 1 + tv$. For $t > 0$ small, u_t is a supporting function of some smooth strictly convex hypersurface, and

$$\sigma_n(\nabla^2 u_t + u_t\sigma) = \sum_{i=1}^n \frac{n!}{i!(n-i)!}\sigma_i\, t^i.$$

Here, and in the rest of this section, we write $\sigma_i = \sigma_i(\nabla^2 v + v\sigma)$. It follows that

$$\int_{\mathbb{S}^n} x_j \sigma_i d\sigma = 0, \quad \forall\, 1 \le j \le n+1, \;\; 1 \le i \le n \tag{1.26}$$

since

$$\int_{\mathbb{S}^n} x_j \sigma_n (\nabla^2 u_t + u_t \sigma) d\sigma = 0, \quad \forall\, 1 \le j \le n+1$$

for all $t > 0$ sufficiently small.

For a fixed k $(1 \le k < n)$, by straightforward calculation we see that

$$\sigma_{n,k}(\nabla^2 u_t + u_t\sigma) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + O(t^4) \tag{1.27}$$

where

$$\begin{aligned} a_1 &= (n-k)\sigma_1, \\ a_2 &= \frac{n-k}{2}[(n+k-1)\sigma_2 - 2k\sigma_1^2], \\ a_3 &= \frac{k(n-k)}{2}[2k\sigma_1^3 - (n+2k-2)\sigma_1\sigma_2] + a\sigma_3, \end{aligned} \tag{1.28}$$

for some constant a depending only on k and n.

From this we compute, for any $m \in \mathbf{R}$, the coefficients of the Taylor expansion

$$[\sigma_{n,k}(\nabla^2 u_t + u_t\sigma)]^m = 1 + b_1 t + b_2 t^2 + b_3 t^3 + O(t^4) \tag{1.29}$$

to obtain

$$\begin{aligned} b_1 &= m(n-k)\sigma_1, \\ b_2 &= \frac{m(n-k)}{2}[(n+k-1)\sigma_2 + (m(n-k) - n - k)\sigma_1^2] \end{aligned} \tag{1.30}$$

and, when $m = \frac{n+k}{n-k}$,

$$b_3 = \frac{nk(n+k)}{6}(3\sigma_1\sigma_2 - 2\sigma_1^3) + b\sigma_3 \tag{1.31}$$

where b is a constant. We are now in a position to prove the following result.

Proposition 1.2. *For every integer k, $1 \le k < n$, and any $m \in \mathbf{R}$, $m \ne 0$ there exists $v \in C^\infty(\mathbb{S}^n)$ such that the function $u_t = 1 + tv$ satisfies*

$$\int_{\mathbb{S}^n} x\, [\sigma_{n,k}(\nabla^2 u_t + u_t\sigma)]^m d\sigma \ne 0 \tag{1.32}$$

for all $t > 0$ sufficiently small.

Proof. We use the spherical coordinates on $\mathbb{S}^n$

$$\begin{aligned} x_1 &= \cos\theta_1, \\ x_j &= \sin\theta_1 \cdots \sin\theta_{j-1}\cos\theta_j, \;\; 1 < j \le n, \\ x_{n+1} &= \sin\theta_1 \cdots \sin\theta_{n-1}\sin\theta_n, \\ d\sigma_{\mathbb{S}^n} &= \sin^{n-1}\theta_1 \sin^{n-2}\theta_2 \cdots \sin\theta_{n-1} d\theta_1 \cdots d\theta_n, \end{aligned} \tag{1.33}$$

where $0 \le \theta_j \le \pi$, $1 \le j \le n-1$; $0 \le \theta_n \le 2\pi$. Let

$$g(x) = \eta(\cos^2\theta_1)\cdots\eta(\cos^2\theta_{n-1})(\cos 2\theta_n + \sin 3\theta_n) \tag{1.34}$$

where η is a smooth cut-off function satisfying $0 \le \eta \le 1$; $\eta(t) = 1$ if $|t| < \frac{1}{2}$ and $\eta(t) = 0$ if $|t| > \frac{3}{4}$. One finds that

$$\int_{\mathbb{S}^n} x_j g(x) = 0, \ \ \forall\, 1 \le j \le n+1, \quad \int_{\mathbb{S}^n} x_{n+1} g^2(x) \ne 0. \tag{1.35}$$

Note that the linear elliptic operator L defined by $L(v) = \sigma_1(\nabla^2 v + v\sigma)$ is self-adjoint with kernel $K_1 = \mathrm{Span}(x_1, \ldots, x_{n+1})$. As g is orthogonal to the kernel of L, there exists $v \in C^\infty(\mathbb{S}^n)$ satisfying the equation

$$\sigma_1(\nabla^2 v + v\sigma) = g \ \text{ on } \mathbb{S}^n. \tag{1.36}$$

By (1.35), we see from (1.26)-(1.30) that $u_t = 1 + tv$ satisfies (1.32) for all $t > 0$ sufficiently small, provided that $m \ne \frac{n+k}{n-k}$.

Turning to the case $m = \frac{n+k}{n-k}$, we take $v = x_1^l$ where $l > 1$ is an odd integer. For $t > 0$ sufficiently small, the function $u_t = 1 + tv$ then is the supporting function of a surface of revolution. For convenience we write $\theta = \theta_1$ and, therefore, $x_1 = \cos\theta$, $0 \le \theta \le \pi$. Using a formula in [**47**] with some simplification, we obtain

$$\begin{aligned}
\sigma_1 &= \frac{1-l}{n}(n\cos^2\theta - l\sin^2\theta)\cos^{l-2}\theta,\\
\sigma_2 &= \frac{(1-l)^2}{n}(n\cos^2\theta - 2l\sin^2\theta)\cos^{2l-2}\theta.
\end{aligned}$$

It follows that

$$3\sigma_1\sigma_2 - 2\sigma_1^3 = \frac{(1-l)^3}{n^3}(n^3\cos^6\theta - 3n^2 l\cos^4\theta\sin^2\theta + 2l^3\sin^6\theta)\cos^{3l-6}\theta.$$

We calculate

$$\begin{aligned}
\int_{\mathbb{S}^n} x_1(3\sigma_1\sigma_2 - 2\sigma_1^3)d\sigma &= c_1\int_0^\pi (3\sigma_1\sigma_2 - 2\sigma_1^3)\sin^{n-1}\theta\cos\theta d\theta\\
&= c_2\int_0^\pi (n^3\cos^6\theta - 3n^2 l\cos^4\theta\sin^2\theta + 2l^3\sin^6\theta)\cos^{3l-5}\theta\sin^{n-1}\theta d\theta\\
&= n^2 c_2\int_0^\pi (n\cos^{3l+1}\theta\sin^{n-1}\theta - 3l\cos^{3l-1}\theta\sin^{n+1}\theta)d\theta\\
&\quad + 2l^3 c_2\int_0^\pi \cos^{3l-5}\theta\sin^{n+5}\theta d\theta\\
&= 2l^3 c_2\int_0^\pi \cos^{3l-5}\theta\sin^{n+5}\theta d\theta < 0
\end{aligned}$$

since

$$\int_0^\pi (n\cos^{3l+1}\theta\sin^{n-1}\theta - 3l\cos^{3l-1}\theta\sin^{n+1}\theta)d\theta = \cos^{3l}\theta\sin^n\theta\Big|_0^\pi = 0$$

and $l > 1$ is an odd integer, where c_1 is a positive constant (equal to the volume of $\mathbb{S}^{n-1}$) and $c_2 = \frac{c_1(1-l)^3}{n^3} < 0$. ¿From (1.26)-(1.31) it follows that u_t satisfies (1.32) for all $t > 0$ sufficiently small. ■

REMARK 1.1. *In the case $m = \frac{n+k}{n-k}$, u_t constructed in the proof of Proposition 1.2 is the support function of a surface of revolution. Similar construction can also be done for $m \neq \frac{n+k}{n-k}$. It follows from the proof of Proposition 1.2 that the linearized operator L_{u_t} of $S^m_{n,k}$ at u_t is not self-adjoint with respect to the standard metric on $\mathbb{S}^n$. We complement this with the following observation. Suppose w is a positive function defined on $\mathbb{S}^n$ such that*

$$\int_{\mathbb{S}^n} x_j w(x)[\sigma_{n,k}(\nabla^2 u + u\sigma)]^m = 0 \tag{1.37}$$

for all $u \in C^\infty(\mathbb{S}^n)$ with $\{\nabla^2 u + u\sigma\} > 0$, where $1 \leq j \leq n$, $1 \leq k < n$ and $m \in \mathbf{R}$, $m \neq 0$ (all are fixed). Then, for any $v \in C^2(\mathbb{S}^n)$, as the function $u_t = 1 + tv$ satisfies (1.37) for all $t > 0$ sufficiently small, we have

$$\int_{\mathbb{S}^n} x_j w(x)\sigma_1(\nabla^2 v + v\sigma) = 0$$

by (1.29) and (1.30). This implies $\Delta(x_j w) + n x_j w = 0$ on $\mathbb{S}^n$. Since the kernel of $\Delta + n$ is the linear span of $x_1, \ldots, x_n$, we see that $w \equiv const$.

Notes

If assuming the convexity, the main results in this chapter are classical, see [**3**], [**45**] and [**36**]. They were extended to k-convex case in [**69**].

Our proof of Theorem 1.2 follows the similar arguments of Alexandrov's second proof of Alexandrov-Fenchel inequality in [**3**] (see also [**81**]), which in turn is adapted from Hilbert's proof of the Brunn-Minkowski inequality in the case $n = 3$. Instead of using Alexandrov's inequality for mixed discriminants in his original proof, we make use of the hyperbolicity of the elementary symmetric functions as in [**81**].

Proposition 1.2 was proved in [**57**], where the existence and uniqueness for the prescribing Weingarten curvatures on outer normals were studied. the problem was proposed by Alexandrov [**4**] and Chern [**36**].

We note that if D is convex, D is polar centrized if and only if the Steiner point of the polar of D is the origin. If D is convex, $V_l^*(D)$ in Definition 1.2 is the is the reciprocal of the lth quermassintegral of the polar of D. The geometric quantities of D and its polar D^* in this case are related by some important inequalities, like Blascke-Santalo inequality, Mahler's conjecture. When D is a centrally symmetric convex body and $l = n + 1$, by the work of [**18**], $V(D)V(D^*) \geq c_n$ for some positive constant c_n depending only on the dimensionality.

CHAPTER 2

The Minkowski Problem

The Minkowski problem is the main source for the study of Monge-Ampère equation. The work of Nirenberg, Cheng-Yau and Pogorelov on the Minkowski problem led to the late development of the theory of fully nonlinear equations.

Suppose M is a closed strongly convex hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, the Gauss map $\vec{n} : M \to \mathbb{S}^n$ is a diffeomorphism, where at any point $p \in M$, $\vec{n}(p)$ is the unit outer normal at p. In this way, the Gauss curvature can be viewed as a positive function $k(\vec{n}^{-1}(x))$ on $\mathbb{S}^n$. Let us denote $\kappa = (\kappa_1, \cdots, \kappa_n)$ be the principal curvatures and $K = \kappa_1 \cdots \kappa_n$ the Guass curvature of M respectively. The Minkowski problem is a problem of prescribing Gauss curvature on the outer normals of convex hypersurfaces. To be more precise, the question is: given a positive function K on $\mathbb{S}^n$, is there a closed strongly convex hypersurface whose Gauss curvature is K as a function on its outer normals? By the Divergence Theorem, K has to satisfy equation

$$\int_{\mathbb{S}^n} \frac{x_i}{K(x)} = \int_M \vec{n} \cdot \vec{E}_i = 0, i = 1, \ldots, n+1, \tag{2.38}$$

where x_i are the coordinate functions and $\vec{E}_i$ is the standard ith coordinate vector of $\mathbb{S}^n$.

A C^2 closed hypersurface M in $\mathbb{R}^{n+1}$ is called strongly convex if its Gauss curvature is positive everywhere. By the Hadamard Theorem, M is a boundary of a convex domain. In turn, M can be parametrized by its inverse Gauss map over $\mathbb{S}^n$ with $y(x) = \vec{n}_M^{-1}(x)$. In this chapter, we prove the following theorem.

THEOREM 2.1. *Suppose $K \in C^2(\mathbb{S}^n), K(x) > 0, \quad \forall\, x \in \mathbb{S}^n$, and K satisfies equation (2.38), then there is a $C^{3,\alpha}(\forall\ 0 < \alpha < 1)$ strongly convex surface M in $\mathbb{R}^{n+1}$, such that $k(\vec{n}_M^{-1}(x)) = K(x)\ \forall\, x \in \mathbb{S}^n$. M is unique up to translations.*

1. Support function

Let M be a closed strongly convex hypersurface. The support function of M is defined as

$$u(x) = \sup_{z \in M} x \cdot z = x \cdot y(x),\ \forall x \in \mathbb{S}^n.$$

We extend u as a homogeneous function of degree one in $\mathbb{R}^{n+1} \setminus \{0\}$. It is easy to check that u is a convex function in $\mathbb{R}^{n+1}$. Since $\frac{\partial y}{\partial x_j}$ is tangent to M for all j, and $x = \vec{n}_M(y)$ is normal to M, we have $x \cdot \frac{\partial y}{\partial x_j} = 0$ for all j. It follows that

$$y(x) = \nabla_{\mathbb{R}^{n+1}} u(x). \tag{2.39}$$

Therefore, M can be recovered completely from u by above equation. The relation $y(x) = \vec{n}_M^{-1}(x)$ and (2.39) yield

$$\nabla_{\mathbb{R}^{n+1}} u(x) = \vec{n}_M^{-1}(x). \tag{2.40}$$

Equation (2.40) implies that u is C^2 if M is C^2 and its Gauss curvature is positive.

Let $e_{n+1} = x$ be the position vector on $\mathbb{S}^n$, let $e_1, \cdots, e_n$ is an orthonormal frame on $\mathbb{S}^n$ so that $e_1, \cdots, e_{n+1}$ is a positive oriented orthonormal frame in $\mathbb{R}^{n+1}$. Let ω^i and ω^i_j be the corresponding dual 1-forms and the connection forms respectively. We have

$$de_j = -\sum_{i=1}^n \omega^i_j e_i, \quad \forall j = 1, 2, \cdots, n, \quad \textit{and} \quad de_{n+1} = \sum_{i=1}^n \omega^i e_i.$$

For each function $u \in C^2(\mathbb{S}^n)$, let u_i be the covariant derivative of u with respect to e_i. We define a vector valued function

$$Y = \sum_{i=1}^n u_i e_i + u e_{n+1}.$$

We note that Y is independent of the choice of the orthonormal frames. We calculate that,

$$\begin{aligned} dY &= \sum_{i=1}^n (du_i e_i + u_i de_i) + due_{n+1} + ude_{n+1} \\ &= \sum_{i=1}^n (\sum_{j=1}^n u_{ij}\omega^j - \sum_{j=1}^n u_j \omega^i_j) e_i + \sum_{i=1}^n (\sum_{\alpha=1}^{n+1} u_i \omega^\alpha_i e_\alpha) \\ &\quad + \sum_{i=1}^n (u_i \omega^i) e_{n+1} + u \sum_{i=1}^n \omega^i e_i \\ &= \sum_{j=1}^n (\sum_{i=1}^n (u_{ij} + \delta_{ij} u) e_i) \omega^j. \end{aligned}$$

In particular, if u is a support function of M, by (2.39) the position vector of M is $y(x) = Y(x)$, that is

$$y(x) = \sum_{i=1}^n u_i e_i + u e_{n+1}.$$

In turn,

$$dy = \sum_{i,j} (u_{ij} + u\delta_{ij}) e_i \otimes \omega_j \tag{2.41}$$

The identity (2.41) indicates that the differential dy maps $T_x(\mathbb{S}^n)$ to itself and it is self-adjoint. dy is sometimes called the reverse Weingarten map. Since the Gauss curvature K is positive, the Gauss map $\vec{n}_M$ is invertible at $y = \vec{n}_M^{-1}(x)$. We have

$$dy = (d\vec{n}_M)^{-1}, \tag{2.42}$$

so that the reverse Weingarten map at x coincides with the inverse of the Weingarten map at y. Since the eigenvaules of the Weingarten map are the principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_n)$ of

M at y, the eigenvalues of reverse Weingarten map at $x = \vec{n}_M(y)$ are exactly the principal radii at y.

Conversely, if $u(x)$ is a C^2 function on $\mathbb{S}^n$ with $(u_{ij} + u\delta_{ij}) > 0$, we claim that there is a strongly convex hypersurface M such that its support function is u. Again, we extend u as a homogeneous function of degree one in $\mathbb{R}^{n+1} \setminus \{0\}$. It is clear that M should be defined as in (2.39), that is,

$$M = \{\nabla_{\mathbb{R}^{n+1}} u(x) | x \in \mathbb{R}^{n+1} \setminus \{0\}\} = \{\sum_{i=1}^{n} u_i(x)e_i(x) + u(x)e_{n+1}(x) | x \in \mathbb{S}^n\}. \tag{2.43}$$

Since $(u_{ij} + u\delta_{ij}) > 0$ is non-singular, we may read off from (2.41) that the tangent space of M in $\mathbb{R}^{n+1}$ at $y(x) = \sum_{i=1}^{n} u_i(x)e_i(x) + u(x)e_{n+1}(x)$ is $span\{e_1, \cdots, e_n\}$. Moreover, from $\det(u_{ij} + u\delta_{ij}) > 0$ and

$$dy \wedge \cdots \wedge dy \wedge e_{n+1} = \det(u_{ij} + u\delta_{ij}) d\omega_1 \wedge \cdots \wedge d\omega_n,$$

we conclude that $e_{n+1} = x$ is a normal vector at $y(x) = \sum_{i=1}^{n} u_i(x)e_i(x) + u(x)e_{n+1}(x)$ of M. This provides a global orientation of M and also gives a global inverse of the map from M (defined in (2.43)) to $\mathbb{S}^n$. That is, the map $y(x) = \sum_{i=1}^{n} u_i(x)e_i(x) + u(x)e_{n+1}(x)$ is globally invertible and M is an embedded hypersurface in $\mathbb{R}^{n+1}$. Equation (2.43) implies $u(x) = x \cdot y(x)$. By (2.42), the principal curvatures at y is exactly the reciprocals of the eigenvalues of $(u_{ij} + u\delta_{ij})$. In particular, the Gauss curvature of M does not vanish. Because M is a compact hypersurface, the Gauss curvature is positive at some point, therefore must be positive at every point. By the Hadamard Theorem, M is strongly convex. And $u(x) = x \cdot y(x) = x \cdot \vec{n}_M^{-1}(x)$ is the support function of M.

In summary, we have proved the following proposition.

Proposition 2.1. *A strongly convex hypersurface M in $\mathbb{R}^{n+1}$ is C^2 if and only if its support function u is in $C^2(\mathbb{S}^n)$ with $(u_{ij} + u\delta_{ij}) > 0$. The eigenvalues of $(u_{ij} + u\delta_{ij})$ are the principal radii of M (parametrized by the inverse Gauss map over $\mathbb{S}^n$). In particular, the Gauss curvature K of M satisfies equation*

$$\det(u_{ij} + u\delta_{ij}) = \frac{1}{K}, \quad on \quad \mathbb{S}^n. \tag{2.44}$$

Furthermore, any function $u \in C^2(\mathbb{S}^n)$ with $(u_{ij} + u\delta_{ij}) > 0$ is a support function of a C^2 strongly convex hypersurface M in $\mathbb{R}^{n+1}$.

From the above discussion, the support function carries all the information of M. Let Ω the convex body bounded by M. The kth quermassintegral $\mathcal{W}_k(\Omega)$ is defined to be the average over the Grassmannian manifold $G(n+1, k)$ of the k-dimensional volume of the projections of Ω into k hyperplanes in $\mathbb{R}^{n+1}$. The Cauchy-Crofton formula (e.g., see [**110**]) yields

$$\mathcal{W}_k(\Omega) = c_{n,k} \int_{\mathbb{S}^n} u\sigma_{k-1}(u_{ij} + u\delta_{ij}), \tag{2.45}$$

where $c_{n,k}$ is a dimensional constant, u is the support function of the boundary of Ω and σ_l is the lth elementary symmetric function. $\mathcal{W}_{n+1}(\Omega)$ and $\mathcal{W}_n(\Omega)$ are the volume of Ω and surface area of M respectively, and $\mathcal{W}_1(\Omega)$ is the mean width of Ω. Moreover, the Alexandrov-Fenchel

quermassintegral inequality in the previous chapter states that for $1 \le l \le k \le n+1$, there is a constant C depending only on l, k, n such that

$$\mathcal{W}_k^{\frac{1}{k}}(\Omega) \le C\mathcal{W}_l^{\frac{1}{l}}(\Omega), \tag{2.46}$$

the equality holds if and only if Ω is a ball.

To conclude this section, we note that we have reduced the Minkowski problem to equation (2.44). The uniqueness part of Theorem 2.1 is implied in Theorem 1.1. Moreover, Proposition 1.1 and the standard Implicit Function Theorem imply the openness of solutions to equation (2.44).

2. A priori estimates

We want to complete the proof of Theorem 2.1 using the continuity method. Here we need to show the closeness, that is, to prove some a priori regularity estimates for equation (2.44). Since equation (2.44) is elliptic at any admissible u, and $\det^{\frac{1}{n}}(W)$ is concave, the higher regularity estimates follow from the Evans-Krylov Theorem and the standard elliptic theory if we have a prior bounds upto the second derivatives of solutions. Therefore, our focus here is to derive C^2 a priori estimates for equation (2.44).

For a solution u of equation (2.44), $u + \sum_{i=1}^{n+1} a_i x_i$ is also a solution. By proper choice of $\{a_i\}_{i=1}^n$, we may assume that u satisfies the following orthogonality condition:

$$\int_{\mathbb{S}^n} x_i u \, dx = 0, \qquad \forall i = 1, 2, ..., n+1. \tag{2.47}$$

If u is a support function of a closed hypersurface M which bounds a convex body Ω, condition (2.47) implies that the Steiner point of Ω coincides with the origin.

We first estimate the extrinsic diameter of M.

Lemma 2.1. *Let $M \in C^2$, M be a closed convex hypersurface in $\mathbb{R}^{n+1}$, and let φ be the k-th surface area function of M. If L is the extrinsic diameter of M, then*

$$L \le c_{n,k} \left(\int_{\mathbb{S}^n} \varphi \right)^{\frac{k+1}{k}} \left(\inf_{y \in \mathbb{S}^n} \int_{\mathbb{S}^n} \max(0, \langle y, x \rangle) \varphi(x) \right)^{-1},$$

where $c_{n,k}$ is a constant depending only on n and k. In particular, if u is a support function of M satisfying (2.44) and (2.47), then

$$0 \le \min u \le \max u \le c_{n,k} \left(\int_{\mathbb{S}^n} \varphi \right)^{\frac{k+1}{k}} \left(\inf_{y \in \mathbb{S}^n} \int_{\mathbb{S}^n} \max(0, \langle y, x \rangle) \varphi(x) \right)^{-1}.$$

Proof. Let $p, q \in M$ such that the line segment joining p and q has length L. We may assume 0 is in the middle of the line segment. Let $\vec{y}$ be a unit vector in the direction of this line. Let v be the support function and $W = \{v_{ij} + v\delta_{ij}\}$. We have $\sigma_k(W) = \varphi$. Now, for $x \in \mathbb{S}^n$, we get

$$v(x) = \sup_{Z \in M} \langle Z, x \rangle \ge \frac{1}{2} L \max(0, \langle y, x \rangle).$$

If we multiply by φ and integrate over $\mathbb{S}^n$, we get

$$L \leq 2\left(\int_{\mathbb{S}^n} v\varphi\right)\left(\int_{\mathbb{S}^n} \max(0,\langle y,x\rangle)\varphi\right)^{-1}.$$

By the Quermassintegral inequality (2.46),

$$(\int_{\mathbb{S}^n} v\sigma_k(W))^{\frac{1}{k+1}} \leq C_{n,k}(\int_{\mathbb{S}^n} v\sigma_{k-1}(W))^{\frac{1}{k}}.$$

On the other hand, from a Minkowski type formula (1.7), we have

$$(n-k+1)\int_{\mathbb{S}^n} v\sigma_{k-1}(W) = k\int_{\mathbb{S}^n}\sigma_k(W) = k\int_{\mathbb{S}^n}\varphi.$$

In turn, we get

$$L \leq c_{n,k}\left(\int_{\mathbb{S}^n}\varphi\right)^{\frac{k+1}{k}}\left(\inf_{y\in\mathbb{S}^n}\int_{\mathbb{S}^n}\max(0,(y,x))\varphi\right)^{-1}.$$

If u satisfies (2.47), the Steiner point of M is the origin. The last inequality is a consequence of the above inequality. ■

PROPOSITION 2.2. *There is a constant $C>0$ depending only on n, k, $\|\varphi\|_{C^2(\mathbb{S}^n)}$ and $\min_{\mathbb{S}^n}\varphi$, such that if u satisfies (2.47) and u is a solution of (2.44), then $\|u\|_{C^2(\mathbb{S}^n)} \leq C$. There is an explicit bound for the function $H := trace(u_{ij}+\delta_{ij}u) = \triangle u + nu$,*

$$\min_{x\in\mathbb{S}^n}(n\tilde{\varphi}(x)) \leq \max_{x\in\mathbb{S}^n} H(x) \leq \max_{x\in\mathbb{S}^n}(n\tilde{\varphi}(x) - \triangle\tilde{\varphi}(x)), \tag{2.48}$$

where $\tilde{\varphi} := \varphi^{\frac{1}{n}}$.

PROOF. Since the entries $|u_{ij}+\delta_{ij}u|$ are controlled by eigenvalues $\{\lambda_i\}_{i=1}^n$ of $W=(u_{ij}+\delta_{ij}u)$. Since $W>0$, $\lambda_i < H, \forall i = 1,\cdots,n$.

By Lemma 2.1, we have a C^0 bound on u. So the $|u_{ij}|$ are controlled by H. C^1 estimates follows from interpolation if we have bounds on the second derivatives. Therefore, we only need to bound H. Assume the maximum value of H is attained at a point $x_0 \in \mathbb{S}^n$. We choose an orthonormal local frame $e_1, e_2, ..., e_n$ near x_0 such that $u_{ij}(x_0)$ is diagonal. We define $G(W) := {\sigma_n}^{\frac{1}{n}}(W)$. Then equation (2.44) becomes

$$G(W) = \tilde{\varphi}. \tag{2.49}$$

For the standard metric on $\mathbb{S}^n$, one may easily check the commutator identity $H_{ii} = \triangle W_{ii} - nW_{ii} + H$. By the assumption that the matrix $W \in \Gamma_k$, so (G^{ij}) is positive definite. Since $(H_{ij}) \leq 0$, and (G^{ij}) is diagonal, by the above commutator identity, it follows that at x_0,

$$0 \geq G^{ij}H_{ij} = G^{ii}(\triangle W_{ii}) - nG^{ii}W_{ii} + H\sum_i^n G^{ii}. \tag{2.50}$$

As G is homogeneous of degree one, we have

$$G^{ii}W_{ii} = \tilde{\varphi}. \tag{2.51}$$

Next we apply the Laplace operator to equation (2.49) to obtain

$$G^{ij}W_{ijk} = \nabla_k \tilde{\varphi}, \quad G^{ij,rs}W_{ijk}W_{rsk} + G^{ij}\triangle W_{ij} = \triangle\tilde{\varphi}.$$

By the concavity of G, at x_o we have

$$G^{ii}\triangle(W_{ii}) \geq \triangle\tilde{\varphi}. \tag{2.52}$$

Combining (2.51), (2.52) and (2.50), we see that

$$0 \geq \triangle\tilde{\varphi} - n\tilde{\varphi} + H\sum_{i=1}^{n} G^{ii}. \tag{2.53}$$

As W is diagonal at the point, we may write $W = (W_{11}, ..., W_{nn})$ as a vector in $\mathbb{R}^n$. A simple calculation yields

$$G^{ii} = \sigma_n(W)^{\frac{1}{n}-1}\sigma_{n-1}(W|i),$$

where $(W|i)$ is the vector given by W with W_{ii} deleted. It follows from the Newton-MacLaurin inequality that

$$\sum_{i=1}^{n} G^{ii} = \sigma_k(W)^{\frac{1}{k}-1}\sigma_{n-1}(W) \geq 1.$$

By (2.53), we have $H \leq n\tilde{\varphi} - \triangle\tilde{\varphi}$. ■

By the Evans-Krylov Theorem and the standard elliptic theory, together with Proposition 2.2, we have the following a priori estimates.

Proposition 2.3. *For each integer $l \geq 1$ and $0 < \alpha < 1$, there exist a constant C depending only on $n, l, \alpha, \min\varphi$, and $||\varphi||_{C^{l,1}}(\mathbb{S}^n)$ such that*

$$||u||_{C^{l+1,\alpha}}(\mathbb{S}^n) \leq C, \tag{2.54}$$

for all admissible solution of (2.44) satisfying the condition (2.47).

We now precede to prove Theorem 2.1 using the method of continuity. By Proposition 2.3. we may assume that K in equation (2.44) is C^∞. Let $H^m(\mathbb{S}^n)$ be the Sobolev space. We pick m sufficient large so that $H^m(\mathbb{S}^n) \subset C^4(\mathbb{S}^n)$. We define

$$\mathcal{S}_m = \{f \in H^m(\mathbb{S}^n) | \int_{\mathbb{S}^n} f(x)x_i = 0, \forall i = 1, \cdots, n+1\},$$

and we define a nonlinear operator on $\mathcal{S}_m$

$$F(u) =: \det(u_{ij} + u\delta_{ij}).$$

For $\forall u \in C^2(\mathbb{S}^n)$ and for each i, if we let $u_1 = x_i$ and $u_j = u$ for $j > 1$, Lemma 1.1 implies that $\int_{\mathbb{S}^n} x_i F(u) = 0$. Therefore,

$$F : \mathcal{S}_m \to \mathcal{S}_{m-2}.$$

For any convex u, let L_u be the linearized operator of F at u. By Proposition 1.1,

$$Range(L_u) = (Ker(L_u^*))^\perp = (Span(x_1, \cdots, x_{n+1}))^\perp.$$

That means that L_u is surjective. The standard Implicit Function Theorem yields that F is locally invertible near u. For $0 \leq t \leq 1$, define

$$E = \{0 \leq t \leq 1 | F(u) = 1 - t + \frac{t}{K} \quad \text{has an admissible solution}\}.$$

So E is open and $E \neq \emptyset$ since $0 \in E$. Proposition 2.3 implies E is closed. Hence $E = [0, 1]$, so the existence of the solution to the Minkowski problem is proved. The uniqueness follows from in Theorem 1.1.

Notes

The Minkowski problem was considered by Minkowski in [**95**] 1897. The differential geometric setting of the problem in this chapter was solved in early 1950s by Nirenberg [**97**] and Pogorelov [**105**] for $n = 2$. The solution of the Minkowski problem in higher dimension came much later in 1970s by Cheng-Yau [**35**] and Pogorelov [**107**]. The proof of Theorem 2.1 in this chapter follows mainly from that of Cheng-Yau in [**35**], see also a forthcoming book "Isometric embeddings".

CHAPTER 3

The Christoffel-Minkowski problem, admissible solutions

We deal with the Christoffel-Minkowski problem in this chapter. For each convex body $\Omega \subset \mathbb{R}^{n+1}$ induces a kth area measure on $\mathbb{S}^n$ by $dA_\Omega = \sigma_k(u_{ij} + u\delta_{ij})d\sigma_{\mathbb{S}^n}$, where u is a support function of Ω and $d\sigma_{\mathbb{S}^n}$ is the standard volume form on $\mathbb{S}^n$. The Christoffel-Minkowski problem is the problem of finding a convex body with its kth area measure is prescribed on $\mathbb{S}^n$. It leads to the following equation on $\mathbb{S}^n$:

$$\sigma_k(u_{ij} + u\delta_{ij}) = \varphi \quad \text{on} \quad \mathbb{S}^n. \tag{3.1}$$

In order to solve Christoffel-Minkowski problem, we want to find a solution of equation (3.1) with the following convex condition:

$$(u_{ij} + u\delta_{ij}) > 0, \quad \text{on} \quad \mathbb{S}^n, \tag{3.2}$$

Our main interest of this chapter is to understand existence and uniqueness of **admissible** solutions. We will treat some general fully nonlinear equations on $\mathbb{S}^n$. In particular, we will establish general existence and uniqueness of **admissible** solutions of equation (3.1).

If $\forall v \in C^2(\mathbb{S}^n)$, it is necessary that

$$\int_{\mathbb{S}^n} x_m \sigma_k(v_{ij}(x) + v(x)\delta_{ij})dx = 0, \qquad \forall m = 1, 2, ..., n+1. \tag{3.3}$$

In order that equation (3.1) to have a solution, it is necessary that

$$\int_{\mathbb{S}^n} x_i \varphi(x)dx = 0, \qquad \forall i = 1, 2, ..., n+1. \tag{3.4}$$

The class of quotient equations is also important: $(0\lambda < k \leq n)$

$$\frac{\sigma_k(W)}{\sigma_l(W)} = \varphi, \quad \text{on} \quad \mathbb{S}^n, \tag{3.5}$$

where $W = (u_{ij} + u\delta_{ij})$. When $l = 0$, equation (3.5) is the same as equation (3.1). In special case $k = n$, the equation is related to the problem of prescribing Weingarten curvature posted by Alexandrov and Chern (see [**57**]). When $1 \leq l < k < n$, like equation (3.1), (3.5) is fully nonlinear. In this aspect, it is similar to the Monge-Ampère equation. But there is an essential difference: the class of *convex* functions is not a natural class of solutions of equation (3.1). By Corollary 13.1, the elementary symmetric functions σ_k are hyperbolic polynomials defined for symmetric matrices. For each σ_k, there is a connect cone Γ_k containing the identity matrix such that σ_k is positive, $(\frac{\partial \sigma_k}{u_{ij}})$ is positive definite and $S_k^{\frac{1}{k}}$ is concave in the cone. Let $\mathcal{S}$ be the space consisting all $n \times n$ symmetric matrices. For any symmetric matrix $A \in \mathcal{S}$, $\sigma_k(A)$ is defined to

be $\sigma_k(\lambda)$, where $\lambda = (\lambda_1, ..., \lambda_n)$ are the eigenvalues of A. Γ_k can be written equivalently as the connected cone in $\mathcal{S}$ containing the identity matrix determined by

$$\Gamma_k = \{A \in \mathcal{S} : \quad \sigma_1(A) > 0, ..., \sigma_k(A) > 0\}. \tag{3.6}$$

We note that k-convex functions are the natural class of functions where equations (3.1) and (3.5) is defined and elliptic.

We now consider existence of admissible solutions of general fully nonlinear equations on $\mathbb{S}^n$. We will establishment of some appropriate estimates for **admissible** solutions of equations under some structural conditions. The existence problem is closely related to the uniqueness of some **particular** constant solution of the equation. Equation (3.1) is among this type of equations. For these equations, the uniqueness in general setting is a difficulty issue and the continuity method does not work well. Instead, degree theory is more suitable in many cases (e.g., see [**57**]). For example, degree theory can be used if one can isolate constant solutions of the equation. This is why the uniqueness of the constant solutions comes in to the picture of the existence.

The following is the existence result for equation (3.1).

THEOREM 3.1. **(Existence)** *Let $\varphi(x) \in C^{1,1}(\mathbb{S}^n)$ be a positive function, suppose φ satisfies (3.4), then equation (3.1) has a solution. More precisely, there exist constant C depending only on $n, \alpha, \min\varphi$, and $||\varphi||_{C^{1,1}}(\mathbb{S}^n)$ and a $C^{3,\alpha}$ ($\forall\ 0 < \alpha < 1$) k-convex solution u of (3.1) such that:*

$$||u||_{C^{3,\alpha}}(\mathbb{S}^n) \leq C. \tag{3.7}$$

Furthermore, if $\varphi(x) \in C^{l,\gamma}(\mathbb{S}^n)$ ($l \geq 2, \gamma > 0$), then u is $C^{2+l,\gamma}$. If φ is analytic, u is analytic.

We first establish the a priori estimates for admissible solutions of equation (3.1). We note that for any solution $u(x)$ of (3.1), $u(x) + l(x)$ is also a solution of the equation for any linear function $l(x) = \sum_{i=1}^{n+1} a_i x_i$. We will confine ourselves to solutions satisfying the following orthogonal condition

$$\int_{\mathbb{S}^n} x_i u \, dx = 0, \qquad \forall i = 1, 2, ..., n+1. \tag{3.8}$$

When u is convex, it is a support function of some convex body Ω. Condition (3.8) implies that the Steiner point of Ω coincides with the origin.

Here we establish a priori estimates for *admissible* solutions. We note equation (3.1) will be uniformly elliptic once C^2 estimates are established for u. By the Evans-Krylov Theorem and the Schauder theory, one may obtain higher derivative estimates for u. Therefore, we only need to get C^2 estimates for u.

In fact, the a priori estimates we will prove are valid for a general class of fully nonlinear elliptic equations on $\mathbb{S}^n$. We consider the following equation:

$$F(u_{ij} + u\delta_{ij}) = \tilde{\varphi} \quad \text{on} \quad \mathbb{S}^n. \tag{3.9}$$

DEFINITION 3.1. *We say a function $u \in C^2(\mathbb{S}^n)$ is Γ-admissible if $W(x) = (u_{ij}(x) + \delta_{ij}u(x)) \in \Gamma$ for all $x \in \mathbb{S}^n$. If u is Γ-admissible and satisfies equation (3.9), we call u an admissible solution of (3.9).*

We now switch our attention to a priori estimates of solutions of equation (3.9).

We will obtain an upper bound on the largest eigenvalue of the matrix $(u_{ij} + \delta_{ij}u)$ first. We then come back to deal with the C^0 bound.

PROPOSITION 3.1. *Suppose F satisfies the structural conditions (13.11)- (13.14), suppose $u \in C^4(\mathbb{S}^n)$ is an admissible solution of equation (3.9), then there is $C > 0$ depending only on $F(I)$ in (13.15), δ in (13.14) and $\|\varphi\|_{C^2}$ such that*

$$0 < \lambda_{\max} \leq C, \tag{3.10}$$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix $(u_{ij} + \delta_{ij}u)$. In particular, for any eigenvalue $\lambda_i(x)$ of $(u_{ij}(x) + \delta_{ij}u(x))$,

$$|\lambda_i(x)| \leq (n-1)C, \quad \forall x \in \mathbb{S}^n. \tag{3.11}$$

Proof. When $F = \sigma_k^{\frac{1}{k}}$ and u is *convex*, this is the Pogorelov type estimates. Here we will deal with general admissible solutions of F under the structure conditions. It seems that the moving frames method is more appropriate for equation (3.9) on $\mathbb{S}^n$.

(3.11) follows from (3.10) and the fact $\Gamma \subset \Gamma_1$. Also the positivity of $\lambda_{\max}$ follows from the assumption that $\Gamma \subset \Gamma_1$. We need to estimate the upper bound of $\lambda_{\max}$. Assume the maximum value of $\lambda_{\max}$ is attained at a point $x_0 \in \mathbb{S}^n$ and in the direction e_1, so we can take $\lambda_{\max} = W_{11}$ at x_0. We choose an orthonormal local frame $e_1, e_2, ..., e_n$ near x_0 such that $u_{ij}(x_0)$ is diagonal, so $W = \{u_{ij} + \delta_{ij}u\}$ is also diagonal at x_0.

For the standard metric on $\mathbb{S}^n$, we have the following commutator identity

$$W_{11ii} = W_{ii11} - W_{ii} + W_{11}.$$

By the assumption, (F^{ij}) is positive definite. Since $W_{11ii} \leq 0$ at x_0, , it follows that at this point

$$0 \geq F^{ii}W_{11ii} = F^{ii}W_{ii11} - F^{ii}W_{ii} + W_{11}F^{ii}. \tag{3.12}$$

By concavity condition (13.13),

$$\sum_i F^{ii}(W)W_{ii} \leq \sum_i F^{ii}(W) + F(W) - F(I) = \sum_i F^{ii}(W) + \tilde{\varphi} - F(I). \tag{3.13}$$

Next we apply the twice differential in the e_1 direction to equation (3.9), we obtain

$$F^{ij}W_{ijk1} = \nabla_1\tilde{\varphi},$$
$$F^{ij,rs}W_{ij1}W_{rs1} + F^{ij}W_{ij11} = \tilde{\varphi}_{11}.$$

By the concavity of F, at x_0 we have

$$F^{ii}W_{ii11} \geq \tilde{\varphi}_{11}. \tag{3.14}$$

Combining (3.13), (3.14) and (3.12), we see that

$$0 \geq \tilde{\varphi}_{11} - \sum_i F^{ii} - \tilde{\varphi} + W_{11}\sum_{i=1}^n F^{ii} + F(I).$$

By assumption, $\tilde{\varphi} \leq M$ for some $M > 0$. From condition (13.14), $\sum_{i=1}^n F^{ii} \geq \delta_M > 0$. It follows that $W_{11} \leq C$. ■

COROLLARY 3.1. *If $u \in C^4(\mathbb{S}^n)$ is an admissible solution of equation (3.1) (so $W(x) = (u_{ij}(x) + u(x)\delta_{ij}) \in \Gamma_k, \forall x \in \mathbb{S}^n$), then $0 < \max_{x\in\mathbb{S}^n} \lambda_{\max}(x) \leq C$.*

In order to obtain a C^2 bound, we need a C^0 bound for u. Here, we use the a priori bounds in Proposition 3.1 to get a C^0 bound for general admissible solutions of equation (3.9).

LEMMA 3.1. *For any Γ-admissible function u, there is a constant C depending only on n, $\max_{x\in\mathbb{S}^n} \lambda_{\max}(x)$ and $\max_{\mathbb{S}^n} |u|$ such that,*

$$\|u\|_{C^2} \leq C. \tag{3.15}$$

Proof. The bound on the second derivatives follows directly the fact $W(x) = (u_{ij}(x)+\delta_{ij}u(x)) \in \Gamma \subset \Gamma_1$. The bound on the first derivatives follows from interpolation. ■

Now we establish the C^0-estimate. The proof is based on a rescaling argument.

PROPOSITION 3.2. *Suppose F satisfies structure conditions (13.11)-(13.14). If u is an admissible solution of equation (3.9) and u satisfies (3.8), then there exists a positive constant C depending only on $n, k, \|\tilde{\varphi}\|_{C^2}$ and F such that,*

$$||u||_{C^2} \leq C. \tag{3.16}$$

Proof. We only need to get a bound on $\|u\|_{C^0}$. Suppose there is no such bound, then $\exists u^l (l = 1, 2, ...)$ satisfying (3.8), there is a constant $\tilde{C}$ independent of l, and $F(W^l) = \tilde{\varphi}^l$ (where $W^l = (u^l_{ij} + \delta_{ij}u^l)$), with $\tilde{\varphi}^l$ satisfies

$$||\tilde{\varphi}^l||_{C^2} \leq \tilde{C}, \quad \sup \tilde{\varphi} \leq 1, \quad ||u^l||_{L^\infty} \geq l.$$

Let $v^l = \frac{u^l}{||u^l||_{L^\infty}}$, then

$$||v^l||_{L^\infty} = 1. \tag{3.17}$$

By Proposition 3.1, we have for any eigenvalue $\lambda_i(W^l(x))$ of $W^l(x)$,

$$|\lambda_i(W^l(x))| \leq (n-1)\lambda_{\max}(W^l) \leq C, \tag{3.18}$$

where $\lambda_{\max}(W^l)$ is the maximum of the largest eigenvalues of W^l on $\mathbb{S}^n$ and the constant C is independent of l. Let $\tilde{W}^l = (v^l_{ij} + \delta_{ij}v^l)$ and from (3.18) v^l satisfies the following estimates

$$|\lambda_i(\tilde{W}^l(x))| \leq (n-1)\lambda_{\max}(\tilde{W}^l) \leq \frac{C}{||u^l||_{L^\infty}} \longrightarrow 0. \tag{3.19}$$

In particular, $\Delta v^l + nv^l \to 0$.

On the other hand, by Lemma 3.1, (3.17) and (3.19), we have

$$||v^l||_{C^2} \leq C.$$

Hence, there exists a subsequence $\{v^{l_i}\}$ and a function $v \in C^{1,\alpha}(\mathbb{S}^n)$ satisfying (3.8) such that

$$v^{l_i} \longrightarrow v \quad \text{in} \quad C^{1,\alpha}(\mathbb{S}^n), \quad \text{with} \quad ||v||_{L^\infty} = 1. \tag{3.20}$$

In the distribution sense we have

$$\Delta v + nv = 0 \quad \text{on} \quad \mathbb{S}^n.$$

By linear elliptic theory, v is in fact smooth. Since v satisfies (3.8), we conclude that, $v \equiv 0$ on $\mathbb{S}^n$. This is a contradiction to (3.20). ■

The higher regularity would follow from the Evans-Krylov Theorem and the Schauder theory if we can ensure the uniform ellipticity for equation (3.9). That can be guaranteed by the following condition,

$$\overline{\lim}_{W\to\partial\Gamma} F(W) = 0. \tag{3.21}$$

THEOREM 3.2. *Suppose F satisfies the structure conditions (13.11)-(13.14) and condition (3.21), and $\tilde{\varphi} > 0$ on $\mathbb{S}^n$, then for each $0 < \alpha < 1$, there exists a constant C depending only on $n, \alpha, \min\tilde{\varphi}, \|\tilde{\varphi}\|_{C^{1,1}}(\mathbb{S}^n)$ and F such that*

$$||u||_{C^{3,\alpha}}(\mathbb{S}^n) \leq C, \tag{3.22}$$

for all admissible solution u of (3.9) satisfying (3.8). If in addition $F \in C^l$ for some $l \geq 2$, then there exists a constant C depending only on $n, l, \alpha, \min\tilde{\varphi}, \|\tilde{\varphi}\|_{C^{l,1}}(\mathbb{S}^n)$ and F such that

$$||u||_{C^{l+1,\alpha}}(\mathbb{S}^n) \leq C. \tag{3.23}$$

In particular, the estimate (3.23) is true for any admissible solution of (3.1) and (3.8) with $\tilde{\varphi} = \varphi^{\frac{1}{k}}$.

Proof. We verify that equation (3.9) is uniformly elliptic. By Proposition 3.2 and condition (3.21), the set $\{W(x) \in \Gamma| \quad F(W(x)) = \tilde{\varphi}(x), \forall x \in \mathbb{S}^n\}$ is compact in Γ. Since $F \in C^1$, equation (3.9) is uniformly elliptic by condition (13.12). ■

We establish existence result for equation (3.1). With the a priori estimates just proved. We will use degree theory argument for the existence. In fact, the argument applies to equation (3.9). In order to compute the degree, we need some uniqueness result. The following uniqueness result is known as when u is a support function of some convex body, e.g., by Alexandrov's moving planes method. But we need to treat the uniqueness problem for general admissible solutions. Here we use a simple a priori estimates argument to obtain a general uniqueness result in this direction.

PROPOSITION 3.3. *Suppose that F satisfies condition (13.12) and (13.13). Assume that*

$$\sum_{i,j} F^{ij}(W)W_{ij} > 0 \quad \text{for each } W \in \Gamma \text{ with } F(W) = F(I). \tag{3.24}$$

If u is an admissible solution of equation of the following equation

$$F(u_{ij} + \delta_{ij}u) = F(I) \quad \text{on} \quad \mathbb{S}^n, \tag{3.25}$$

then $u = 1 + \sum_{j=1}^{n+1} a_j x_j$ for some constants $a_1, \cdots, a_{n+1}$.

Proof. By concavity, for $W = (W_{ij}) \in \Gamma$,

$$F(I) \leq F(W) + \sum_{i,j} F^{ij}(W)(\delta_{ij} - W_{ij}) = F(W) + \sum_{i}^{n} F^{ii}(W) - \sum_{i,j}^{n} F^{ij}(W)W_{ij}. \tag{3.26}$$

Also by the symmetry, $F^{11}(I) = \cdots = F^{nn}(I) = \frac{\sum_{i=1}^n F^{ii}(I)}{n}$.

If u is an admissible solution of (3.25), we know $u \in C^2$ by definition. By the Evans-Krylov Theorem and the Schauder theory, $u \in C^\infty$. Let $W(x) = (u_{ij}(x) + \delta_{ij}u(x))$ and $H(x) = traceW(x) = \Delta u(x) + nu(x)$. Since $F^{jj}(I) = \frac{\sum_{i=1}^n F^{ii}(I)}{n}, \forall j$, by concavity, for all $x \in \mathbb{S}^n$,

$$F(W(x)) \leq F(I) + \sum_{i,j} F^{ij}(I)(W_{ij}(x) - \delta_{ij}) = F(I) + \frac{\sum_{i=1}^n F^{ii}(I)}{n} H(x) - \sum_{i=1}^n F^{ii}(I).$$

As $F(W(x)) = F(I)$ and $\sum_{i=1}^n F^{ii}(I) > 0$, we get

$$H(x) \geq n, \quad \forall x \in \mathbb{S}^n. \tag{3.27}$$

We want to show $H(x) \leq n$ for all $x \in \mathbb{S}^n$. Assume the maximum value of $H(x)$ is attained at a point $x_0 \in \mathbb{S}^n$. We choose an orthonormal local frame $e_1, e_2, ..., e_n$ near x_0 such that $u_{ij}(x_0)$ is diagonal, so $W = \{u_{ij} + \delta_{ij}u\}$ is also diagonal at x_0. For the standard metric on $\mathbb{S}^n$, we have the following commutator identity

$$H_{ii} = \Delta W_{ii} - nW_{ii} + H.$$

Since $F(W(x)) = F(I)$, it follows from (3.26) that $\sum_{i=1}^n F^{ii}(W) \geq \sum_{i=1}^n F^{ii}(W)W_{ii}$. As $H_{ii} \leq 0$ at x_0,

$$\begin{aligned} 0 &\geq \sum_{i=1}^n F^{ii}(W)H_{ii} = \sum_{i=1}^n F^{ii}(W)\Delta W_{ii} - n\sum_{i=1}^n F^{ii}(W)W_{ii} + H\sum_{i=1}^n F^{ii}(W) \\ &\geq \sum_{i=1}^n F^{ii}(W)\Delta W_{ii} - n\sum_{i=1}^n F^{ii}(W)W_{ii} + H\sum_{i=1}^n F^{ii}(W)W_{ii}. \end{aligned} \tag{3.28}$$

Applying Δ to $F(W) = F(I)$, and by the concavity of F, we obtain at x_0,

$$F^{ii}(W)\Delta W_{ii} \geq \Delta F(I) = 0. \tag{3.29}$$

Combining (3.29) and (3.28),

$$n\sum_{i=1}^n F^{ii}(W)W_{ii} \geq H\sum_{i=1}^n F^{ii}(W)W_{ii}.$$

By assumption (3.24), $\sum_{i=1}^n F^{ii}(W)W_{ii} > 0$, we get $n \geq H(x_0)$. Combining (3.27), we conclude that $H(x) = n, \forall x \in \mathbb{S}^n$. Therefore, $u - 1 \in span\{x_1, \cdots, x_{n+1}\}$. ■

Remark 3.1. *By Lemma 13.6, conditions (13.11)-(13.13) and (13.22) imply (3.24). We note that conditions (13.13) and (3.21) implies $F(W) \geq 0$ for all $W \in \Gamma$. Therefore, (3.24) follows from (13.11)-(13.13) and (3.21).*

For $\alpha > 0$, $l \geq 0$ integer, we set,

$$\mathcal{A}^{l,\alpha} = \{f \in C^{l,\alpha}(\mathbb{S}^n) : f \text{ satisfying (3.8)}\}. \tag{3.30}$$

For $R > 0$ fixed, let

$$\mathcal{O}_{\mathcal{R}} = \{w \in \mathcal{A}^{l,\alpha} : w \text{ is } \Gamma\text{-admissible and } \|w\|_{C^{l,\alpha}(\mathbb{S}^n)} < R\}. \tag{3.31}$$

In addition to the structural conditions on F in the previous section, we need some further conditions on F in (3.9) to ensure general existence result. We assume that there is a smooth strictly monotonic positive function Q defined in $R_+ = (0, \infty)$, such that $\forall u \in C^2(\mathbb{S}^n)$ with $W = (u_{ij} + u\delta_{ij}) \in \Gamma_k$, F satisfies the orthogonal condition,

$$\int_{\mathbb{S}^n} Q(F(W(x)))x_m = 0, \forall m = 1, 2..., n+1. \tag{3.32}$$

PROPOSITION 3.4. *Suppose F satisfies the* structural conditions *(13.11)-(13.14), (3.21) and the orthogonal condition (3.32). Then for any positive $\tilde{\varphi} \in C^{1,1}(\mathbb{S}^n)$ with $\varphi(x) = Q(\tilde{\varphi}(x))$ satisfies (3.8), equation (3.9) has an admissible solution $u \in \mathcal{A}^{3,\alpha}, \forall 0 < \alpha < 1$ satisfying*

$$||u||_{C^{3,\alpha}}(\mathbb{S}^n) \leq C,$$

where C is a constant depending only on F, Q, $\alpha, \min\varphi$, and $||\varphi||_{C^{1,1}}(\mathbb{S}^n)$. Furthermore, if $\varphi(x) \in C^{l,\gamma}(\mathbb{S}^n)$ ($l \geq 2, \gamma > 0$), then u is $C^{2+l,\gamma}$.

Proof. For each fixed $0 < \tilde{\varphi} \in C^\infty(\mathbb{S}^n)$ with $\varphi = F(\tilde{\varphi})$ satisfying (3.8), and for $0 \leq t \leq 1$, we define

$$T_t(u) = Q(F(\{u_{ij} + u\delta_{ij}\})) - t\varphi - (1-t)F(I). \tag{3.33}$$

T_t is a nonlinear differential operator which maps $\mathcal{A}^{l+2,\alpha}$ into $\mathcal{A}^{l,\alpha}$. If R is sufficiently large, $T_t(u) = 0$ has no solution on $\partial\mathcal{O}_\mathcal{R}$ by the a priori estimates in Theorem 3.2. Therefore, the degree of T_t is well-defined (e.g., [**93**]). As degree is a homotopic invariant,

$$\deg(T_0, \mathcal{O}_\mathcal{R}, 0) = \deg(T_1, \mathcal{O}_\mathcal{R}, 0).$$

At $t = 0$, by Remark 3.1 and Proposition 3.3, $u = 1$ is the unique solution of (3.9) in $\mathcal{O}_\mathcal{R}$. We may compute the degree using formula

$$\deg(T_0, \mathcal{O}_\mathcal{R}, 0) = \sum_{\mu_j > 0} (-1)^{\beta_j},$$

where μ_j are the eigenvalues of the linearized operator of T_0 and β_j its multiplicity. Since F is symmetric, it is easy to show that the linearized operator of T_0 at $u = 1$ is

$$L = \nu(\Delta + n),$$

for some constant $\nu > 0$. As the eigenvalues of the Beltrami-Laplace operator Δ on $\mathbb{S}^n$ are strictly less than $-n$, except for the first two eigenvalues 0 and $-n$. There is only one positive eigenvalue of L with multiplicity 1, namely $\mu = n\nu$. Therefore,

$$\deg(T_1, \mathcal{O}_\mathcal{R}, 0) = \deg(T_0, \mathcal{O}_\mathcal{R}, 0) = -1.$$

That is, there is an admissible solution of equation (3.9). The regularity and estimates of the solution follows directly from Theorem 3.2. ■

Proof of Theorem 3.1. Theorem 3.1 follows from the above Proposition, since $F(W) = \sigma_k^{\frac{1}{k}}(W)$ satisfies conditions (13.11)-(13.14) and (3.21). The orthogonal condition (3.32) follows from (3.4). ■

REMARK 3.2. *Since the C^2 a priori bound in Proposition 3.2 is independent of the lower bound of $\tilde{\varphi}$ (we note it is used only for the $C^{2,\alpha}$ estimate), Proposition 3.4 can be used to prove existence of $C^{1,1}$ solutions to equation (3.9) in the degenerate case. To be more precise, if F satisfies the* structural conditions *(13.11)-(13.14), (3.21) and the orthogonal condition (3.32). Then for any nonnegative $\tilde{\varphi} \in C^{1,1}(\mathbb{S}^n)$ with $\varphi(x) = F(\tilde{\varphi}(x))$ satisfies (3.8), equation (3.9) has a solution $u \in C^{1,1}(\mathbb{S}^n)$. For equation (3.1), we can do a little better. One can prove that if $\varphi \geq 0$ satisfying (3.4) and $\varphi^{\frac{1}{k-1}} \in C^{1,1}$, then equation (3.1) has a $C^{1,1}$ solution (see* [**59**] *and* [**58**] *for the similar results for the degenerate Monge-Ampère equation). For this, we only need to rework Proposition 3.1. Instead, we estimate $H = \Delta u + nu$. Following the same lines of proof of Proposition 3.1, the desired estimate can be obtained using two facts: (1), for $f = \varphi^{\frac{1}{k-1}}$, we have $|\nabla f(x)|^2 \leq Cf(x)$ for all $x \in \mathbb{S}^n$, where C depending only on $C^{1,1}$ norm of f; (2), for $k > 1$ and $F = \sigma_k^{\frac{1}{k}}$, $\sum_{i=1}^n F^{ii}(W) \geq \frac{1}{k}\sigma_k^{-\frac{1}{k(k-1)}}(W)\sigma_1^{\frac{1}{k-1}}(W)$ (for a proof, see Fact 3.5 on page 1429 in* [**75**]*).*

The *structural conditions* (13.11)-(13.14) and (3.21) are satisfied for the quotient operator $F(W) = (\frac{\sigma_k(W)}{\sigma_l(W)})^{\frac{1}{k-l}}$ with $\Gamma = \Gamma_k$ for any $0 \leq l < k$. Also, constant is the unique solution of $F(W) = 1$ in $\mathcal{A}^{2,\alpha}$ by Proposition 3.3. Unfortunately, the orthogonal condition (3.32) is not valid in general by some simple examples in Proposition 1.2. Nevertheless,we have the following existence result.

PROPOSITION 3.5. *Suppose F satisfies the* structural conditions *(13.11)-(13.14) and (3.21). Assume $\tilde{\varphi} \in C^{l,1}(\mathbb{S}^n)$ ($l \geq 1$) is a positive function. Suppose there is an automorphism group $\mathcal{G}$ of $\mathbb{S}^n$ which has no fixed points. If $\tilde{\varphi}$ is invariant under $\mathcal{G}$, i.e., $\tilde{\varphi}(g(x)) = \tilde{\varphi}(x)$ for all $g \in \mathcal{G}$ and $x \in \mathbb{S}^n$. Then there exists a $\mathcal{G}$-invariant admissible function $u \in C^{l+2,\alpha}$ ($\forall 0 < \alpha < 1$), such that u satisfies equation (3.9). Moreover, there is a constant C depending only on $\alpha, \min\tilde{\varphi}$, and $\|\tilde{\varphi}\|_{C^{l,1}}(\mathbb{S}^n)$, such that*

$$||u||_{C^{l+1,\alpha}}(\mathbb{S}^n) \leq C.$$

In particular, for any positive $\mathcal{G}$-invariant positive $\varphi \in C^{1,1}(\mathbb{S}^n)$, equation (3.5) has a k-convex $\mathcal{G}$-invariant solution.

Proof. We only sketch the main arguments of the proof. Since any $\mathcal{G}$-invariant function is orthogonal to $span\{x_1, ..., x_{n+1}\}$ by [**57**]. Therefore, $u = 1$ is the unique $\mathcal{G}$-invariant solution of (3.9) by Proposition 3.3. We again use degree theory. This time, we consider $\mathcal{G}$-invariant function spaces:

$$\tilde{\mathcal{A}}^{l,\alpha} = \{f \in C^{l,\alpha}(\mathbb{S}^n) : f \text{ is } \mathcal{G}\text{-invariant}\},$$

and

$$\tilde{\mathcal{O}_{\mathcal{R}}} = \{w \text{ is } k\text{-convex}, w \in \tilde{\mathcal{A}}^{l,\alpha} : \|w\|_{C^{l,\alpha}(\mathbb{S}^n)} < R\}.$$

One may compute that the degree of F is not vanishing as in the proof of Theorem 3.4. ■

THEOREM 3.3. *Suppose there is an automorphism group $\mathcal{G}$ of $\mathbb{S}^n$ which has no fixed points. Suppose $\varphi \in C^\infty(\mathbb{S}^n)$ is positive and $\mathcal{G}$-invariant, then equation (3.5) has a $\mathcal{G}$-invariant convex smooth solution u.*

We remark that the reason to impose group invariant condition in Theorem 3.3 is that, since for $l \neq 0$, equation (3.5) does not have variational structure. For this reason, it is found in ?? that condition (3.4) is *neither* sufficient, *nor* necessary for the existence of admissible solutions of (3.5).

Proof Theorem 3.3. For $0 \leq t \leq 1$, we define $\varphi_t = (1 - t + t\varphi^{\frac{-1}{k-l}})^{-k+l}$. Certainly φ_t is $\mathcal{G}$-invariant and $\{(\varphi_t{}^{\frac{-1}{k-l}})_{ij} + \varphi_t{}^{\frac{-1}{k-l}}\delta_{ij}\}$ is semi-positive definite everywhere on $\mathbb{S}^n$. We consider equation

$$\frac{\sigma_k}{\sigma_l}(u^t_{ij} + u^t\delta_{ij}) = \varphi_t. \tag{3.34}$$

Applying degree theory as in the proof of Proposition 3.5, there exists admissible solution u^t of equation (3.34) for each $0 \leq t \leq 1$. ■

Notes

When $k = n$, equation (3.1) is the Monge-Ampère equation corresponding to the Minkowski problem:

$$\det(u_{ij} + u\delta_{ij}) = \varphi, \quad \text{on} \quad \mathbb{S}^n. \tag{3.35}$$

In this case, our Existence Theorem was proved in the works of Nirenberg [**97**] (for $n = 2$), Cheng-Yau [**35**] and Pogorelov [**107**]. For the other extremal case $k = 1$, equation corresponds to the Christoffel problem. In this case, equation (3.1) has the following simple form:

$$\Delta u + nu = \varphi, \quad \text{on} \quad \mathbb{S}^n, \tag{3.36}$$

where Δ is the spherical Laplacian of the round unit sphere. The operator $L = \Delta + n$ is linear and self-adjoint. In this case, our Existence Theorem follows easily from the linear elliptic theory. The general form of the Existence Theorem was proved in [**68**].

Some general form of fully nonlinear geometric equations on $\mathbb{S}^n$ were studied by Alexandrov [**4**] and Pogorelov [**106**]. In particular, uniqueness problem was considered in [**4**] and existence problem was addressed in [**106**] under various structural conditions. Their attentions were mainly drawn to solutions which may represented as support functions of some convex bodies. The results for admissible solutions were obtained in [**68**].

CHAPTER 4

The Christoffel-Minkowski problem, the issue of convexity

In this chapter, we discuss when an admissible solution of equation (3.1) is convex. The convexity of equation (3.1) is important since it is related to the geometric problem: the Christoffel-Minkowski problem.

We will establish a general convexity principle for solutions of fully nonlinear partial differential equations. The existence of convex solutions is usually obtained by the continuity method or flow method. The basic philosophy of this type of deformation lemma is to show the strict convexity is preserved in the process. Here, we prove a convexity principle under some general simple structure conditions.

Let us fix some notations. Let $\Psi \subset \mathbb{R}^n$ be an open symmetric domain, denote $Sym(n) = \{n \times n \quad \text{real symmetric matrices}\}$, set

$$\tilde{\Psi} = \{A \in Sym(n) : \lambda(A) \in \Psi\}. \tag{4.1}$$

We will assume

$$f \in C^2(\Psi) \quad \text{symmetric and } f_{\lambda_i}(\lambda) = \frac{\partial f}{\partial \lambda_i}(\lambda) > 0, \forall i = 1, \cdots, n, \quad \forall \lambda \in \Psi. \tag{4.2}$$

extend it to $F : \tilde{\Psi} \to R$ by $F(A) = f(\lambda(A))$. We define $\tilde{F}(A) = -F(A^{-1})$ whenever $A^{-1} \in \tilde{\Psi}$, we also assume

$$\tilde{F} \quad \text{is locally concave.} \tag{4.3}$$

THEOREM 4.1. *Under conditions (4.1)-(4.3), if u is a C^3 convex solution of the following equation in a domain Ω in $\mathbb{R}^n$*

$$F(u_{ij}(x)) = \varphi(x, u(x), \nabla u(x)), \quad \forall x \in \Omega, \tag{4.4}$$

for some $\varphi \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. If $\varphi(x, u, p)$ is concave in $\Omega \times \mathbb{R}$ for any fixed $p \in \mathbb{R}^n$, then the Hessian (u_{ij}) has constant rank in Ω.

We now turn to fully nonlinear equations arising from classical differential geometry.

Let M be an oriented immersed connect hypersurface in $\mathbb{R}^{n+1}$ with a nonnegative definite second fundamental form. Let $\kappa(X) = (\kappa_1(X), \cdots, \kappa_n(X))$ be the principal curvature at $X \in M$. We consider the following curvature equation

$$f(\kappa(X)) = \varphi(X, \vec{n}(X)), \quad \forall X \in M, \tag{4.5}$$

where $\vec{n}(X)$ the unit normal of M at X.

THEOREM 4.2. *Suppose f and F as in Theorem 4.1. Suppose $\Sigma \subset \mathbb{R}^{n+1} \times \mathbb{S}^n$ is a bounded open set and $\varphi \in C^{1,1}(\Gamma)$ and $\varphi(X, y)$ is locally concave in X variable for any $y \in \mathbb{S}^n$. Let M be an oriented immersed connect hypersurface in $\mathbb{R}^{n+1}$ with a nonnegative definite second fundamental form. If $(X, \vec{n}(X)) \in \Sigma$ for each $X \in M$ and the principal curvatures of M satisfies equation (4.5), then the second fundamental form of M is of constant rank. If in addition M is compact, then M is the boundary of a strongly convex bounded domain in $\mathbb{R}^{n+1}$.*

We next consider the Christoffel-Minkowski type equation,

$$F(u_{ij} + u\delta_{ij}) = \varphi \quad \text{on} \quad \Omega \subset \mathbb{S}^n, \tag{4.6}$$

where u_{ij} are the second covariant derivatives of u with respect to orthonormal frames on $\mathbb{S}^n$.

THEOREM 4.3. *Let f and F as in Theorem 4.1, and assume f is of homogeneous degree -1 and Ω is an open domain in $\mathbb{S}^n$. If $0 > \varphi \in C^{1,1}(\Omega)$ and $(\varphi_{ij} + \varphi\delta_{ij}) \leq 0$ on Ω, if u is a solution of equation (4.6) with $u_{ij} + u\delta_{ij}$ is nonnegative, then $(u_{ij} + u\delta_{ij})$ of constant rank. If $\Omega = \mathbb{S}^n$, then $(u_{ij} + u\delta_{ij})$ is positive definite everywhere on $\mathbb{S}^n$.*

1. Equations in flat domains in $\mathbb{R}^n$

We first present proof of Theorem 4.1 to illustrate the main idea to establish a local differential inequality (4.11) near the point where the minimum rank of the Hessian (u_{ij}) is attained. One of the key property we will use is the symmetry of u_{ijk} with respect to indices i, j, k. The proof of Theorem 4.2 and Theorem 4.3 will be given in the next section. The main arguments also work for equations on Codazzi tensors in Riemannian manifolds, which we will discuss in the last section.

We define $\dot{f}^k = \frac{\partial f}{\partial \lambda_k}$, $\ddot{f}^{kl} = \frac{\partial^2 f}{\partial \lambda_k \partial \lambda_l}$, $F^{\alpha\beta} = \frac{\partial F}{\partial A_{\alpha\beta}}$ and $F^{\alpha\beta,rs} = \frac{\partial^2 F}{\partial A_{\alpha\beta} \partial A_{rs}}$.

Proof of Theorem 4.1. We set $\tilde{\varphi}(x) = \varphi(x, u(x), \nabla u(x))$ and $W = (W_{ij})$ with $W_{ij} = u_{ij}$. We rewrite (4.4) in the following form

$$F(W(x)) = \tilde{\varphi}(x), \quad \forall x \in \Omega. \tag{4.7}$$

Suppose $z_0 \in \Omega$ is a point where W is of minimal rank l. We pick an open neighborhood O of z_0, for any $z \in O$, let $\lambda_1 \leq \lambda_2 ... \leq \lambda_n$ be the eigenvalues of W at z. There is a positive constant $C > 0$ depending only on $\|u\|_{C^3}$, $\|\varphi\|_{C^2}$ and n, such that $\lambda_n \geq \lambda_{n-1} ... \geq \lambda_{n-l+1} \geq C$. Let $G = \{n-l+1, n-l+2, ..., n\}$ and $B = \{1, ..., n-l\}$ be the "good" and "bad" sets of indices respectively. Let $\Lambda_G = (\lambda_{n-l+l}, ..., \lambda_n)$ be the "good" eigenvalues of W at z, for the simplicity of the notations, we also write $G = \Lambda_G$ if there is no confusion.

Since F is elliptic and W is continuous, if O is sufficiently small, we may pick a positive constant A such that

$$\min_{\alpha} F^{\alpha\alpha}(W(x)) \geq \frac{2}{A} \sum_{\alpha,\beta,r,s} |F^{\alpha\beta,rs}(W(x))|, \quad \forall x \in O. \tag{4.8}$$

Set (with the convention that $\sigma_j(W) = 0$ if $j < 0$ or $j > n$)

$$\phi(x) = \sigma_{l+1}(W) + A\sigma_{l+2}(W). \tag{4.9}$$

Following the notations in [**24**], for two functions defined in an open set $O \subset \Omega$, $y \in O$, we say that $h(y) \lesssim k(y)$ provided there exist positive constants c_1 and c_2 such that

$$(h-k)(y) \leq (c_1|\nabla\phi| + c_2\phi)(y). \tag{4.10}$$

We also write $h(y) \sim k(y)$ if $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$. Next, we write $h \lesssim k$ if the above inequality holds in O, with the constant c_1, and c_2 depending only on $||u||_{C^3}$, $||\tilde{\varphi}||_{C^2}$, n and C_0 (independent of y and O). Finally, $h \sim k$ if $h \lesssim k$ and $k \lesssim h$. In the following, all calculations are at the point z using the relation "$\lesssim$", with the understanding that the constants in (4.10) are under control.

We shall show that

$$\frac{1}{\sigma_l(G)} \sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim \sum_{i\in B} \tilde{\varphi}_{ii}. \tag{4.11}$$

To prove (4.11), we may assume $u \in C^4$ by approximation. For each $z \in O$ fixed, we can rotate coordinate so that W is diagonal at z, and $W_{ii} = \lambda_i, \forall i = 1, ..., n$. We note that since W is diagonal at z, $(F^{\alpha\beta})$ is also diagonal at z and $F^{\alpha\beta,rs} = 0$ unless $\alpha = \beta, r = s$ or $\alpha = r, \beta = s$.

Now we compute ϕ and its first and second derivatives in the direction x_α. The following computations follow mainly from [**67**]. As W is diagonal at z, $\sigma_{l+2}(W) \leq C\sigma_{l+1}^{\frac{l+2}{l+1}}(W)$, we obtain

$$0 \sim \phi(z) \sim \sigma_{l+1}(W) \sim (\sum_{i\in B} W_{ii})\sigma_l(G) \sim \sum_{i\in B} W_{ii}, \quad (\text{so} \quad W_{ii} \sim 0, \quad i \in B), \tag{4.12}$$

Let W be a $n \times n$ diagonal matrix, we denote $(W|i)$ to be the $(n-1)\times(n-1)$ matrix with ith row and ith column deleted, and denote $(W|ij)$ to be the $(n-2)\times(n-2)$ matrix with i, jth rows and i, jth columns deleted. We also denote $(G|i)$ be the subset of G with λ_i deleted. Since $\sigma_{l+1}(W|i) \lesssim 0$, we have

$$0 \sim \phi_\alpha \sim \sigma_l(G) \sum_{i\in B} W_{ii\alpha} \sim \sum_{i\in B} W_{ii\alpha} \tag{4.13}$$

(4.12) yields that, for $1 \leq m \leq l$,

$$\sigma_m(W) \sim \sigma_m(G), \quad \sigma_m(W|j) \sim \begin{cases} \sigma_m(G|j), & \text{if } j \in G; \\ \sigma_m(G), & \text{if } j \in B. \end{cases} \tag{4.14}$$

$$\sigma_m(W|ij) \sim \begin{cases} \sigma_m(G|ij), & \text{if } i, j \in G, i \neq j; \\ \sigma_m(G|j), & \text{if } i \in B, j \in G; \\ \sigma_m(G), & \text{if } i, j \in B, i \neq j. \end{cases}$$

Since W is diagonal, it follows from (4.12) and Proposition 13.1,

$$\frac{\partial\sigma_{l+1}(W)}{\partial W_{ij}} \sim \begin{cases} \sigma_l(G), & \text{if } i = j \in B, \\ 0, & \text{otherwise}, \end{cases} \tag{4.15}$$

and for $1 \le m \le n$,

$$\frac{\partial^2 \sigma_m(W)}{\partial W_{ij}\partial W_{rs}} = \begin{cases} \sigma_{m-2}(W|ir), & \text{if } i=j, r=s, i\neq r; \\ -\sigma_{m-2}(W|ij), & \text{if } i\neq j, r=j, s=i; \\ 0, & \text{otherwise.} \end{cases} \tag{4.16}$$

From (4.13)-(4.16), we have

$$\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(W|ij)W_{ii\alpha}W_{jj\alpha} \sim (\sum_{j\in G}\sigma_{l-1}(G|j)W_{jj\alpha})\sum_{i\in B} W_{ii\alpha} \sim 0, \tag{4.17}$$

$$\sum_{\substack{i,j\in B\\ i\neq j}} \sigma_{l-1}(W|ij)W_{ii\alpha}W_{jj\alpha} \sim -\sigma_{l-1}(G)\sum_{i\in B} W_{ii\alpha}^2, \tag{4.18}$$

$$\sum_{j\in G, i\in B} \sigma_{l-1}(W|ij)W_{ij\alpha}^2 \sim \sum_{i\in B, j\in G} \sigma_{l-1}(G|j)W_{ij\alpha}^2, \tag{4.19}$$

and if $l \le n-2$ (that is $|B| \ge 2$)

$$\begin{aligned} \sum_{i,j=1}^{n} \frac{\partial^2 \sigma_{l+2}(W)}{\partial W_{ij}\partial W_{rs}} W_{ij\alpha}W_{rs\alpha} &\sim \sum_{i\neq j\in B} \sigma_l(G)W_{ii\alpha}W_{jj\alpha} - \sum_{i\neq j\in B}\sigma_l(G)W_{ij\alpha}^2 \\ &\sim -\sum_{i\in B}\sigma_l(G)W_{ii\alpha}^2 - \sum_{i\neq j\in B}\sigma_l(G)W_{ij\alpha}^2 \\ &\sim -\sigma_l(G)\sum_{i,j\in B} W_{ij\alpha}^2. \end{aligned} \tag{4.20}$$

We note that if $l = n-1$, we have $|B| = 1$, (4.20) still holds since $w_{ii\alpha} \sim 0$ by (4.13).

By (4.14)-(4.19), $\forall \alpha \in \{1, 2, ..., n\}$

$$\begin{aligned} \phi_{\alpha\alpha} &= A\sigma_{l+2}(W)_{\alpha\alpha} + (\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}})\sigma_{l-1}(W|ij)W_{ii\alpha}W_{jj\alpha} \\ &- (\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}})\sigma_{l-1}(W|ij)W_{ij\alpha}^2 + \sum_i \frac{\partial \sigma_{l+1}(W)}{\partial W_{ii}} W_{ii\alpha\alpha} \\ &\sim \sigma_l(G)\sum_{i\in B} W_{ii\alpha\alpha} + A\sum_{i=1}^{n}\sigma_{l+1}(W|i)W_{ii\alpha\alpha} - 2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W_{ij\alpha}^2 \\ &\quad -(\sigma_{l-1}(G) + A\sigma_l(G))\sum_{i,j\in B} W_{ij\alpha}^2. \end{aligned} \tag{4.21}$$

Since $F^{\alpha\beta}$ is diagonal at z, we have

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \quad &\sim \quad A\sum_{\alpha=1}^{n}\sum_{i=1}^{n} F^{\alpha\alpha}\sigma_{l+1}(W|i)W_{ii\alpha\alpha} + \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sigma_l(G)(\sum_{i\in B} W_{ii\alpha\alpha} - A\sum_{i,j\in B} W_{ij\alpha}^2) \\ &-\sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2 - 2\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)W_{ij\alpha}^2].\end{aligned} \tag{4.22}$$

By equation (4.7),

$$\tilde{\varphi}_i = \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}W_{\alpha\beta i}, \quad \tilde{\varphi}_{ii} = \sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi} + \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}W_{\alpha\beta ii}.$$

So for any $i \in B$, we have

$$\sum_{\alpha=1}^{n} F^{\alpha\alpha}W_{\alpha\alpha ii} \sim \tilde{\varphi}_{ii} - \sum_{\alpha,\beta,r,s} F^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi} \tag{4.23}$$

As $W_{\alpha\alpha ii} = W_{ii\alpha\alpha}$ and $\sigma_{l+1}(W|i) \sim 0$, from (4.22) and (4.23)

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \quad &\sim \quad \sigma_l(G)[\sum_{i\in B}\tilde{\varphi}_{ii} - \sum_{i\in B}\sum_{\alpha,\beta,r,s} F^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi} - A\sum_{\alpha}\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2] \\ &-\sigma_{l-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2 - 2\sum_{\alpha=1}^{n}\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)F^{\alpha\alpha}W_{ij\alpha}^2.\end{aligned} \tag{4.24}$$

In order to study terms in (4.24), we may assume the eigenvalues of W are distinct at z (if necessary, we perturb W then take limit). In the following we let $\lambda_i = W_{ii}$.

Using (13.8), (13.9) and (4.24), we obtain

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \quad &\sim \quad \sigma_l(G)\sum_{i\in B}\tilde{\varphi}_{ii} - \sigma_l(G)\sum_{i\in B}[\sum_{\alpha,\beta=1}^{n} \ddot{f}^{\alpha\beta}W_{\alpha\alpha i}W_{\beta\beta i} + 2\sum_{\alpha<\beta}\frac{\dot{f}^{\alpha}-\dot{f}^{\beta}}{\lambda_\alpha-\lambda_\beta}W_{\alpha\beta i}^2] \\ &-(\sigma_{l-1}(G) + A\sigma_l(G))\sum_{\alpha=1}^{n}\sum_{i,j\in B} \dot{f}^{\alpha}W_{ij\alpha}^2 - 2\sum_{\alpha=1}^{n}\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)\dot{f}^{\alpha}W_{ij\alpha}^2.\end{aligned} \tag{4.25}$$

As W_{ijk} is symmetric with respect to i, j, k (*here the symmetry of W_{ijk} is essential!*),

$$\begin{aligned}\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \quad \sim \quad & \sum_{i\in B}\tilde{\varphi}_{ii} - \sum_{i\in B}[\sum_{\alpha,\beta\in B} + \sum_{\alpha,\beta\in G} + 2\sum_{\substack{\alpha\in G\\ \beta\in B}}]\ddot{f}^{\alpha\beta}W_{\alpha\alpha i}W_{\beta\beta i} \\ & -2\sum_{i\in B}[\sum_{\substack{\alpha,\beta\in G\\ \alpha<\beta}} + \sum_{\substack{\alpha\in G\\ \beta\in B}} + \sum_{\substack{\alpha,\beta\in B\\ \alpha<\beta}}]\frac{\dot{f}^{\alpha}-\dot{f}^{\beta}}{\lambda_\alpha-\lambda_\beta}W^2_{\alpha\beta i} - A\sum_{\alpha=1}^{n}\sum_{i,j\in B}\dot{f}^{\alpha}W^2_{ij\alpha} \\ & -2\sum_{i\in B}[\sum_{\alpha,\beta\in G} + \sum_{\substack{\alpha\in B\\ \beta\in G}}]\frac{\dot{f}^{\alpha}}{\lambda_\beta}W^2_{\alpha\beta i} - \sum_{i\in B}[\sum_{\alpha,\beta\in B} + \sum_{\substack{\alpha\in G\\ \beta\in B}}](\sum_{k=n-l+1}^{n}\frac{1}{\lambda_k})\dot{f}^{\alpha}W^2_{\alpha\beta i}.\end{aligned} \tag{4.26}$$

Now we divide (4.26) into three parts according to sum $\sum_{\alpha,\beta\in G}$, $\sum_{\substack{\alpha\in G\\ \beta\in B}}$ and $\sum_{\alpha,\beta\in B}$. Then (4.26) becomes

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \sim \sum_{i\in B}\tilde{\varphi}_{ii} - \sum_{i\in B}(I_i + II_i + III_i) - A\sum_{\alpha\in G}\sum_{i,j\in B}\dot{f}^{\alpha}W^2_{ij\alpha}, \tag{4.27}$$

where

$$I_i = \sum_{\alpha,\beta\in G}\ddot{f}^{\alpha\beta}W_{\alpha\alpha i}W_{\beta\beta i} + 2\sum_{\substack{\alpha,\beta\in G\\ \alpha<\beta}}\frac{\dot{f}^{\alpha}-\dot{f}^{\beta}}{\lambda_\alpha-\lambda_\beta}W^2_{\alpha\beta i} + 2\sum_{\alpha,\beta\in G}\frac{\dot{f}^{\beta}}{\lambda_\alpha}W^2_{\alpha\beta i},$$

$$II_i = \sum_{\substack{\alpha\in G\\ \beta\in B}}[2\ddot{f}^{\alpha\beta}W_{\alpha\alpha i}W_{\beta\beta i} + 2\frac{\dot{f}^{\alpha}-\dot{f}^{\beta}}{\lambda_\alpha-\lambda_\beta}W^2_{\alpha\beta i} + 2\frac{\dot{f}^{\beta}}{\lambda_\alpha}W^2_{\alpha\beta i} + (\sum_{k=n-l+1}^{n}\frac{1}{\lambda_k})\dot{f}^{\alpha}W^2_{\alpha\beta i}],$$

$$III_i = \sum_{\alpha,\beta\in B}\ddot{f}^{\alpha\beta}W_{\alpha\alpha i}W_{\beta\beta i} + 2\sum_{\substack{\alpha,\beta\in B\\ \alpha<\beta}}\frac{\dot{f}^{\alpha}-\dot{f}^{\beta}}{\lambda_\alpha-\lambda_\beta}W^2_{\alpha\beta i} + \sum_{\alpha,\beta\in B}(\sum_{k=n-l+1}^{n}\frac{\dot{f}^{\alpha}}{\lambda_k} + A\dot{f}^{i})W^2_{\alpha\beta i}.$$

We need the following lemma.

LEMMA 4.1. *If f and F satisfy conditions (4.1)-(4.3), (W_{ij}) satisfies (4.12)-(4.13), and A defined as in (4.8), then*

$$I_i \gtrsim 0, \quad II_i \gtrsim 0, \quad III_i \gtrsim 0, \quad \forall i \in B. \tag{4.28}$$

Since (u_{ij}) is diagonal at the point,

$$\sum_{i\in B}\tilde{\varphi}_{ii} = \sum_{i\in B}(\varphi_{x_ix_i} + 2\varphi_{x_iu}u_i + \varphi_{uu}u_i^2) + \sum_{i\in B}u_{ii}(2\varphi_{x_ip_i} + \varphi_{p_ip_i}u_{ii} + \varphi_u + 2\varphi_{up_i}u_i) + \sum_j\varphi_{p_j}\sum_{i\in B}u_{iij}.$$

By our assumption on φ, (4.12) and (4.13),

$$\sum_{i\in B}\tilde{\varphi}_{ii} \lesssim 0.$$

By Lemma 4.1,

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim \sum_{i\in B}\tilde{\varphi}_{ii} \lesssim 0.$$

Theorem 4.1 then follows from the strong minimum principle. ■

Proof of Lemma 4.1. $I_i \gtrsim 0$ follows from (13.10) in Corollary 13.3.

For II_i, we note for $\beta,\gamma\in B, \alpha\in G$, $\ddot{f}^{\alpha\beta}\sim\ddot{f}^{\alpha\gamma}$. Thus from (4.13)

$$2\sum_{\substack{\alpha\in G\\ \beta\in B}}\ddot{f}^{\alpha\beta}W_{\alpha\alpha i}W^2_{\beta\beta i}\sim\sum_{\alpha\in G}\ddot{f}^{\alpha\beta}W_{\alpha\alpha i}(\sum_{\beta\in B}W_{\beta\beta i})\sim 0.$$

And for $\alpha\in G,\beta\in B$, $\lambda_\beta\sim 0$, we have

$$\frac{\dot{f}^\alpha-\dot{f}^\beta}{\lambda_\alpha-\lambda_\beta}+\frac{\dot{f}^\beta}{\lambda_\alpha}\sim\frac{\dot{f}^\alpha}{\lambda_\alpha}.$$

In turn,

$$II_i \sim 2\sum_{\substack{\alpha\in G\\ \beta\in B}}\frac{\dot{f}^\alpha}{\lambda_\alpha}W^2_{\alpha\beta i}+\sum_{\substack{\alpha\in G\\ \beta\in B}}(\sum_{k=n-k+1}^{n}\frac{1}{\lambda_k})\dot{f}^\alpha W^2_{\alpha\beta i}\gtrsim 0.$$

Finally $III_i\gtrsim 0$ by our choice of A in (4.8) and Lemma 7.1. The proof of Lemma 4.1 is complete. ■

2. Curvature equations of hypersurfaces in $\mathbb{R}^{n+1}$

In this section, we convexity problem of fully nonlinear curvature equations of hypersurfaces in $\mathbb{R}^{n+1}$. We prove Theorem 4.3 first.

Proof of Theorem 4.3. We work on spherical Hessian $W=(u_{ij}+u\delta_{ij})$ in place of standard Hessian (u_{ij}) in the proof of Theorem 4.1.

As in the proof of Theorem 4.1, let $z_0\in\Omega$ be a point where W is of minimum rank and O is a small open neighborhood of z_0. For any $z\in O\subset\Omega$, we divide eigenvalues of W at z into G and B, the "good" and "bad" sets of indices respectively. Define ϕ as in (4.9). We may assume at the point, W is diagonal under some local orthonormal frames. We want to show that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n}F^{\alpha\alpha}\phi_{\alpha\alpha}\lesssim\sum_{i\in B}[\varphi_{ii}+\varphi] \tag{4.29}$$

The same arguments in the proof of Theorem 4.1 yield (4.12)-(4.13) for $W=(u_{ij}+u\delta_{ij})$, and

$$\sum_{\alpha=1}^{n}F^{\alpha\alpha}\phi_{\alpha\alpha}\sim\sum_{\alpha=1}^{n}F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B}W_{ii\alpha\alpha}-\sigma_{l-1}(G)\sum_{i,j\in B}W^2_{ij\alpha}-2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W^2_{ij\alpha}]. \tag{4.30}$$

Since f is of homogeneous degree of -1, $\sum_{\alpha=1}^n F^{\alpha\alpha}W_{\alpha\alpha} = -\varphi$, we get

$$\begin{aligned}\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha} &\sim \sum_{\alpha=1}^n F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B}(W_{\alpha\alpha ii}+W_{ii}-W_{\alpha\alpha})-\sigma_{l-1}(G)W_{ij\alpha}^2-2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W_{ij\alpha}^2]\\ &\sim \sum_{\alpha=1}^n F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B}W_{\alpha\alpha ii}+(n-l)\sigma_l(G)\varphi\\ &\quad -\sigma_{l-1}(G)\sum_{i,j\in B}W_{ij\alpha}^2-2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W_{ij\alpha}^2]. \end{aligned} \tag{4.31}$$

Since W_{ijk} is symmetric respect to indices $\{ijk\}$ (which is used in the derivation from (4.25) to (4.26) in the proof of Theorem 4.1), as in (4.27), we reduce that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha}\sim\sum_{i\in B}[\varphi_{ii}+\varphi]-\sum_{i\in B}I_i-\sum_{i\in B}II_i-\sum_{i\in B}III_i, \tag{4.32}$$

where I_i, II_i, III_i defined similarly as in (4.27). Therefore, (4.29) follows from Lemma 4.1. The condition $(\varphi_{ij}+\varphi\delta_{ij})\le 0$ yields

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha}\lesssim 0., \tag{4.33}$$

It follows from strong minimum principle that W is of constant rank in Ω. If $\Omega=\mathbb{S}^n$, the Minkowski integral formula implies W is of full rank (e.g., see argument in [**67**]). ■

We now precede to treat curvature equation (4.5). Let W be the second fundamental form of M, equation (4.5) can be rewritten as

$$F(W(X))=\varphi(X,\vec{n}),\quad \forall X\in M. \tag{4.34}$$

Proof of Theorem 4.2. We let $\tilde{\varphi}(X)=\varphi(X,\vec{n}(X))$. We work on second fundamental form $W=(h_{ij})$ in place of standard Hessian (u_{ij}) in the proof of Theorem 4.1.

As in the proof of Theorem 4.1, let $O\subset M$ be an open neighborhood of some point z_0 where the minimum rank of W is attained. For any $z\in O$, we choose a local orthonormal frame $\{e_A\}$ in the neighborhood of z in M with $\{e_1,e_2,...,e_n\}$ tangent to M and $e_{n+1}(=\vec{n})$ is the normal so that the second fundamental form (W_{ij}) is diagonal at z, we divide eigenvalues of W at z into G and B, the "good" and "bad" sets of indices respectively. Set $\phi=\sigma_{l+1}(W)$. As in the proof of Theorem 4.3, we want to show

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha}\lesssim\sum_{i\in B}\tilde{\varphi}_{ii} \tag{4.35}$$

The same arguments in the proof of Theorem 4.1 yield (4.12)-(4.13) for $W = (h_{ij})$, and

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \quad &\sim \quad \sigma_l(G)\sum_{\alpha=1}^{n}\sum_{i\in B} F^{\alpha\alpha}W_{ii\alpha\alpha} - \sigma_{l-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2 \\ &-2\sum_{\alpha=1}^{n}\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)F^{\alpha\alpha}W_{ij\alpha}^2. \end{aligned} \tag{4.36}$$

It follows from the Gauss equation and (4.12) that

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \quad &\sim \quad \sigma_l(G)\sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sum_{i\in B}(W_{\alpha\alpha ii} + W_{ii}W_{\alpha\alpha}^2 - W_{ii}^2 W_{\alpha\alpha}) \\ &\quad -\sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2 - 2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W_{ij\alpha}^2] \\ &\sim \quad \sigma_l(G)\sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sum_{i\in B} W_{\alpha\alpha ii} - \sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2 - 2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W_{ij\alpha}^2].\end{aligned}$$

Since by Codazzi formula W_{ijk} is symmetric respect to indices $\{ijk\}$, as in (4.27), we reduce that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\varphi_{\alpha\alpha} \sim \sum_{i\in B}\tilde{\varphi}_{ii} - \sum_{i\in B} I_i - \sum_{i\in B} II_i - \sum_{i\in B} III_i, \tag{4.37}$$

where I_i, II_i, III_i defined similarly as in (4.27). Now (4.35) follows from the Lemma 4.1 in the proof of Theorem 4.1.

We now compute $\tilde{\varphi}_{ii}$. $\forall \quad i \in \{1, 2, ..., n\}$,

$$\begin{aligned}\tilde{\varphi}(X)_i \quad &= \sum_{A=1}^{n+1}\varphi_{X_A}e_i^A + \varphi_{e_{n+1}}(e_{n+1})_i, \\ \tilde{\varphi}(X)_{ii} \quad &= \sum_{A,C=1}^{n+1}\varphi_{X_A X_C}e_i^A e_i^C + \sum_{A=1}^{n+1}\varphi_{X_A}X_{ii}^A + 2\sum_{A=1}^{n+1}\varphi_{X_A e_{n+1}}e_i^A(e_{n+1})_i \\ &\quad +\varphi_{e_{n+1},e_{n+1}}(e_{n+1})_i(e_{n+1})_i + \varphi_{e_{n+1}}(e_{n+1})_{ii}.\end{aligned}$$

By the Gauss formula and the Weingarten formula for hypersurfaces, it follows that,

$$\sum_{i\in B}\tilde{\varphi}(X)_{ii} \backsim \sum_{i\in B}\sum_{A,C=1}^{n+1}\varphi_{X_A X_C}e_i^A e_i^C. \tag{4.38}$$

By our assumption on φ, we conclude that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim 0. \tag{4.39}$$

The strong minimum principle implies W is of constant rank l. If M is compact, there is at least one point that its second fundamental form is positive definite. Therefore it is positive definite everywhere and M is the boundary of some strongly convex bounded domain in $\mathbb{R}^{n+1}$. ■

We note the proof of Theorem 4.3 is of local nature, there is a corresponding local statement of constant rank result for $W = (u_{ij} + u\delta_{ij})$ as in Theorem 4.2. If $\Omega = \mathbb{S}^n$, the condition on φ in Theorem 4.3 is equivalent to say that $\varphi(x)$ is concave in $\mathbb{R}^{n+1}$ after being extended as a homogeneous function of degree 1. Theorem 4.3 can deduce a positive upper bound on principal curvatures of M if it satisfies (4.6).

Corollary 4.1. *In addition to the conditions on F in Theorem 4.3, we assume that F is concave and*

$$\lim_{\lambda\to\partial\Psi} f(\lambda) = -\infty.$$

For any constant $\beta \geq 1$, there exist positive constants $\gamma > 0, \vartheta > 0$ such that if $0 > \varphi(x) \in C^{1,1}(\mathbb{S}^n)$ is a negative function with $\inf_{\mathbb{S}^n}(-\varphi) = 1, \|\varphi\|_{C^{1,1}(\mathbb{S}^n)} \leq \beta$, and $(\varphi_{ij} + (\varphi - \gamma)\delta_{ij}) \leq 0$ on $\mathbb{S}^n$, if u satisfies (4.6) on $\mathbb{S}^n$ with $(u_{ij} + u\delta_{ij}) \geq 0$, then $(u_{ij} + u\delta_{ij}) \geq \frac{1}{\vartheta}I$ on $\mathbb{S}^n$. That is, the principal curvature of convex hypersurface M with u as its support function is bounded from above by ϑ.

Proof of Corollary 4.1. We argue by contradiction. If the result is not true, for some $\beta \geq 1$, there are sequences functions $0 \geq \varphi^l \in C^{1,1}(\mathbb{S}^n)$ and $u^l \in C^2(\mathbb{S}^n)$, with $\sup_{\mathbb{S}^n} \varphi^l = -1$, $\|\varphi^l\|_{C^{1,1}(\mathbb{S}^n)} \leq \beta$, $(\varphi_{ij} + (\varphi - \frac{1}{l})\delta_{ij}) \leq 0$, $W^l = (u^l_{ij} + u^l\delta_{ij}) \geq 0$ on $\mathbb{S}^n$ and its minimum eigenvalue $\lambda^l_m(x_l) \leq \frac{1}{l}$ at some point $x_l \in \mathbb{S}^n$. Since equation (4.6) is invariant if we transfer $u(x)$ to $u(x) + \sum_{i=1}^{n+1} a_i x_i$, we may assume that

$$\int_{\mathbb{S}^n} u(x)x_j = 0, \quad \forall j = 1, \cdots, n+1.$$

It follows [**57, 67, 68**] that

$$\|u^l\|_{C^{1,1}(\mathbb{S}^n)} \leq C,$$

independent of l. By the assumption that

$$\lim_{\lambda\to\partial\Psi} f(\lambda) = -\infty,$$

W^l stay in a fixed compact subset of Ψ for all l, and F is uniformly elliptic. By the Evans-Krylov Theorem and Schauder theory,

$$\|u^l\|_{C^{2,\alpha}(\mathbb{S}^n)} \leq C,$$

independent of l. Therefore, there exist subsequences, we still denote φ_l and u^l,

$$\varphi_l \to \varphi \quad \text{in} \quad C^{1,\alpha}(\mathbb{S}^n), \quad u^l \to u \quad \text{in} \quad C^{3,\alpha}(\mathbb{S}^n),$$

for $0 > \varphi \in C^{1,1}(\mathbb{S}^n)$ with $\sup_{\mathbb{S}^n} \varphi = -1$, $(\varphi_{ij} + \varphi\delta_{ij}) \leq 0$ on $\mathbb{S}^n$, u satisfies equation (4.6) and the smallest eigenvalue of $(u_{ij}(x) + u(x)\delta_{ij})$ vanishes at some point x. On the other hand, Theorem 4.3 ensures $(u_{ij} + u\delta_{ij}) > 0$. This is a contradiction. ■

We also have the corresponding consequence of Theorem 4.2

COROLLARY 4.2. *In addition to the conditions on f and F in Theorem 4.2, we assume that F is concave and*

$$\lim_{\lambda\to\partial\Psi}|f(\lambda)|=\infty.$$

For any constant $\beta\geq 1$, there exist positive constants $\gamma>0,\vartheta>0$ such that if $\|\varphi(x)\|_{C^{1,1}(\Gamma)}\leq\beta$, and $\varphi(X,p)-\gamma$ is locally concave in X for any $p\in\mathbb{S}^n$ fixed, if M is a compact convex hypersurface satisfying (4.5) with $\|M\|_{C^2}\leq\beta$, then $\kappa_i(X)\geq\vartheta$ for all $X\in M$ and $i=1,\cdots,n$.

The proof of Corollary 4.2 is similar to the proof of Corollary 4.1, we won't repeat it here.

3. Codazzi tensors on Riemannian manifolds

Let (M,g) be a Riemannian manifold, a symmetric 2-tensor W is call a Codazzi tensor if W is closed (viewed as a TM-valued 1-form). W is Codazzi if and only if

$$\nabla_X W(Y,Z)=\nabla_Y W(X,Z),$$

for all tangent vectors X,Y,Z, where ∇ is the Levi-Civita connection. In local orthonormal frame, the condition is equivalent to w_{ijk} is symmetric with respect to indices i,j,k. Codazzi tensors arise naturally from differential geometry. We refer Chapter 16 in [**15**] for general discussions on Codazzi tensors in Riemannian geometry. The followings are some important examples.

(1) The second fundamental form of a hypersurface is a Codazzi tensor, implied by the Codazzi equation.
(2) If (M,g) is a space form of constant curvature c, then for any $u\in C^\infty(M)$, $W_u=Hess(u)+cug$ is a Codazzi tensor.
(3) If (M,g) has harmonic Riemannian curvature, then the Ricci tensor Ric_g is a Coddazi tensor and its scalar curvature R_g is constant.
(4) If (M,g) has harmonic Weyl tensor, the Schouten tensor S_g is a Codazzi tensor.

The convexity principle we established in the previous sections can be generalized to Codazzi tensors on Riemannian manifolds. Let (M,g) be a connected Riemannian manifold, for each $x\in M$, let $\tau(x)$ be the minimum of sectional curvatures at x.

PROPOSITION 4.1. *Let F as in Theorem 4.3, and (M,g) is a connected Riemannian manifold. Suppose $\varphi\in C^2(M)$ with $Hess(\varphi)(x)+\tau(x)\varphi(x)g(x)\leq 0$ for every $x\in M$. If W is a semi-positive definite Codazzi tensor on M satisfying equation*

$$F(g^{-1}W)=\varphi\quad on\ M, \tag{4.40}$$

then W is of constant rank.

Proof. The proof goes the similar way as in the proof of Proposition 4.3. We sketch here some necessary modifications.

We work on a small neighborhood of $z_0\in M$ be a point where $W(z_0)$ is of minimum rank l. Set $\phi(x)=\sigma_{l+1}(W(x))$ for $x\in O$. For any $z\in O\subset M$, we choose a local orthonormal frame so that at the point W is diagonal. As in the proof of Theorem 4.3, we may divide eigenvalues of

W at z into G and B, the "good" and "bad" sets of indices respectively with $|G| = l, |B| = n-l$. As before, (4.12)-(4.13) hold for our Codazzi tensor W. We want to show that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim \sum_{i\in B}[\varphi_{ii} + \tau\varphi] \tag{4.41}$$

Our condition on φ implies

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim 0.$$

Theorem 4.1 would follow from the strong minimum principle.

We now prove (4.41). The Codazzi condition implies W_{ijk} is symmetric respect to indices $\{ijk\}$. The same computation for $\phi = \sigma_{l+1}(W)$ in the proof of Theorem 4.1 can carry through to deduce the same formula (4.30) for our Codazzi tensor W.

Since W is diagonal at the point, it follows from Ricci identity, (4.12), (4.30) and homogeneity of F,

$$\begin{aligned}
\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} &\sim \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B}(W_{\alpha\alpha ii} + R_{i\alpha i\alpha}(W_{ii} - W_{\alpha\alpha})) - \sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2] \\
&\lesssim \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B}(W_{\alpha\alpha ii} - \tau W_{\alpha\alpha}) - \sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2 - 2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W_{ij\alpha}^2 \\
&= \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B} W_{\alpha\alpha ii} + (n-l)\tau\sigma_l(G)\varphi - \sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2 \\
&\quad -2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W_{ij\alpha}^2]. \qquad (4.42)
\end{aligned}$$

As in (4.27), we reduce that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim \sum_{i\in B}[\varphi_{ii} + \tau\varphi] - \sum_{i\in B} I_i - \sum_{i\in B} II_i - \sum_{i\in B} III_i, \tag{4.43}$$

where I_i, II_i, III_i defined similarly as in (4.27). (4.41) now follows directly from Lemma 4.1. ■

COROLLARY 4.3. *Suppose (M, g) is a connected Riemannian manifold with nonnegative harmonic Riemannian curvature, then the Ricci tensor is of constant rank.*

Proof. When M is compact, there is a stronger assertion that Ric_g is parallel by Weitzenböck formula and Stokes Theorem (e.g., [**15**]). Corollary 4.3 is of local nature, M is not assumed to be compact. Since (M, g) has nonnegative harmonic Riemannian curvature, Ric_g is a Codazzi tensor and it is semi-positive definite and the scalar curvature R_g is constant. Let $W = Ric_g$ and $F(W) = \sigma_1(W)$. W satisfies

$$F(g^{-1}W) = c. \tag{4.44}$$

The Corollary 4.3 now follows from Proposition 4.1. ■

The same argument also works for manifolds with non-positive harmonic curvature.

PROPOSITION 4.2. *Suppose (M, g) is a connected Riemannian manifold with non-positive harmonic Riemannian curvature, then the Ricci tensor is of constant rank.*

Proof. We work on $W = -Ric_g$. Since (M, g) has non-positive harmonic Riemannian curvature, Ric_g is a Codazzi tensor and it is semi-negative definite and the scalar curvature R_g is constant. So W is semi-positive definite and $\sigma_1(g^{-1}W) = c$ is a nonnegative constant. Let $F(W) = \sigma_1(W)$. W satisfies

$$F(g^{-1}W) = c. \tag{4.45}$$

Suppose $z_0 \in M$ is the point where W attains the minimal rank l. We choose a small neighborhood O of z_0, set $\phi(x) = \sigma_{l+1}(W(x))$ for $x \in O$. For any $z \in O$, we choose a local orthonormal frame so that at the point W is diagonal. As in the proof of Theorem 4.3, we may divide eigenvalues of W at z into G and B, the "good" and "bad" sets of indices respectively with $|G| = l, |B| = n - l$. As before, the proposition will follow, if we can show

$$\frac{1}{\sigma_l(G)} \sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim 0. \tag{4.46}$$

Following the same computation in the proof of Theorem 4.1, since W is diagonal at the point, it follows from Ricci identity, (4.12) and (4.30),

$$\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \sim \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B}(W_{\alpha\alpha ii} + R_{i\alpha i\alpha}(W_{ii} - W_{\alpha\alpha})) - \sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2]. \tag{4.47}$$

Since $R_{i\alpha i\alpha} \leq 0$, we have $|R_{i\alpha i\alpha}| \leq W_{ii}$. Again by (4.12), (4.47) becomes

$$\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B}(W_{ii\alpha\alpha} - \sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2]. \tag{4.48}$$

As in (4.27), we reduce that

$$\frac{1}{\sigma_l(G)} \sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim \sum_{i\in B}\varphi_{ii} - \sum_{i\in B} I_i - \sum_{i\in B} II_i - \sum_{i\in B} III_i, \tag{4.49}$$

where I_i, II_i, III_i defined similarly as in (4.27) and $\varphi = c$. (4.46) now follows directly from Lemma 4.1. ■

We note that if the Ricci tensor in Corollary 4.3 or in Proposition 4.2 is not of full rank, then the eigenspace distribution $V_0(x)$ corresponding to the zero eigenvalue of the Ricci tensor is of constant dimension and it is integrable (e.g., Proposition 16.11 in [**15**]). In fact, since the sectional curvature in both cases has a fixed sign, the nullity space $T_0(x)$ is the same as $V_0(x)$ for every $x \in M$. Therefore, T_0 is integrable and totally geodesic (e.g., Proposition 2, page 349, [**85**]).

Notes

The results in this chapter appeared in [**25**]. The deformation argument for the convexity by estimating a lower bound for problems in geometry used by Singer-Wang-Yau-Yau in [**118**] (see also [**106**] for the Christoffel-Minkowski problem). The argument here traces back to [**24**] where Caffarelli-Friedman treated semilinear equation in plane domains. Their result was generalized to domains in $\mathbb{R}^n$ by Korevaar-Lewis in [**86**]. A sufficient condition for solution of the Christoffel-Minkowski problem was found in [**67**] via this convexity approach, generalizing results in [**24, 86**] to equation (3.1). The corresponding results for σ_k (or quotient of elementary symmetric functions) of principal curvatures or principal radii were treated in [**63, 68**].

The constant rank results in Theorems 4.1-4.3 are of local nature in the sense that there is no global or boundary condition imposed on the solutions. Conditions (4.1)-(4.3) are natural, there is a large class of functions satisfying them. Some well known examples are: $f(\lambda) = \sigma_k^{\frac{1}{k}}(\lambda)$, $f(\lambda) = (\frac{\sigma_k}{\sigma_l})^{\frac{1}{k-l}}(\lambda)$, $f(\lambda) = -\sigma_k^{-\frac{1}{k}}(\lambda)$, $f(\lambda) = -(\frac{\sigma_k}{\sigma_l})^{-\frac{1}{k-l}}(\lambda)$ with $\Psi = \Gamma_k$, where $0 \leq l < k \leq n$, σ_j the jth elementary symmetric function and $\Gamma_k = \{\lambda \in \mathbb{R}^n | \quad \sigma_j(\lambda) > 0, \forall 1 \leq j \leq k\}$. The results in [**24, 86, 67, 63, 68**] should be interpreted as $f(\lambda) = -\sigma_k^{-\frac{1}{k}}(\lambda)$. We choose this form for the sake of a simple statement of the condition on φ. We also note that homogeneity assumption is not imposed in Theorem 4.1 and Theorem 4.2.

The condition (4.3) was first appeared in [**7**], where they treated the existence of convex viscosity solutions under state constraints boundary conditions and the assumption of a comparison principle for the state constraints problem. The conditions in Theorem 4.1, together with some proper convex cone condition on Ψ and concavity condition on f, were also used in [**9**] on pinching estimates of evolving closed convex hypersurfaces in $\mathbb{R}^{n+1}$. We also note that concavity condition on $\frac{1}{F(A^{-1})}$ was used in [**125**] for the related work on curvature flow of closed convex hypersurfaces in $\mathbb{R}^{n+1}$.

Combining the results in previous and this chapters, one funds a sufficient condition for solution of the Christoffel-Minkowski problem. Set $\mathcal{C}_{-\frac{1}{k}} = \{0 < \phi \in C^2(S^n) | (\phi_{ij}^{-\frac{1}{k}} + \delta_{ij}\phi^{-\frac{1}{k}}) \geq 0\}$.

THEOREM 4.4. *Let $\phi \in \mathcal{C}_{-\frac{1}{k}}$, then the Christoffel-Minkowski problem has a unique convex solution up to translations.*

The Theorem was first proved in Guan-Ma [**67**] under further assumption that ϕ is connected to 1 in $\mathcal{C}_{-\frac{1}{k}}$. It turns out this extra condition is redundant as $\mathcal{C}_{-\frac{1}{k}}$ is indeed connected. This fact was first proved in the joint work of Andrews-Ma [**10**] via curvature flow approach. More recently, this fact was also verified directly by Sheng-Trudinger-Wang [**116**].

CHAPTER 5

Weingarten curvature equations

In this chapter, we study the curvature equations of radial graphs over $\mathbb{S}^n$. Our main concern is the existence of hypersurface with prescribed Weingarten curvature on radial directions. For a compact hypersurface M in $\mathbb{R}^{n+1}$, the kth Weingarten curvature at $x \in M$ is defined as

$$W_k(x) = \sigma_k(\kappa_1(x), \kappa_2(x), \cdots, \kappa_n(x))$$

where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n)$ the principal curvatures of M, and σ_k is the kth elementary symmetry function. If the surface is starshaped about the origin, it follows that the surface can be parametrized as a graph over $\mathbb{S}^n$:

$$X = \rho(x)x, \qquad x \in \mathbb{S}^n, \tag{5.1}$$

where ρ is the radial function. In this correspondence, the Weingarten curvature can be considered as a function on $\mathbb{S}^n$ or in $\mathbb{R}^{n+1}$. There is an extensive literature on the problem of prescribing curvature functions. For example, given a positive function F in $\mathbb{R}^{n+1} \setminus \{0\}$, one would like to find a starshaped hypersurface M about the origin such that its kth Weingarten curvature is F. The problem is equivalent to solve the following equation

$$\sigma_k(\kappa_1, \kappa_2, ..., \kappa_n)(X) = F(X) \quad \text{for any} \quad X \in M. \tag{5.2}$$

Definition 5.1. *For $1 \le k \le n$, A C^2 surface M is called k-admissible if at every point $X \in M$, $(\kappa_1, \kappa_2, ..., \kappa_n) \in \Gamma_k$.*

Theorem 5.1. *Let $F(X)$ be a smooth positive function in $r_1 \le |X| \le r_2$, $r_1 < 1 < r_2$, satisfying*

$$F(X)^{\frac{1}{k}} \ \ge (C_n^k)^{\frac{1}{k}} \frac{1}{r_1} \quad \textit{for} \quad |X| = r_1, \quad F(X)^{\frac{1}{k}} \ \le (C_n^k)^{\frac{1}{k}} \frac{1}{r_2} \quad \textit{for} \quad |X| = r_2. \tag{5.3}$$

and

$$\frac{\partial}{\partial \rho}(\rho^k F(X)) \le 0, \quad \textit{where} \quad \rho = |X|. \tag{5.4}$$

Then there is a C^∞ k-admissible hypersurface M satisfying

$$\sigma_k(\kappa_1, \kappa_2, ..., \kappa_n)(X) = F(X). \tag{5.5}$$

Any two solutions are endpoints of a one-parameter family of homothetic dilations, all of which are solutions.

As a consequence of Theorem 4.2, we have the following existence of convex hypersurface with prescribed Weingarten curvature.

COROLLARY 5.1. *Suppose F as in Theorem 5.1, if in addition $F(X)^{-\frac{1}{k}}$ is a convex function in the region $r_1 < |X| < r_2$. Then the k-admissible solution in Theorem 5.1 is strictly convex.*

We also consider homogeneous Weingarten curvature problem. If M is a starshaped hypersurface about the origin in $\mathbb{R}^{n+1}$, by dilation property of the curvature function, the kth Weingarten curvature can be considered as a function of homogeneous degree $-k$ in $\mathbb{R}^{n+1} \setminus \{0\}$. The **homogeneous Weingarten curvature problem** is: given a homogeneous function F of degree $-k$ in $\mathbb{R}^{n+1} \setminus \{0\}$, does there exist a starshaped hypersurface M such that its kth Weingarten curvature is at $x \in M$ is equal to $F(x)$? If F is of homogeneous degree $-k$, then the barrier condition (5.3) will never be valid unless the function is constant. Therefore Theorem 5.1 is not applicable, the problem needs a different treatment. In fact, the problem is a nonlinear eigenvalue problem for the curvature equation.

THEOREM 5.2. *Suppose $n \geq 2$, $\quad 1 \leq k \leq n$ and f is a positive smooth function on $\mathbb{S}^n$. If $k < n$, assume further that f satisfies*

$$\sup_{\mathbb{S}^n} \frac{|\nabla f|}{f} < 2k, \tag{5.6}$$

Then there exist a unique constant $\gamma > 0$ with

$$\frac{C_n^k}{\max_{S^n} f} \leq \gamma \leq \frac{C_n^k}{\min_{S^n} f} \tag{5.7}$$

and a smooth k-admissible hypersurface M satisfying

$$\sigma_k(k_1, k_2, ..., k_n)(X) = \gamma f(\frac{X}{|X|})|X|^{-k}, \quad \forall X \in M, \tag{5.8}$$

and solution is unique up to homothetic dilations. Furthermore, for $1 \leq k < n$, if in addition $|X| f(\frac{X}{|X|})^{-\frac{1}{k}}$ is convex in $\mathbb{R}^{n+1} \setminus \{0\}$, then M is strictly convex.

For the simplicity of notations, the summation convention is always used. Covariant differentiation will simply be indicated by indices.

We first recall some identities for the relevant geometric quantities of a smooth closed compact starshaped hypersurfaces $M \subset \mathbb{R}^{n+1}$ about the origin. We assume the origin is not on M.

Since M is starshaped with respect to origin, the position vector X of M can be written as in (5.1). For any local orthonormal frame on $\mathbb{S}^n$, let ∇ be the gradient on $\mathbb{S}^n$ and covariant differentiation will simply be indicated by indices. Then in term of ρ the metric g_{ij} and its inverse g^{ij} on M are given by

$$g_{ij} = \rho^2 \delta_{ij} + \rho_i \rho_j. \quad g^{ij} = \rho^{-2}(\delta_{ij} - \frac{\rho_i \rho_j}{1 + |\nabla \rho|^2}).$$

The second fundamental form of M is

$$h_{ij} = (\rho^2 + |\nabla \rho|^2)^{-\frac{1}{2}} (\rho^2 \delta_{ij} + 2\rho_i \rho_j - \rho \rho_{ij}).$$

and the unit outer normal of the hypersurface M in $\mathbb{R}^{n+1}$ is $\mathbf{N} = \frac{\rho x - \nabla\rho}{\sqrt{\rho^2+|\nabla\rho|^2}}$. The principal curvature of M are the eigenvalue of the second fundamental form with respect to the metric and therefore are the solutions of

$$\det(h_{ij} - kg_{ij}) = 0.$$

Equivalently they satisfy

$$\det(A_{ij} - k\delta_{ij}) = 0.$$

where the symmetric matrix $\{A_{ij}\}$ is given by

$$\{A_{ij}\} = \{g^{ik}\}^{\frac{1}{2}} h_{kl} \{g^{lj}\}^{\frac{1}{2}}. \tag{5.9}$$

Let $\{g^{ij}\}^{\frac{1}{2}}$ be the positive square root of $\{g^{ij}\}$ and is given

$$[g^{ij}]^{\frac{1}{2}} = \rho^{-1}[\delta_{ij} - \frac{\rho_i\rho_j}{\sqrt{\rho^2+|\nabla\rho|^2}(1+\sqrt{\rho^2+|\nabla\rho|^2})}].$$

We may also work on orthonormal frame on M directly. We choose an orthonormal frame $\{e_A\}$ such that $\{e_1, e_2, ..., e_n\}$ are tangent to M and e_{n+1} is normal. Let the corresponding coframe be denoted by $\{\omega_A\}$ and the connection forms by $\{\omega_{A,B}\}$. The pull-backs of those through the immersion will still be denoted by $\{\omega_A\}$,$\{\omega_{A,B}\}$ if there is no confusion. Therefore $\omega_{n+1} = 0$ on M. The second fundamental form is defined by the symmetric matrix $\{h_{ij}\}$ with

$$\omega_{i,n+1} = h_{ij}\omega_j. \tag{5.10}$$

The following fundamental formulas are well known for hypersurfaces in $\mathbb{R}^{n+1}$.

$$X_{ij} = -h_{ij}e_{n+1}, \quad \text{(Gauss formula)} \tag{5.11}$$

$$(e_{n+1})_i = h_{ij}e_j, \quad \text{(Weingarten equation)} \tag{5.12}$$

$$h_{ijk} = h_{ikj}, \qquad \text{(Codazzi formula)} \tag{5.13}$$

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} \quad \text{(Gauss equation)}, \tag{5.14}$$

where R_{ijkl} is the curvature tensor. Using (5.13), (5.14) and the rule for interchanging the orders of derivatives, we observe the following commutation formula

$$h_{ijkl} = h_{klij} + (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} + (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}. \tag{5.15}$$

From (5.11)-(5.12)

$$(e_{n+1})_{ii} = h_{iij}e_j - h_{ij}^2 e_{n+1}. \tag{5.16}$$

Then $\sigma_k(\kappa_1, \kappa_2, ..., \kappa_n) = \sigma_k(\lambda\{h_{ij}\})$. We consider the following curvature equation

$$\sigma_k(\lambda\{h_{ij}\})(X) = f(X, e_{n+1}), \quad \forall X \in M, \tag{5.17}$$

where f is a positive function defined in $U \times \mathbb{S}^n$ for some neighborhood of M in $\mathbb{R}^{n+1}$.

Proof of Corollary 5.1. For $0 \le t \le 1$ and $0 < \epsilon < 1$, set

$$F(t, X) = [(1-t)(C_n^k)^{-\frac{1}{k}}|X|^{1+\epsilon} + tF^{-\frac{1}{k}}(X)]^{-k}.$$

Consider

$$\sigma_k(\kappa_1, \kappa_2, ..., \kappa_n)(X^t) = F(t, X^t), \tag{5.18}$$

Following the same lines of the proof using continuity method in [**26**], there is a unique M^t when $t \in [0, 1]$ solving (5.18) with C^4 norm under control. Using continuity method as a deformation process, M^t is strictly convex is preserved for $t \in [0, 1]$ by Theorem 4.2. ■

1. Homogeneous Weingarten curvature equation

We consider the homogeneous Weingarten curvature problem in this subsection. Since equation (5.2) is invariant under dilations, there is no C^0 bound in general. To solve the equation, we need to establish the Harnack inequality for solutions of (5.2). This is the main part of the proof in this section. We will follow ideas in [**61**] to consider the following auxiliary equation first

$$\sigma_k(\kappa_1, \kappa_2, ..., \kappa_n)(X) = f(\frac{X}{|X|})|X|^{-p}, \quad \forall X \in M, \quad 1 \le k \le n-1, \tag{5.19}$$

where f is a prescribed positive function on $\mathbb{S}^n$ and M is a starlike hypersurface in $\mathbb{R}^{n+1}$. Since M is starshaped, let ρ be the radial function as in (5.2). The following is the equation for ρ.

$$\sigma_k(\kappa_1, ..., \kappa_n)(x) = f(x)\rho^{-p} \quad \text{on} \quad \mathbb{S}^n, \tag{5.20}$$

We first derive an upper bound of $|\nabla^2\rho|$ estimates for the k-admissible solution ρ of equation (5.20) for any $p \in [k, k+1]$ assuming C^1 boundedness.

Lemma 5.1. *If M is a starlike hypersurface in $\mathbb{R}^{n+1}$ respect to the origin, f is a C^2 positive function on $\mathbb{S}^n$, $k > 1, p \in [k, k+1]$, if M is a C^4 k-admissible solution of equation (5.19). Then we have the mean curvature $H \le C$ for some constant C depends only on $k, n, \frac{|\nabla f|}{f}, \frac{|\nabla^2 f|}{f}$, $\|\rho\|_{C^1}$ and $\|\frac{1}{\rho}\|^0_C$ (independent of p). In turn, $\max_{x\in\mathbb{S}^n} |\nabla^2\rho(x)| \le C$.*

Proof: Let $F(X) = f(\frac{X}{|X|})$ and $\varphi(X) = [|X|^{-p}F(X)]^{\frac{1}{k}}$. The equation in Lemma 5.1 become

$$G(\lambda\{h_{ij}\})(X) = [\sigma_k(\lambda\{h_{ij}\})]^{\frac{1}{k}}(X) = \varphi(X), \quad \text{on} \quad M. \tag{5.21}$$

Assume the function $P = \log H - \log < X, e_{n+1} >$ attains its maximum at $X_o \in M$, then at X_o we have

$$P_i = \frac{H_i}{H} - \frac{< X, e_{n+1} >_i}{< X, e_{n+1} >} = 0, \quad P_{ii} = \frac{H_{ii}}{H} - \frac{< X, e_{n+1} >_{ii}}{< X, e_{n+1} >}.$$

Let $G^{ij} = \frac{\partial G(\lambda\{h_{ij}\})}{\partial h_{ij}}$, and choose the suitable $\{e_1, e_2, ..., e_n\}$ on the neighborhood of $X_o \in M$ such that at X_o the matrix $\{h_{ij}\}$ is diagonal. Then at X_o, the matrix $\{G^{ij}\}$ is also diagonal and positive definitive. At X_o

$$\sum_{ij=1}^{n} G^{ij}P_{ij} = \frac{\sum_{i=1}^n G^{ii}H_{ii}}{H} - \frac{\sum_{i=1}^n G^{ii} < X, e_{n+1} >_{ii}}{< X, e_{n+1} >} \le 0, \tag{5.22}$$

from this inequality we shall obtain an upper bound of H.

We set $|A|^2 = \sum_{i=1}^n h_{ii}^2$. From (5.14), we have

$$\sum_{i=1}^n G^{ii} H_{ii} = \sum_{i=1}^n G^{ii}(\sum_{l=1}^n h_{llii}) = \sum_{i=1}^n G^{ii} \sum_{l=1}^n (h_{iill} + h_{ii}h_{ll}^2 - h_{ll}h_{ii}^2)$$
$$= \sum_{il=1}^n G^{ii} h_{iill} + |A|^2 \sum_{i=1}^n G^{ii} h_{ii} - H \sum_{i=1}^n G^{ii} h_{ii}^2 \geq \sum_{l=1}^n \varphi_{ll} + |A|^2 \varphi - H \sum_{i=1}^n G^{ii} h_{ii}^2.$$

And from (5.10) and (5.15)

$$\sum_{i=1}^n G^{ii} < X, e_{n+1} >_{ii} = \sum_{i=1}^n G^{ii}[\sum_{l=1}^n h_{iil} < X, e_l > + h_{ii} - h_{ii}^2 < X, e_{n+1} >]$$
$$= \sum_{l=1}^n (\sum_{i=1}^n G^{ii} h_{iil}) < X, e_l > + \sum_{i=1}^n G^{ii} h_{ii} - < X, e_{n+1} > \sum_{i=1}^n G^{ii} h_{ii}^2$$
$$= \sum_{l=1}^n \varphi_l < X, e_l > + \varphi - < X, e_{n+1} > \sum_{i=1}^n G^{ii} h_{ii}^2.$$

So from (5.22), at X_o we have the following inequality

$$|A|^2 + \sum_{l=1}^n \frac{\varphi_{ll}}{\varphi} - \sum_{l=1}^n \frac{H\varphi_l}{\varphi < X, e_{n+1}} < X, e_l > - \frac{H}{< X, e_{n+1} >} \leq 0, \tag{5.23}$$

Let F_A, F_{AB} are the ordinary Euclidian differential in $\mathbb{R}^{n+1}$. Since

$$\frac{\varphi_l}{\varphi} = \frac{1}{k}[-p|X|^{-2} < X, e_l > + \sum_{A=1}^{n+1} \frac{F_A}{F} X_l^A],$$

$$\begin{aligned}
\sum_{l=1}^n \frac{\varphi_{ll}}{\varphi} &= H[\frac{p}{k}|X|^{-2} < X, e_{n+1} > - \frac{1}{k} \sum_{A=1}^{n+1} \frac{F_A}{F} e_{n+1}^A] \\
&+ \frac{1}{k} \sum_{l=1}^n \sum_{A,B=1}^{n+1} \frac{F_{AB}}{F} X_l^A X_l^B + \sum_{l=1}^n \sum_{AB=1}^{n+1} \frac{F_A F_B}{F^2} X_l^A X_l^B \\
&+ \frac{p}{k}[1 + \frac{p}{k}]|X|^{-2} - \frac{p}{k}[2 + \frac{p}{k}]|X|^{-4} < X, e_{n+1} >^2 \\
&- \frac{2p}{k^2}|X|^{-2} \sum_{l=1}^n \sum_{A=1}^{n+1} \frac{F_A}{F} X_l^A < X, e_l > .
\end{aligned}$$

As $|A|^2 \geq \frac{1}{n}H^2$, by (5.23) there exist a positive constant C depends only on the k, n, $\frac{|\nabla f|}{f}$, $\frac{|\nabla^2 f|}{f}$ such that $H(X_o) \leq C$. Again from C^1 bound, we have $\max H \leq C$. The proof of the Lemma is complete. ■

One may also derive C^1-estimates if C^0 bound is assumed. Instead, we will derive the Harnack inequality directly, that will imply C^0 and C^1 bounds. It is convenient to introduce a new function $v = -\log\rho$. Then the first and second fundamental forms become

$$g_{ij} = e^{-2v}[\delta_{ij} + v_i v_j],$$
$$h_{ij} = e^{-v}(1+|\nabla v|^2)^{-\frac{1}{2}}[\delta_{ij} + v_i v_j + v_{ij}].$$

and

$$[g^{ij}]^{\frac{1}{2}} = e^{v}[\delta_{ij} - \frac{v_i v_j}{\sqrt{1+|\nabla v|^2}(1+\sqrt{1+|\nabla v|^2})}].$$

So if we let

$$\overline{g}^{ij} = [\delta_{ij} - \frac{v_i v_j}{\sqrt{1+|\nabla v|^2}(1+\sqrt{1+|\nabla v|^2})}],$$
$$\overline{h}_{lm} = \delta_{lm} + v_l v_m + v_{lm},$$
$$a_{ij} = \overline{g}^{il}\overline{h}_{lm}\overline{g}^{mj}. \tag{5.24}$$

Then the matrix in (5.9) become

$$A_{ij} = e^{v}(1+|\nabla v|^2)^{-\frac{1}{2}} a_{ij}, \tag{5.25}$$

and equation (5.20) turns into

$$\sigma_k(\lambda\{a_{ij}\}) = e^{(p-k)v}(1+|\nabla v|^2)^{\frac{k}{2}} f(x) \quad \text{on} \quad \mathbb{S}^n. \tag{5.26}$$

First we have the easy case $p > k$.

Proposition 5.1. *Suppose $p > k$. For any $f(x) \in C^2(\mathbb{S}^n), n \geq 2, \quad f > 0$, there exist a unique k-admissible starlike hypersurface M satisfies (5.19). If in addition to f satisfies*

$$|X|^{\frac{p}{k}} f(\frac{X}{|X|})^{-\frac{1}{k}} \quad \textit{is a convex function in} \quad \mathbb{R}^{n+1}\setminus\{0\}, \tag{5.27}$$

then M is a strictly convex hypersurface.

Proof of Proposition 5.1: For any positive function $f \in C^2(\mathbb{S}^n)$, for $0 \leq t \leq 1$, set $f_t = [1 - t + tf^{-\frac{1}{k}}]^{-k}$. We consider the equation

$$\sigma_k(\kappa_1, ..., \kappa_n)(x) = f_t(x)\rho^{-p} \quad \text{on} \quad \mathbb{S}^n, \tag{5.28}$$

Set $I = \{t|(5.28) \quad \text{solvable}\}$.

We first consider $C^0 -$ **estimates**. let

$$l = \min_{\mathbb{S}^n}\rho \quad \text{and} \quad L = \max_{\mathbb{S}^n}\rho,$$

If $x_o \in \mathbb{S}^n$ such that $\rho(x_o) = L$. Then at x_o

$$\nabla\rho = 0, \quad \text{and} \quad \{\rho_{ij}\} \leq 0.$$

It follows that at x_o,

$$\kappa_i(x_o) \geq L^{-1}, \quad \forall \quad 1 \leq i \leq n.$$

Evaluating (5.28) at x_o, using the above, we have $L \leq [\frac{\max_{\mathbb{S}^n} f_t}{C_n^k}]^{\frac{1}{p-k}}$. The similar argument also yields $l \geq [\frac{\min_{\mathbb{S}^n} f_t}{C_n^k}]^{\frac{1}{p-k}}$.

With the C^0-estimates, the arguments in [**26**] immediately yield the C^1-estimates. Together with Lemma 5.1, we have

$$||\rho||_{C^2(\mathbb{S}^n)} \leq C \quad \text{and} \quad ||\frac{1}{\rho}||_{C^2(\mathbb{S}^n)} \leq C, \tag{5.29}$$

where C depends only on $p, k, n, ||f||_{C^2(\mathbb{S}^n)}$ and $\min_{\mathbb{S}^n} f$ (in the case $k = 1$, (5.29) follows from the standard quasilinear theory. The regularity assumption on f can also be reduced).

Now the Evens-Krylov theorem and the Schauder theorem imply that I is closed. The openness is from the implicit function theorem since the linearized operator of (5.26) is invertible when $p > k$. The method of continuity yields the existence. The uniqueness follows easily from the Strong Maximum Principle and the dilation property of equation (5.19) for $p > k$.

Since f_t satisfies the convexity condition (5.27) in Theorem 5.1 for $0 \leq t \leq 1$, the strict convexity from Theorem 4.2. ■

We now deal equation (5.19) for the case $p = k$ in the rest of this section. Equation is in the following form,

$$\sigma_k(\kappa_1, \kappa_2, ..., \kappa_n)(x) = f(x)\rho^{-k}, \quad \forall x \in \mathbb{S}^n, \tag{5.30}$$

In order to bound $\frac{\max \rho}{\min \rho}$, we turn to estimate $|\nabla \log \rho| = |\nabla v|$. We follow an argument in [**61**] to make use of the result in Proposition 5.1 with some refined estimates for ρ_r with $p_r = k + \frac{1}{r}$. We hope to get the convergence of ρ_r as r tends to infinity. It turn out the limit of ρ_r will satisfies equation (5.30) but with f replaced by γf for some positive γ. We will show the constant γ is unique.

Lemma 5.2. *For $1 \leq k \leq n$ and f is a positive C^1 function on $\mathbb{S}^n$. Suppose ρ is a C^3 k-admissible solution of equation (5.20) with $p \in [k, k+1]$. If $k < n$, we further assume that f satisfies*

$$\delta_f =: \min_{x \in \mathbb{S}^n, d_1 \leq s \leq d_2} \{k((\frac{(n-k)s}{nf(x)})^{\frac{1}{k}} + (\frac{nf(x)}{(n-k)s})^{\frac{1}{k}}) - \frac{|\nabla f(x)|}{f(x)}\} > 0, \tag{5.31}$$

where $d_1 = \min f$, $d_2 = \max f$. Then $\max_{S^n} |\nabla \log \rho(x)| \leq C$, for some constant C depending only on k, n, δ_f, $\max \frac{|\nabla f|}{f}$ (and independent of p). In particular,

$$1 \leq \frac{\max \rho}{\min \rho} \leq C.$$

Remark 5.1. *If $k = p$, from the proof below, the gradient estimate Lemma 5.2 can be established under simpler and weaker condition*

$$\min_{x \in \mathbb{S}^n} \{k(\frac{(C_{n-1}^k)^{\frac{1}{k}}}{f^{\frac{1}{k}}} + \frac{f^{\frac{1}{k}}}{(C_{n-1}^k)^{\frac{1}{k}}}) - \frac{|\nabla f|}{f}\} > 0.$$

From the counter-example in Treibergs, it can be shown that this condition is sharp for the gradient estimate of equation (5.26) when $1 \le k \le n-1$.

Proof: We work on equation (5.26) to get gradient estimates for v. Let $P = |\nabla v|^2$ attains its maximum at $x_o \in \mathbb{S}^n$, then

$$P_i = \sum_{k=1}^{n} v_k v_{ki} = 0, \quad \text{at} \quad x_o. \tag{5.32}$$

Let $\{e_1, e_2, ..., e_n\}$ be the standard orthonormal frame at the neighborhood of x_o, take e_1 such that

$$v_1 = |\nabla v|, \quad v_i = 0, \quad i \ge 2, \tag{5.33}$$

and $e_2, ..., e_n$ such that $\{v_{ij}\}(x_o)$ is diagonal, it follows that at x_o

$$v_{11} = 0, \qquad v_{ij} = 0, \quad i \ne j,$$

so the matrices $\{\overline{g}^{ij}\}, \{\overline{h}^{ij}\}$ and $\{a_{ij}\}$ are diagonal at the point, and $\overline{g}^{11} = \frac{1}{\sqrt{1+|\nabla v|^2}}$, $\overline{h}_{11} = 1 + |\nabla v|^2$, $a_{11} = 1$; and for all $i > 1$, $\overline{g}^{ii} = 1$, $\overline{h}_{ii} = a_{ii} = 1 + v_{ii}$.

Let $F^{ij} = \frac{\partial \sigma_k}{\partial a_{ij}}$, so $\{F^{ij}\}$ is diagonal at x_o. Differentiating equation (5.26) to get

$$F^{ij} a_{ijs} = e^{(p-k)v}(1 + |\nabla v|^2)^{\frac{k}{2}}[(p-k)v_s f + f_s]. \tag{5.34}$$

From (5.24),

$$a_{ijs} = (\overline{g}^{il}\overline{h}_{lm}\overline{g}^{mj})_s, \quad v_s \overline{g}^{mj}_s = 0 = v_s \overline{g}^{il}_s,$$

we have

$$v_s a_{ijs} = \overline{g}^{il} v_s v_{lms} \overline{g}^{mj}. \tag{5.35}$$

Couple (5.34) and (5.35)

$$v_s F^{ij} a_{ijs} = F^{ij}\overline{g}^{il} v_s v_{lms} \overline{g}^{mj} = e^{(p-k)v}(1 + |\nabla v|^2)^{\frac{k}{2}} v_s[(p-k)v_s f + f_s]. \tag{5.36}$$

On the other hand

$$\begin{aligned} v_s F^{ij} a_{ijs} &= F^{ij}\overline{g}^{il} v_s v_{lms} \overline{g}^{mj} = F^{lm}\overline{g}^{il} v_s v_{ijs} \overline{g}^{mj} \\ &= F^{lm}\overline{g}^{il}\overline{g}^{mj} v_s[v_{sij} - v_s\delta_{ij} + v_j\delta_{si}] \\ &= F^{lm}\overline{g}^{il}\overline{g}^{mj} v_s v_{sij} - |\nabla v|^2 \sum_{ilm} F^{lm}\overline{g}^{il}\overline{g}^{mi} + \sum_{ijlm} F^{lm}\overline{g}^{il}\overline{g}^{mj} v_i v_j. \end{aligned}$$

Let $\overline{F}^{ij} = \sum_{lm} F^{lm}\overline{g}^{il}\overline{g}^{mj}$, so at x_o, $\overline{F}^{ij}$ is diagonal with $\overline{F}^{11} = \frac{F^{11}}{1+v_1^2}$ and $\overline{F}^{ii} = F^{ii}$ for $i > 1$. Then we have

$$\sum_{ijs} v_s F^{ij} a_{ijs} = \sum_{ijs} \overline{F}^{ij} v_s v_{sij} - |\nabla v|^2 \sum_i \overline{F}^{ii} + \sum_{ij} \overline{F}^{ij} v_i v_j.$$

From (5.36), (5.37) and (5.33)

$$\sum_{ijs} \overline{F}^{ij} v_s v_{sij} = e^{(p-k)v}(1 + |\nabla v|^2)^{\frac{k}{2}} v_s[(p-k)v_s f + f_s] + |\nabla v|^2 \sum_{i=2}^{n} \overline{F}^{ii}. \tag{5.37}$$

For $\overline{F}^{ij}$ is positive definite and

$$P_{ij} = \sum_s v_{si}v_{sj} + \sum_s v_s v_{sij},$$

thus at x_o

$$\overline{F}^{ij}P_{ij} = \sum_{ijs} \overline{F}^{ij} v_{si}v_{sj} + \sum_{ijs} \overline{F}^{ij} v_s v_{sij} \leq 0, \tag{5.38}$$

From (5.37) and (5.38) it follows that at x_o

$$\sum_{i=2}^n \overline{F}^{ii}(v_1^2 + v_{ii}^2) + e^{(p-k)v}(1+|\nabla v|^2)^{\frac{k}{2}}[(p-k)v_1^2 f + v_1 f_1] \leq 0, \tag{5.39}$$

i.e., we obtain the following inequality

$$\sum_{i=2}^n \overline{F}^{ii}(v_1^2 + v_{ii}^2) + e^{(p-k)v}(1+|\nabla v|^2)^{\frac{k}{2}} v_1 f_1 \leq 0. \tag{5.40}$$

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ be the eigenvalues of the matrix $\{a_{ij}\}$, at the point,

$$\lambda_1 = 1, \quad \lambda_2 = 1 + v_{22}, ... \quad \lambda_n = 1 + v_{nn}; \tag{5.41}$$

and for $i \geq 2$

$$\overline{F}^{ii} = \sigma_{k-1}(\lambda|i), \quad v_{ii}^2 = \lambda_i^2 - 2\lambda_i + 1. \tag{5.42}$$

Then equation (5.26) becomes

$$\sigma_k(\lambda) = e^{(p-k)v}(1+|\nabla v|^2)^{\frac{k}{2}} f(x) \quad \text{on} \quad \mathbb{S}^n. \tag{5.43}$$

From (5.40) and (5.42) we have

$$\begin{aligned} &(1+v_1^2)\sum_{i=2}^n \sigma_{k-1}(\lambda|i) + \sum_{i=2}^n \lambda_i^2 \sigma_{k-1}(\lambda|i) \\ &-2\sum_{i=2}^n \lambda_i \sigma_{k-1}(\lambda|i) + e^{(p-k)v}(1+|\nabla v|^2)^{\frac{k}{2}} v_1 f_1 \leq 0. \end{aligned} \tag{5.44}$$

Since

$$\sum_{i=2}^n \sigma_{k-1}(\lambda|i) = (n-k)\sigma_{k-1}(\lambda) + \sigma_{k-2}(\lambda|1), \tag{5.45}$$

and

$$\begin{aligned} &\sum_{i=2}^n \lambda_i^2 \sigma_{k-1}(\lambda|i) - 2\sum_{i=2}^n \lambda_i \sigma_{k-1}(\lambda|i) \\ &= \sum_{i=1}^n \lambda_i^2 \sigma_{k-1}(\lambda|i) - 2\sum_{i=1}^n \lambda_i \sigma_{k-1}(\lambda|i) + \sigma_{k-1}(\lambda|1) \\ &= \sigma_1(\lambda)\sigma_k(\lambda) - (k+1)\sigma_{k+1}(\lambda) - 2k\sigma_k(\lambda) + \sigma_{k-1}(\lambda|1). \end{aligned} \tag{5.46}$$

Put (5.45) and (5.46) to (5.44), it follows that

$$\begin{aligned}&(1+v_1^2)(n-k)\sigma_{k-1}(\lambda)+\sigma_1(\lambda)\sigma_k(\lambda)-(k+1)\sigma_{k+1}(\lambda)\\&+e^{(p-k)v}v_1f_1(1+v_1^2)^{\frac{k}{2}}-2k\sigma_k(\lambda)+(1+v_1^2)\sigma_{k-2}(\lambda|1)+\sigma_{k-1}(\lambda|1)\leq 0.\end{aligned}\tag{5.47}$$

We also note that if x_1 and x_2 are minimum and maximum points of v respectively, from equation (5.43),

$$e^{(p-k)v(x_1)}\geq\frac{C_n^k}{f(x_1)}\geq\frac{C_n^k}{\max f},\quad e^{(p-k)v(x_2)}\leq\frac{C_n^k}{f(x_2)}\leq\frac{C_n^k}{\min f}.\tag{5.48}$$

So $\forall x$,

$$C_n^k\frac{\max f}{\min f}\geq e^{(p-k)v}f\geq C_n^k\frac{\min f}{\max f}.\tag{5.49}$$

This fact will be used in late on.

We divide into two cases.

Case 1: $k=n$**.**

As $\sigma_{n+1}(\lambda)=0$, and both $\sigma_{n-2}(\lambda|1)$ and $\sigma_{n-1}(\lambda|1)$ are positive, the above inequality takes a simpler form

$$\sigma_1(\lambda)\sigma_n(\lambda)+e^{(p-n)v}v_1f_1(1+v_1^2)^{\frac{n}{2}}\leq 2n\sigma_n(\lambda).$$

Since $\lambda_1=1$, $\sigma_n(\lambda)=\sigma_{n-1}(\lambda|1)$. By the Newton-MacLaurin inequality,

$$\sigma_1(\lambda)>\sigma_1(\lambda|1)\geq(n-1)\sigma_{n-1}(\lambda|1)^{\frac{1}{n-1}}=(n-1)\sigma_n(\lambda)^{\frac{1}{n-1}}.$$

In turn, we get

$$(n-1)\sigma_n(\lambda)^{\frac{n}{n-1}}-e^{(p-n)v}v_1|f_1|(1+v_1^2)^{\frac{n}{2}}\leq 2n\sigma_n(\lambda).\tag{5.50}$$

(5.43), (5.49) and (5.50) yield that at the point,

$$(n-1)(1+v_1^2)^{\frac{n}{2(n-1)}}(\frac{\min f}{\max f})^{\frac{1}{n-1}}-(1+v_1^2)^{\frac{1}{2}}\frac{|\nabla f|}{f}\leq 2n.$$

Since $\frac{n}{2(n-1)}>\frac{1}{2}$ and $\frac{\min f}{\max f}$ is bounded from below by a positive constant (depending only on the upper bound of $\frac{|\nabla f|}{f}$), we obtain an upper bound for $|\nabla v|$.

Case 2: $k<n$**.**

Claim:

$$(k+1)\sigma_{k+1}(\lambda)\leq(k+1)\sigma_k(\lambda)+(n-k-1)(C_{n-1}^k)^{-\frac{1}{k}}\sigma_k(\lambda)^{\frac{1}{k}+1}.\tag{5.51}$$

Proof of Claim:

If $\sigma_{k+1}(\lambda)\leq 0$, it is automatic. We may assume If $\sigma_{k+1}(\lambda)>0$. As $\lambda\in\Gamma_k$, we get $\lambda\in\Gamma_{k+1}$. In turn $(\lambda|1)\in\Gamma_k$. We have

$$\sigma_{k+1}(\lambda)=\sigma_{k+1}(\lambda|1)+\sigma_k(\lambda|1)\leq\sigma_{k+1}(\lambda|1)+\sigma_k(\lambda).\tag{5.52}$$

If $\sigma_{k+1}(\lambda|1)\leq 0$, we are done. Thus we may assume $\sigma_{k+1}(\lambda|1)>0$. Again as $(\lambda|1)\in\Gamma_k$, this gives $(\lambda|1)\in\Gamma_{k+1}$.

By the Newton-MacLaurin inequality,

$$\begin{aligned}\sigma_{k+1}(\lambda|1) \le C_{n-1}^{k+1}(C_{n-1}^k)^{-\frac{k+1}{k}}(\sigma_k(\lambda|1))^{\frac{k+1}{k}} &\le C_{n-1}^{k+1}(C_{n-1}^k)^{-\frac{k+1}{k}}\sigma_k(\lambda)^{\frac{k+1}{k}} \\ &= \frac{n-k-1}{k+1}(C_{n-1}^k)^{-\frac{1}{k}}\sigma_k^{\frac{1}{k}+1}(\lambda). \end{aligned} \tag{5.53}$$

The **Claim** now follows from (5.52)-(5.53).

Now back to the proof of the lemma. If $\sigma_k(\lambda|1) \le 0$, we will have $\sigma_{k-1}(\lambda) \ge \sigma_k(\lambda)$. From (5.51), (5.47) and the Newton-MacLaurin inequality, we get

$$(1+v_1^2)(n-k)\sigma_k(\lambda) - \frac{|\nabla f|}{f}(1+v_1^2)^{\frac{1}{2}}\sigma_k(\lambda) - (3k+1)\sigma_k(\lambda) \le 0.$$

¿From this we obtain an upper bound of $|\nabla v|$.

We may now assume $\sigma_k(\lambda|1) > 0$, i.e., $(\lambda|1) \in \Gamma_k$ in the rest of the proof. From the Newton-MacLaurin inequality,

$$\sigma_1(\lambda) > \sigma_1(\lambda|1) \ge (n-1)(C_{n-1}^k)^{\frac{-1}{k}}\sigma_k^{\frac{1}{k}}(\lambda|1),$$

similarly,

$$\sigma_1(\lambda|1) \ge (n-1)(C_{n-1}^{k-1})^{\frac{-1}{k-1}}\sigma_{k-1}^{\frac{1}{k-1}}(\lambda|1).$$

¿From this, we get

$$\begin{aligned}(\sigma_1(\lambda) + \frac{n-1}{n-k})^k &\ge \sigma_1^k(\lambda) + \frac{k(n-1)}{n-k}\sigma_1^{k-1}(\lambda) \\ &\ge \frac{(n-1)^k}{C_{n-1}^k}(\sigma_k(\lambda|1) + \sigma_{k-1}(\lambda|1)) = \frac{(n-1)^k}{C_{n-1}^k}\sigma_k(\lambda).\end{aligned}$$

That is

$$\sigma_1(\lambda) > (n-1)(C_{n-1}^k)^{\frac{-1}{k}}\sigma_k^{\frac{1}{k}}(\lambda) - \frac{n-1}{n-k}. \tag{5.54}$$

Since

$$(n-k)\sigma_{k-1}(\lambda) + \sigma_{k-2}(\lambda|1) = (n-k)\sigma_{k-1}(\lambda|1) + (n-k+1)\sigma_{k-2}(\lambda|1),$$

and $\sigma_k(\lambda) = \sigma_k(\lambda|1) + \sigma_{k-1}(\lambda|1)$, we get

$$[(n-k)\sigma_{k-1}(\lambda) + \sigma_{k-2}(\lambda|1)]^k = \sum_{0\le j\le k} C_k^j(n-k)^{k-j}(n-k+1)^j\sigma_{k-1}^{k-j}(\lambda|1)\sigma_{k-2}^j(\lambda|1),$$

$$k^kC_{n-1}^k\sigma_k^{k-1}(\lambda) = \sum_{0\le j\le k-1} k^kC_{n-1}^kC_{k-1}^j\sigma_k^{k-1-j}(\lambda|1)\sigma_{k-1}^j(\lambda|1).$$

Again using the Newton-MacLaurin inequality on $\sigma_l(\lambda|1)$, it is elementary to check that for $0 \le j \le k-1$,

$$C_k^j(n-k)^{k-j}(n-k+1)^j\sigma_{k-1}^{k-j}(\lambda|1)\sigma_{k-2}^j(\lambda|1) \ge k^kC_{n-1}^kC_{k-1}^j\sigma_k^{k-1-j}(\lambda|1)\sigma_{k-1}^j(\lambda|1),$$

that is

$$(n-k)\sigma_{k-1}(\lambda) + \sigma_{k-2}(\lambda|1) \ge k(C_{n-1}^k)^{\frac{1}{k}}\sigma_k^{\frac{k-1}{k}}(\lambda). \tag{5.55}$$

Combining (5.51), (5.55), (5.54), (5.43) and (5.47), we obtain

$$(1+v_1^2)^{\frac{1}{2}}(k(A+A^{-1})-\frac{|\nabla f|}{f})\leq C, \tag{5.56}$$

where $A=e^{\frac{(k-p)v}{k}}(\frac{C_{n-1}^k}{f})^{\frac{1}{k}}$ and C is a constant under control.

In view of condition (5.31), and by (5.48), we get $(1+v_1^2)^{\frac{1}{2}}\delta_f\leq C$. The proof is complete. ■

Since (5.6) implies (5.31), Theorem 5.2 is a consequence of the following proposition.

PROPOSITION 5.2. *Suppose $n\geq 2$, $1\leq k\leq n$, suppose f is a positive smooth function on $\mathbb{S}^n$. If $k<n$, we assume f satisfies condition (5.31). Then there exist a unique constant $\gamma>0$ satisfying (5.7) and a smooth k-admissible hypersurface M satisfying equation (5.8). The solution is unique up to homothetic dilations. Furthermore, for $1\leq k<n$, if in addition $|X|f(\frac{X}{|X|})^{-\frac{1}{k}}$ is convex in $\mathbb{R}^{n+1}\setminus\{0\}$, then M is strictly convex.*

Proof of Proposition 5.2:

First we deal with the existence of solution and γ. For all $r\in\mathbf{Z}^+$, from Proposition 5.1, we let $\rho_r=|X_r|$ be the unique solution of equation (5.20) with $p=k+\frac{1}{r}$. We rescale ρ, let $\tilde{\rho}_r=\frac{\rho_r}{l_r}$, with $l_r=\min\rho_r$. Now $\tilde{\rho}_r$ satisfies

$$\sigma_k(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n)(x)=\tilde{\rho}^{-k-\frac{1}{r}}\tilde{f}_r(x),\quad \text{on}\quad \mathbb{S}^n,$$

where $\tilde{f}={l_r}^{-\frac{1}{r}}f$. ¿From (5.48), $\frac{C_n^k\min_{\mathbb{S}^n}f}{\max_{\mathbb{S}^n}f}\leq\tilde{f}\leq\frac{C_n^k\max_{\mathbb{S}^n}f}{\min_{\mathbb{S}^n}f}$.

if f satisfies the conditions in the proposition, by Lemmas 5.2 5.1, there exists a positive constant C independent of r, such that $1\leq\tilde{\rho}_r$ and $\|\tilde{\rho}_r\|_{C^2}\leq C$. The Evans-Krylov theorem gives $||\tilde{\rho}_r||_{C^{l,\alpha}}\leq C_{l,\alpha}$, with $C_{l,\alpha}$ $(l\geq 2, 0<\alpha<1)$ independent of r. So, there is a subsequence $r_j\to\infty$, such that $\tilde{\rho_{r_j}}\to\rho$ in $C^{l,\alpha}(\mathbb{S}^n)$, and ${l_{r_j}}^{-\frac{1}{r_j}}\to\gamma$ for some positive constant γ. (5.48) implies (5.7) and the radial graph of ρ satisfies (5.8). The higher regularity of ρ follows from the standard elliptic theory.

We now turn to the uniqueness. Let $M(\rho)=\sigma_k(\kappa_1,\kappa_2,...,\kappa_n)\rho^k$ and suppose $\exists\gamma_0$, γ_1, $\rho_0>0$ and $\rho_1>0$ satisfying (5.8) respectively. We may assume $\gamma_0\geq\gamma_1$, so we have

$$M(\rho_0)-M(\rho_1)=(\gamma_0-\gamma_1)f\geq 0.$$

Since M is invariant under scaling, we may assume $\rho_0\leq\rho_1$, and $\rho_0(x_o)=\rho_1(x_o)$ at some point $x_o\in\mathbb{S}^n$. Let $\rho_t=t\rho_1+(1-t)\rho_0$. Since $\rho_t=\rho_0$ and $\nabla\rho_t=\nabla\rho_0$ at x_o. So the first fundamental forms of ρ_t are same at x_o for all $0\leq t\leq 1$. Therefore ρ_t is k-admissible for all $0\leq t\leq 1$ at x_o. By the continuity of the second derivatives, there is a neighborhood of x_o such that ρ_t is

k-admissible for all $0\leq t\leq 1$. We have, in the neighborhood of x_o,

$$\begin{aligned}M(\rho_1)-M(\rho_0)&=\int_0^1\frac{\partial}{\partial t}M_t dt\\&=\sum_{i,j=1}^n b^{ij}(\rho_1,\rho_0)(\rho_1-\rho_0)_{ij}+\sum_{i=1}^n c^i(\rho_1,\rho_0)(\rho_1-\rho_0)_i+d(\rho_1,\rho_0)(\rho_1-\rho_0).\end{aligned}$$

By the Strong Maximum Principle, $\rho_1 = \rho_0$ everywhere and $\gamma_1 = \gamma_0$.

Finally we discuss the *convexity*. It is easy to check that the convexity of $|X| f^{-\frac{1}{k}}(\frac{X}{|X|})$ implies the convexity of $|X|^{\frac{p}{k}} f^{-\frac{1}{k}}(\frac{X}{|X|})$ for any $p \geq k$. When $1 \leq k \leq n-1$, from Proposition 5.1, we know the solution $M = \{\rho(x)x : \mathbb{S}^n \to \mathbb{R}^{n+1}\}$ is convex if f satisfies the convex condition in Theorem 5.2. The strict convexity follows from Theorem 4.2. ■

Notes

The equations we treated in this chapter were first considered by Alexandrov [**6**] and Aeppli [**1**], they studied the uniqueness question of starshaped hypersurfaces with prescribed curvature.

Theorem 5.1 was proved by Caffarelli-Nirenberg-Spruck in [**26**] (in the case $k = 1$, by Bakelman-Kantor [**12**], Treibergs-Wei [**121**]). The question of convexity of solution in Theorem 5.1 was treated by Chou [**37**] (see also [**133**]) for the mean curvature case under concavity assumption on F, and by Gerhardt [**52**] for general Weingarten curvature case under concavity assumption on $\log F$, see also [**53**] for the work on general Riemannian manifolds. The convexity results for hypersurfaces in this chapter were proved in [**62**].

When $k = n$, then equation (5.2) can be expressed as a Monge-Ampère equation of radial function ρ on $\mathbb{S}^n$, the problem was studied by Delanoë [**41**]. The case $k = 1$ was considered by Treibergs in [**120**]. Here we give a uniform treatment for $1 \leq k \leq n$ here. Condition (5.6) in Theorem 5.2 can be weakened as in Proposition 5.2.

CHAPTER 6

Problem of prescribed curvature measure

Curvature measure is one of the basic notion in the theory of convex bodies. Together with surface area measures, they play fundamental roles in the study of convex bodies. They are closely related to the differential geometry and integral geometry of convex hypersurfaces. Let Ω is a bounded convex body in $\mathbb{R}^{n+1}$ with C^2 boundary M, the corresponding curvature measures and surface area measures of Ω can be defined according to some geometric quantities of M. Let $\kappa = (\kappa_1, \cdots, \kappa_n)$ be the principal curvatures of M at point x, let $W_k(x) = \sigma_k(\kappa(x))$ be the k-th Weingarten curvature of M at x (where σ_k the k-th elementary symmetric function). In particular, W_1 is the mean curvature, W_2 is the scalar curvature, and W_n is the Gauss-Kronecker curvature. The k-th curvature measure of Ω is defined as

$$\mathcal{C}_k(\Omega, \beta) := \int_{\beta \cap M} W_{n-k} dF_n,$$

for every Borel measurable set β in $\mathbb{R}^{n+1}$, where dF_n is the volume element of the induced metric of $\mathbb{R}^{n+1}$ on M. Since M is convex, M is star-shaped about some point. We may assume that the origin is inside of Ω. Since M and $\mathbb{S}^n$ is diffeomorphic through radial correspondence R_M. Then the k-th curvature measure can also be defined as a measure on each Borel set β in $\mathbb{S}^n$:

$$\mathcal{C}_k(M, \beta) = \int_{R_M(\beta)} W_{n-k} dF_n.$$

We note that $\mathcal{C}_k(M, \mathbb{S}^n)$ is the k-th quermasintegral of Ω. Similarly, if M is strictly convex, let $r_1, ..., r_n$ be the principal radii of curvature of M, $P_k = \sigma_k(r_1, \cdots, r_n)$. The k-th surface area measure of Ω then can be defined as

$$\mathcal{S}_k(\Omega, \beta) := \int_\beta P_k d\sigma_n,$$

for every Borel set β in $\mathbb{S}^n$.

Curvature measure problem: Given a C^2 positive function f on $\mathbb{S}^n$. For each $0 \le k < n$, find a convex hypersurface M as a graph over $\mathbb{S}^n$, such that $\mathcal{C}_{n-k}(M, \beta) = \int_\beta f d\sigma$ for each Borel set β in $\mathbb{S}^n$, where $d\sigma$ is the standard volume element on $\mathbb{S}^n$.

The problem is equivalent to solve certain curvature equation on $\mathbb{S}^n$. If M is of class C^2, then

$$\mathcal{C}_{n-k}(M, \beta) = \int_{R_M(\beta)} \sigma_k dF = \int_\beta \sigma_k g d\sigma. \tag{6.57}$$

where g is the density of dF respect to standard volume element $d\sigma$ on $\mathbb{S}^n$. The problem of prescribing $(n-k)$-th curvature measure can be reduced to the following curvature equation

$$\sigma_k(\kappa_1, \kappa_2, ..., \kappa_n) = \frac{f(x)}{g(x)}, \quad 1 \le k \le n \quad \text{on} \quad \mathbb{S}^n \tag{6.58}$$

Here we encounter a difficulty issue around equation (6.58): the lack of some appropriate a priori estimates for admissible solutions due to the appearance of $g(x)$ (which implicitly involves the gradient of solution) make the matter very delicate.

Since equation (6.58) is originated in geometric problem in the theory of convex bodies, the purpose of this paper is to find convex hypersurface M (as a graph over $\mathbb{S}^n$) satisfying equation (6.58). The followings are our main results.

THEOREM 6.1. *Suppose $f(x) \in C^2(\mathbb{S}^n)$, $\quad f > 0$, $n \ge 2$, $1 \le k \le n-1$. If f satisfies the condition*

$$|X|^{\frac{n+1}{k}} f(\frac{X}{|X|})^{-\frac{1}{k}} \quad \textit{is a strictly convex function in} \quad \mathbb{R}^{n+1} \setminus \{0\}, \tag{6.59}$$

then there exists a unique strictly convex hypersurface $M \in C^{3,\alpha}, \alpha \in (0,1)$ such that it satisfies (6.58).

When $k = 1$ or 2, the strict convex condition (6.59) can be weakened.

THEOREM 6.2. *Suppose $k = 1, or \quad 2$ and $k < n$, and suppose $f(x) \in C^2(\mathbb{S}^n)$ is a positive function. If f satisfies*

$$|X|^{\frac{n+1}{k}} f(\frac{X}{|X|})^{-\frac{1}{k}} \quad \textit{is a convex function in} \quad \mathbb{R}^{n+1} \setminus \{0\}, \tag{6.60}$$

then there exists unique strictly convex hypersurface $M \in C^{3,\alpha}, \alpha \in (0,1)$ such that it satisfies equation (6.58).

We first recall some relevant geometric quantities of a smooth closed hypersurface $M \subset \mathbb{R}^{n+1}$, which we suppose the origin is not contained in M.

$A, B, ...$ will be from 1 to $n+1$ and Latin from 1 to n, the repeated indices denote summation over the indices. Covariant differentiation will simply be indicated by indices.

Let M^n be a n-dimension closed hypersurface immersed in $\mathbb{R}^{n+1}$. We choose an orthonormal frame in $\mathbb{R}^{n+1}$ such that $\{e_1, e_2, ..., e_n\}$ are tangent to M and e_{n+1} is the outer normal. Let the corresponding coframe be denoted by $\{\omega_A\}$ and the connection forms by $\{\omega_{A,B}\}$. The pull back of their through the immersion are still denoted by $\{\omega_A\}$,$\{\omega_{A,B}\}$ in the abuse of notation. Therefore on M

$$\omega_{n+1} = 0.$$

The second fundamental form is defined by the symmetry matrix $\{h_{ij}\}$ with

$$\omega_{i,n+1} = h_{ij}\omega_j. \tag{6.61}$$

Since M is starshaped with respect to origin, the position vector X of M can be written as $X(x) = \rho(x)x, \quad x \in \mathbb{S}^n$, where ρ is a smooth function on $\mathbb{S}^n$. Let $\{e_1, ..., e_n\}$ be smooth local

orthonormal frame field on $\mathbb{S}^n$, let ∇ be the gradient on $\mathbb{S}^n$ and covariant differentiation will simply be indicated by indices. Then in term of ρ the metric of M is given by

$$g_{ij} = \rho^2\delta_{ij} + \rho_i\rho_j.$$

So the area factor

$$g = (\det g_{ij})^{\frac{1}{2}} = \rho^{1-n}(\rho^2 + |\nabla\rho|^2)^{-\frac{1}{2}}.$$

The second fundamental form of M is

$$h_{ij} = (\rho^2 + |\nabla\rho|^2)^{-\frac{1}{2}}(\rho^2\delta_{ij} + 2\rho_i\rho_j - \rho\rho_{ij}).$$

and the unit outer normal of the hypersurface M in $\mathbb{R}^{n+1}$ is

$$\mathbf{N} = \frac{\rho x - \nabla\rho}{\sqrt{\rho^2 + |\nabla\rho|^2}}. \tag{6.62}$$

The principal curvature $(\kappa_1, \kappa_2, ..., \kappa_n)$ of M are the eigenvalue of the second fundamental form respect to the metric and therefore are the solutions of

$$\det(h_{ij} - kg_{ij}) = 0.$$

Equation (6.58) can be expressed as a differential equations on the radial function ρ and position vector X respectively.

$$\sigma_k(\kappa_1, \kappa_2, ..., \kappa_n) = f\rho^{1-n}(\rho^2 + |\nabla\rho|^2)^{-1/2}, \quad \text{on} \quad \mathbb{S}^n, \tag{6.63}$$

where $f > 0$ is the given function. From (6.62) we have

$$< X, N >= \rho^2(\rho^2 + |\nabla\rho|^2)^{-1/2}.$$

$$\sigma_k(\kappa_1, \kappa_2, ..., \kappa_n)(X) = |X|^{-(n+1)} f(\frac{X}{|X|}) < X, N >, \quad \forall X \in M. \tag{6.64}$$

Definition 6.1. *For $1 \leq k \leq n$, let Γ_k be a cone in $\mathbb{R}^n$ determined by*

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \quad \sigma_1(\lambda) > 0, ..., \quad \sigma_k(\lambda) > 0\}.$$

A C^2 surface M is called k-admissible if at every point $X \in M$, $(\kappa_1, \kappa_2, ..., \kappa_n) \in \Gamma_k$.

The following three lemmas had been proved in [**60**], for the completeness we provide the proofs here.

Lemma 6.1. *If M satisfies (6.64), then*

$$(\frac{\min_{\mathbb{S}^n} f}{C_n^k})^{1/(n-k)} \leq \min_{\mathbb{S}^n} |X| \leq \max_{\mathbb{S}^n} |X| \leq (\frac{\max_{\mathbb{S}^n} f}{C_n^k})^{1/(n-k)}.$$

In particular, if M is convex and ρ is the radial function of M, then there is a constant C depending only on $\max f$ *and* $\min f$ *such that*

$$max_{\mathbb{S}^n}|\nabla\rho| \leq C. \tag{6.65}$$

Proof: Let $B_R(o)$ be a ball of smallest radius so that $M \subset B_R(o)$, then at the maximum point X_1 of $|X|$, $R = |X_1|$. Through some geometrical considerations, we have

$$f(\frac{X_1}{|X_1|}) \geq C_n^k |X_1|^{n-k}.$$

This is

$$\max_{\mathbb{S}^n} |X| \leq (\frac{\max_{\mathbb{S}^n} f}{C_n^k})^{1/(n-k)}.$$

The first half inequality can be shown in a similar way.

The gradient estimates follows from C^0 estimates and convexity. In fact, the gradient estimates for general admissible solutions are also true, which was proved in [**60**]. ■

Set $F = \sigma_k^{1/k}$, equation (6.63) is written as

$$F(\lambda) \equiv F(\lambda_1, ..., \lambda_n) = f^{1/k} \rho^{(1-n)/k} (\rho^2 + |\nabla \rho|^2)^{-1/(2k)} \equiv K(x, \rho, \nabla \rho).$$

The following is the uniqueness result of the problem.

Lemma 6.2. *Suppose $1 \leq k < n$, $\lambda(\rho_i) \in \Gamma_k$, $i = 1, 2$. Suppose ρ_1, ρ_2 are solutions of (6.63). Then $\rho_1 \equiv \rho_2$.*

Proof Suppose the contrary, $\rho_2 > \rho_1$ somewhere on $\mathbb{S}^n$. Take $t \geq 1$ such that

$$t\rho_1 \geq \rho_2 \quad \text{on } \mathbb{S}^n, \qquad t\rho_1 = \rho_2 \quad \text{at some point } P \in \mathbb{S}^n.$$

Obviously, $\lambda(t\rho_1) = t^{-1}\lambda(\rho_1)$, and therefore $F(\lambda(t\rho_1)) = t^{-1}F(\lambda(\rho_1))$. It is clear that

$$\begin{aligned} &K(x, t\rho_1, \nabla(t\rho_1)) = t^{-n/k} K(x, \rho_1, \nabla \rho_1) \\ &= t^{-n/k} F(\lambda(\rho_1)) \leq t^{-1} F(\lambda(\rho_1)) = F(\lambda(t\rho_1)). \end{aligned}$$

It follows that

$$F(\lambda(t\rho_1)) - K(x, t\rho_1, \nabla(t\rho_1)) \geq 0, \qquad F(\lambda(\rho_2)) - K(x, \rho_2, \nabla \rho_2) = 0.$$

Hence

$$L(t\rho_1 - \rho_2) \geq 0,$$

where L is the linearized operator. Applying the strong maximum principle, we have $t\rho_1 - \rho_2 \equiv 0$ on $\mathbb{S}^n$. Since $n > k$, from equation (6.63), we conclude that $t = 1$. ■

The following lemma will also be used in this paper.

Lemma 6.3. *Let L denote the linearized operator of $F(\lambda) - K(x, \rho, \nabla \rho)$ at a solution ρ of (6.63), w satisfies $Lw = 0$ on $\mathbb{S}^n$. Then $w \equiv 0$ on $\mathbb{S}^n$.*

Proof Writing $F(x, \rho, \nabla \rho, \nabla^2 \rho) \equiv F(\lambda)$, we have

$$F(x, t\rho, \nabla(t\rho), \nabla^2(t\rho)) = F(\lambda(t\rho)) = F(\lambda(\rho)/t).$$

Applying $\frac{d}{dt}\big|_{t=1}$, we have

$$F_{\nabla^2 \rho} \nabla^2(\rho) + F_{\nabla \rho} \nabla \rho + F_\rho \rho = -\sum_i \lambda_i F_{\lambda_i} = -F.$$

It is easy to see that

$$K(x, t\rho, \nabla(t\rho)) = t^{-n/k} K(x, \rho, \nabla\rho).$$

Applying $\frac{d}{dt}\big|_{t=1}$, we have

$$K_{\nabla\rho}\nabla\rho + F_\rho \rho = -n/k K(x, \rho, \nabla\rho).$$

It follows from and that

$$L\rho = -F(\lambda) + n/k K(x, \rho, \nabla) = (n/k - 1)K(x, \rho, \nabla\rho) > 0.$$

Set $w = z\rho$. We know that

$$0 = Lw = L(z\rho) \equiv L'z + zL\rho,$$

where $L'z = \rho F_{\nabla^2\rho}\nabla^2 z$+first order term in z. Notice that $L\rho > 0$, we derive from the maximum principle that $z \equiv 0$, namely, $w \equiv 0$. ■

We first prove C^2 estimates for equation (6.58) under the convexity of solution. For the mean curvature measure case ($k = 1$), a gradient bound is enough for a C^2 a priori bound by the standard theory of quasilinear elliptic equations. For the rest of this section, we assume $k > 1$.

For the C^2 estimates for admissible solutions of (6.58), it is equivalent to estimate the upper bounds of principal curvatures. If the hypersurface is strictly convex, it is simple to observe that a positive lower bound on the principal curvatures implies an upper bound of the principal curvatures. This follows from equation (6.58) and the Newton-Maclaurin inequality,

$$\sigma_n{}^{\frac{1}{n}}(\lambda) \leq [\frac{\sigma_k}{C_n^k}]^{\frac{1}{k}}(\lambda).$$

This is the starting point of our approach here. To achieve such a lower bound, we shall use the inverse Gauss map and consider the equation for the support function of the hypersurface. The role of the Gauss map here should be compared with the role of the Legendre transformation on the graph of convex surface in a domain in $\mathbb{R}^n$. Since M is curved and compact, the Gauss map fits into the picture neatly. This way, we can make use some special features of the support function. We note that a lower bound on the principal curvature is an upper bound on the principal radii. And the principal radii are exactly the eigenvalues of the spherical hessians of the support function. Therefore, we are led to get a C^2 bound on the support function of M.

Let $X : M \to \mathbb{R}^{n+1}$ be a closed strictly convex smooth hypersurface in $\mathbb{R}^{n+1}$. We may assume the X is parametrized by the inverse Gauss map

$$X : \quad \mathbb{S}^n \quad \to \quad \mathbb{R}^{n+1}.$$

The support function of X is defined by

$$u(x) =< x, X(x) >, \quad \text{at} \quad x \in \mathbb{S}^n.$$

Let $e_1, e_2, ..., e_n$ be a smooth local orthonormal frame field on $\mathbb{S}^n$, we know that the inverse second fundamental form of X is

$$h_{ij} = u_{ij} + u\delta_{ij},$$

and the metric of X is

$$g_{ij} = \sum_{l=1}^{n} h_{il} h_{jl}.$$

The principal radii of curvature are the eigenvalues of matrix

$$W_{ij} = u_{ij} + u\delta_{ij}.$$

Equation (6.63) can be rewritten as an equation on support function u.

$$F(W_{ij}) = [\frac{\det W_{ij}}{\sigma_{n-k}(W_{ij})}]^{\frac{1}{k}}(x) = G(X)u^{-\frac{1}{k}} \quad \text{on} \quad \mathbb{S}^n, \tag{6.66}$$

where X is position vector of hypersurface, and

$$G(X) = |X|^{\frac{n+1}{k}} f^{-\frac{1}{k}}(\frac{X}{|X|}).$$

Equation (ref3.2) is similar to the equation in [**57**], where a problem of prescribing Weingarten curvature was considered. The position function and the support function have the following explicit form.

$$X(x) = \sum_{i=1}^{n} u_i e_i + ux, \quad \text{on} \quad x \in \mathbb{S}^n.$$

It follows from some straightforward computations,

$$X_l = u_{il} e_i + u_i (e_i)_l + u_l x + u x_l = u_{il} e_i - x u_i \delta_{il} + u_l x + u e_l = W_{il} e_i, \tag{6.67}$$

$$\begin{aligned} \sum_{l=1}^{n} X_{ll} &= \sum_{i,l=1}^{n} [W_{ill} e_i + W_{il} (e_i)_l] \\ &= \sum_{i=1}^{n} [\sum_{l=1}^{n} W_{ll}]_i e_i - \sum_{i,l=1}^{n} W_{il}(-x\delta_{il}) = \sum_{i=1}^{n} [\sum_{l=1}^{n} W_{ll}]_i e_i - x \sum_{l=1}^{n} W_{ll}. \end{aligned} \tag{6.68}$$

The following is a key lemma.

Lemma 6.4. *If $G(X)$ is strictly convex function in $\mathbb{R}^{n+1} \setminus \{o\}$, then*

$$\max(\Delta u + nu) \leq C, \tag{6.69}$$

where the constant C depends only on $n, \max_{\mathbb{S}^n} f, \min_{\mathbb{S}^n} f$ and $|\nabla f|_{C^0}$ and $|\nabla^2 f|_{C^0}$. In turn,

$$|\nabla^2 \rho| \leq C. \tag{6.70}$$

Proof: Since we already obtained C^1 bound in Lemma 6.1, to get (6.70), we only need to prove (6.69). Let

$$H = \sum_{l=1}^{n} = \Delta u + nu$$

and assume the maximum of H attains at some point $x_o \in \mathbb{S}^n$. We choose an orthonormal frame $e_1, e_2, ..., e_n$ near x_o such that $u_{ij}(x_o)$ is diagonal (so is $W_{ij} = u_{ij} + u\delta_{ij}$ at x_o). The following formula for commuting covariant derivatives are elementary:

$$(\Delta u)_{ii} = \Delta(u_{ii}) + 2\Delta u - 2nu_{ii}.$$

So we have

$$H_{ii} = (\Delta u)_{ii} + nu_{ii} = \Delta(W_{ii}) - nW_{ii} + H. \tag{6.71}$$

Let $F^{ij} = \frac{\partial F(W)}{\partial W_{ij}}$. At x_o the matrix F^{ij} is positive definite, diagonal. Setting the eigenvalues of W_{ij} at x_o as $\lambda(W_{ij}) = (\lambda_1, \lambda_2, ..., \lambda_n))$,

$$F^{ii} = \frac{1}{k}(\frac{\sigma_n}{\sigma_{n-k}})^{\frac{1}{k}}[\frac{\sigma_{n-1}(\lambda|i)}{\sigma_{n-k}} - \frac{\sigma_n\sigma_{n-k-1}(\lambda|i)}{{\sigma_{n-k}}^2}].$$

The following facts are true (e.g., see [**57**]).

$$\sum_{i=1}^{n} F^{ii}W_{ii} = F, \quad \sum_{i=1}^{n} F^{ii} \geq (C_n^{n-k})^{-\frac{1}{k}}.$$

Now at x_o, we have

$$H_i = 0, \qquad H_{ij} \leq 0 \tag{6.72}$$

Through this section the repeated upper indices denote summation over the indices, and our calculation will do at x_o. Using the above calculations we have

$$\begin{aligned} 0 \quad &\geq \sum_{i,j=1}^{n} F^{ij}H_{ij} = \sum_{i=1}^{n} F^{ii}H_{ii} = \sum_{i=1}^{n} F^{ii}\Delta(W_{ii}) - n\sum_{i=1}^{n} F^{ii}W_{ii} + H\sum_{i=1}^{n} F^{ii} \\ &\geq \sum_{i=1}^{n} F^{ii}\Delta(W_{ii}) - nF + (C_n^{n-k})^{-\frac{1}{k}}H. \end{aligned} \tag{6.73}$$

From the equation (6.66)

$$F^{ij}W_{ijl} = [G(X)u^{-\frac{1}{k}}]_l, \quad F^{ij}W_{ijll} + F^{ij,st}W_{ijl}W_{stl} = [G(X)u^{-\frac{1}{k}}]_{ll}.$$

From the concavity of F, we get

$$\sum_{i=1}^{n} F^{ii}\Delta(W_{ii}) \geq \sum_{l=1}^{n}[G(X)u^{-\frac{1}{k}}]_{ll},$$

combining this with (6.73) we have the following inequality at x_o

$$\sum_{l=1}^{n}[G(X)u^{-\frac{1}{k}}]_{ll} - nF + (C_n^{n-k})^{-\frac{1}{k}}H \leq 0. \tag{6.74}$$

Now we treat the term $[G(X)u^{-\frac{1}{k}}]_{ll}$,in the following the repeated indices on α, β denote summation over the indices from $1, 2, ...n+1$. Denote $G_\alpha = \frac{\partial G}{\partial X^\alpha}$, $G_{\alpha\beta} = \frac{\partial^2 G}{\partial X^\alpha \partial X^\beta}$.

$$\begin{aligned} &[G(X)u^{-\frac{1}{k}}]_l = G_\alpha X_l^\alpha u^{-\frac{1}{k}} + G(X)(-\frac{1}{k})u^{-\frac{1}{k}-1}u_l, \\ &\sum_{l=1}^{n}[G(X)u^{-\frac{1}{k}}]_{ll} = G_{\alpha,\beta}X_l^\alpha X_l^\beta u^{-\frac{1}{k}} + G_\alpha X_{ll}^\alpha u^{-\frac{1}{k}} \\ &-\frac{2}{k}G_\alpha X_l^\alpha u^{-\frac{1}{k}-1}u_l + \frac{1}{k}(\frac{1}{k}+1)G(X)u^{-\frac{1}{k}-2}|Du|^2 - \frac{1}{k}G(X)u^{-\frac{1}{k}-1}u_{ll}. \end{aligned}$$

Using (6.67) and (6.68), it follows that at x_o

$$\sum_{l=1}^{n}[G(X)u^{-\frac{1}{k}}]_{ll} = G_{\alpha,\beta}e_l^\alpha e_l^\beta W_{ll}^2 u^{-\frac{1}{k}} - [G_\alpha x^\alpha u^{-\frac{1}{k}} + \frac{1}{k}G(X)u^{-\frac{1}{k}-1}]H$$

$$-\frac{2}{k}(G_\alpha e_l^\alpha u_l W_{ll})u^{-\frac{1}{k}-1} + \frac{1}{k}(\frac{1}{k}+1)G(X)u^{-\frac{1}{k}-2}|Du|^2 + +\frac{n}{k}G(X)u^{-\frac{1}{k}}. \tag{6.75}$$

Using (6.75), at x_o (6.74) becomes

$$G_{\alpha,\beta}e_l^\alpha e_l^\beta W_{ll}^2 u^{-\frac{1}{k}} - [G_\alpha x^\alpha u^{-\frac{1}{k}} + \frac{1}{k}G(X)u^{-\frac{1}{k}-1}]H - nF + (C_n^{n-k})^{-\frac{1}{k}}H$$

$$-\frac{2}{k}(G_\alpha e_l^\alpha u_l W_{ll})u^{-\frac{1}{k}-1} + \frac{1}{k}(\frac{1}{k}+1)G(X)u^{-\frac{1}{k}-2}|Du|^2 + +\frac{n}{k}G(X)u^{-\frac{1}{k}} \leq 0. \tag{6.76}$$

If $G(X)$ is strictly convex in $\mathbb{R}^{n+1}\setminus\{o\}$, then exist a uniform constant $c_o > 0$ such that

$$\sum_{\alpha,\beta=1}^{n} G_{\alpha,\beta}e_l^\alpha e_l^\beta \geq c_o, \quad l = 1, 2, ...n.$$

Since $\sum_{l=1}^{n} W_{ll}^2 \geq \frac{H^2}{n}$, we obtain $H(x_o) \leq C$. ■

Proof of existence theorem I: For any positive function $f \in C^2(\mathbb{S}^n)$, for $0 \leq t \leq 1$ and $1 \leq k \leq n-1$, set $f_t(x) = [1 - t + tf^{-\frac{1}{k}}(x)]^{-k}$. We consider the equation

$$\sigma_k(\kappa_1, \kappa_2, ..., \kappa_n)(x) = f_t(x)\rho^{1-n}(\rho^2 + |\nabla\rho|^2)^{-1/2}, \quad \text{on} \quad \mathbb{S}^n, \tag{6.77}$$

where $n \geq 2$. We find the hypersurface in the class of strictly convex surface. Let $I = \{t \in [0,1] : \text{such that (6.77) is solvable}\}$. Since $\rho = [C_n^k]^{-\frac{1}{n-2}}$ is a solution for $t = 0$, I is not empty. By Lemma 6.1 and Lemma 6.70, $\rho \in C^{1,1}(\mathbb{S}^n)$ and is bound below. That is equation (6.77) is elliptic. By the Evans-Krylov theorem $\rho \in C^{2,\alpha}(\mathbb{S}^n)$ and

$$||\rho||_{C^{2,\alpha}(\mathbb{S}^n)} \leq C, \tag{6.78}$$

Where C depends only on $n, \max_{\mathbb{S}^n} f, \min_{\mathbb{S}^n} f$ and $|\nabla f|_{C^0}$ and $|\nabla^2 f|_{C^0}$, and α. The a priori estimates guarantee I is closed. The openness is from Lemma 6.3 and the implicit function theorem So we have the existence. The uniqueness of the solution for $t \in [0,1]$ is from Lemma 6.2. This complete the proof of Theorem 6.1. ■

Remark 6.1. *We suspect the strict convexity condition (6.59) can be weakened. For the cases $k = 1, 2$, this is verified in Theorem 6.2. The proof of Theorem 6.2 is different from the proof of Theorem 6.1 in this section. Due to the weakened condition, we are not able to obtained a positive lower bound for the principal curvatures directly. Instead, we will use special structure of the elementary symmetric function σ_2 to get an upper bound of principal curvatures for convex solutions of (6.58).*

In the rest of this section, we will prove the C^2 estimate for the scalar curvature case under the convexity assumption of the solution. We shall make use of some explicit structure of σ_2.

We consider the following prescribed scalar curvature measure equation

$$\sigma_2(\lambda\{h_{ij}\})(X) = |X|^{-(n+1)} f(\frac{X}{|X|}) < X, N >, \quad \forall X \in M. \tag{6.79}$$

Now we state the mean curvature estimate for the above equation on the convexity of solution surface.

LEMMA 6.5. *Let f be a C^2 positive function on $\mathbb{S}^n$ and M be a starshaped hypersurface in $\mathbb{R}^{n+1}$ respect to the origin, if M is a convex solution surface of equation (6.79) and for the function $\rho = |X|$ on $\mathbb{S}^n$ the following estimates hold*

$$\|\rho\| \leq C, \tag{6.80}$$

where the constant C depends only on $n, k, \min_{\mathbb{S}^n} f$ and $\|f\|_{C^2}$.

Proof: C^1 estimates were already obtained in Lemma 6.1 in the section 2. We only need to get an upper bound of the mean curvature H.

Let

$$F(X) = f(\frac{X}{|X|}), \quad \phi(X) = |X|^{-(n+1)} F(X), \tag{6.81}$$

then the equation (6.79) becomes

$$\sigma_2(\kappa_1, \kappa_2, ..., \kappa_n)(X) = \phi(X) < X, e_{n+1} >, , \quad \text{on} \quad M, \tag{6.82}$$

Assume the function $P = H + \frac{a}{2}|X|^2$ attains its maximum at $X_o \in M$, where a is a constant will be determined later. Then at X_o we have

$$P_i = H_i + a < X, e_i >= 0, \tag{6.83}$$

$$P_{ii} = H_{ii} + a[1 - h_{ii} < X, e_{n+1} >]. \tag{6.84}$$

Let $F^{ij} = \frac{\partial \sigma_2(\lambda\{h_{ij}\})}{\partial h_{ij}}$, and choose a suitable orthonormal frame $\{e_1, e_2, ..., e_n\}$ in a neighborhood of $X_o \in M$ such that at X_o the matrix $\{h_{ij}\}$ is diagonal. Then at X_o, the matrix $\{F^{ij}\}$ is also diagonal and positive definitive. At X_o

$$\sum_{ij=1}^{n} F^{ij} P_{ij} = \sum_{i=1}^{n} F^{ii} H_{ii} + a \sum_{i=1}^{n} F^{ii} - a < X, e_{n+1} > \sum_{i=1}^{n} F^{ii} h_{ii} \leq 0, \tag{6.85}$$

from this inequality we shall obtain the mean curvature estimate.

In what follows, all the calculations will be done at $x_o \in M$.

First we deal with the term $\sum_{i=1}^{n} F^{ii} H_{ii}$. From (6.83) and (5.15), we have

$$\begin{aligned}\sum_{i=1}^{n} F^{ii} H_{ii} &= \sum_{i=1}^{n} F^{ii} (\sum_{j=1}^{n} h_{jjii}) = \sum_{i=1}^{n} F^{ii} \sum_{j=1}^{n} (h_{iijj} + h_{ii} h_{jj}^2 - h_{jj} h_{ii}^2) \\ &= \sum_{ij=1}^{n} F^{ii} h_{iijj} + |A|^2 \sum_{i=1}^{n} F^{ii} h_{ii} - H \sum_{i=1}^{n} F^{ii} h_{ii}^2,\end{aligned}$$

where $|A|^2 = \sum_{i=1}^{n} h_{ii}^2$.

Then we treat the term $\sum_{ij=1}^n F^{ii}h_{iijj}$. Differentiate equation (6.82) twice, by (5.11)-(5.14),

$$\begin{aligned}\sum_{ij=1}^n F^{ii}h_{iijj} &= \sum_{j=1}^n [\phi(X) < X, e_{n+1} >]_{jj} + \sum_{j,k\neq l} h_{jkl}^2 - \sum_{j,k\neq l} h_{jkk}h_{jll} \\ &= \Delta\phi < X, e_{n+1} > +2\sum_{j=1}^n \phi_j h_{jj} < X, e_j > +\phi \sum_{j=1}^n < X, e_{n+1} >_{jj} \\ &+ \sum_{j,k\neq l} h_{jkl}^2 - \sum_{j,k,l} h_{jkk}h_{jll} + \sum_{j,k} h_{jkk}^2.\end{aligned}$$

Now use (5.11)-(5.16), we have

$$\begin{aligned}\sum_{i=1}^n < X, e_{n+1} >_{ii} &= \sum_{i,l=1}^n [h_{il} < X, e_l >]_i \\ &= \sum_{i=1}^n [\sum_{l=1}^n h_{iil} < X, e_l > +h_{ii} - h_{ii}^2 < X, e_{n+1} >] \\ &= \sum_{l=1}^n H_l < X, e_l > +H - |A|^2 < X, e_{n+1} > \\ &= -a\sum_{i=1}^n < x, e_i >^2 +H - |A|^2 < X, e_{n+1} > .\end{aligned}$$

In turn, by equation (6.82) we have the following estimate

$$\begin{aligned}&\sum_{ij=1}^n F^{ii}h_{iijj} \geq -|A|^2\sigma_2(h_{ij}) + \phi H + \Delta\phi < X, e_{n+1} > \\ &+2\sum_{j=1}^n \phi_j h_{jj} < X, e_j > -a\phi \sum_{i=1}^n < x, e_i >^2 -a^2 \sum_{i=1}^n < x, e_i >^2 .\end{aligned} \tag{6.86}$$

It is easy to compute that

$$\begin{aligned}&\sum_{i=1}^n F^{ii} = (n-1)H, \quad \sum_{i=1}^n F^{ii}h_{ii} = 2\sigma_2(h_{ij}), \\ &\sum_{i=1}^n F^{ii}h_{ii}^2 = H\sigma_2(h_{ij}) - 3\sigma_3(h_{ij}), \quad |A|^2 = H^2 - 2\sigma_2(h_{ij}).\end{aligned} \tag{6.87}$$

Combining the (6.85)-(6.87), at x_o we get the following

$$\begin{aligned}&a(n-1)H + \phi H + 2\sum_{i=1}^n \phi_i h_{ii} < X, e_i > +\Delta\phi < X, e_{n+1} > +3H\sigma_3(h_{ij}) \\ &\leq 2\sigma_2(h_{ij})^2 + 2a < X, e_{n+1} > \sigma_2(h_{ij}) + [a\phi + a^2]\sum_{i=1}^n < X, e_i >^2 .\end{aligned} \tag{6.88}$$

Let F_A, F_{AB} be the ordinary Euclidean differential in $\mathbb{R}^{n+1}$, use (5.11)-(5.14), we compute

$$\phi_i = -(n+1)|X|^{-(n+3)} < X, e_i > F(X) + |X|^{-(n+1)} \sum_{A=1}^{n+1} F_A X_i^A$$

$$\Delta\phi = \sum_{i=1}^{n} \phi_{ii} = H[(n+1)|X|^{-(n+3)} < X, e_{n+1} > F - |X|^{-(n+1)} \sum_{A=1}^{n+1} F_A e_{n+1}^A]$$

$$-2(n+1)|X|^{-(n+3)} \sum_{i=1}^{n} \sum_{A=1}^{n+1} < X, e_i > F_A X_i^A - n(n+1)|X|^{-(n+3)} F$$

$$+|X|^{-(n+1)} \sum_{A,B=1}^{n+1} \sum_{i=1}^{n} F_{AB} X_i^A X_i^B + (n+1)(n+3)|X|^{-(n+5)} F \sum_{i=1}^{n} < X, e_i >^2 .$$

Now we use the convexity of the solution, we have

$$\sigma_3(h_{ij}) \geq 0, \quad 0 \leq h_{ii} \leq H.$$

If a is suitable large, we get the following mean curvature estimate

$$\max H \leq C(n, \max_{\mathbb{S}^n} f, \min_{\mathbb{S}^n} f, |\nabla f|_{C^0}, |\nabla^2 f|_{C^0}). \tag{6.89}$$

This finishes the proof of the Lemma. ■

Since C^2 estimates in Lemma 6.5 only valid for convex solutions, in order to carry on the method of continuity, we need to show the convexity is preserved during the process. This in fact is a consequence of Theorem 4.2. We state it as

THEOREM 6.3. *Suppose M is a convex hypersurface and satisfies equation (6.64) for $k < n$ with the second fundamental form $W = \{h_{ij}\}$ and $|X|^{\frac{n+1}{k}} f(\frac{X}{|X|})$ is convex in $\mathbb{R}^{n+1} \setminus \{0\}$, then W is positive definite.*

We now use Theorem 6.3 to prove Theorem 6.2.

Proof of Theorem 6.2. The proof is the same as in the proof of Theorem 6.1 by the method of continuity, here we make use of Theorem 6.3. The openness and uniqueness have already treated in the proof of Theorem 6.1. The closeness follows from a priori estimates in Lemma 6.1 and quasilinear elliptic theory in the case of $k = 1$ and the a priori estimates in Lemma 6.5 in the case of $k = 2$, and the preservation of convexity in Theorem 6.3. ■

Notes

For the curvature measures, the problem of prescribing C_0 is called the Alexandrov problem, which can be considered as a counterpart to Minkowski problem. The existence and uniqueness were obtained by Alexandrov [**5**]. The regularity of the Alexandrov problem in elliptic case was proved by Pogorelov [**106**] for $n = 2$ and by Oliker [**100**] for higher dimension case. The general regularity results (degenerate case) of the problem were obtained in [**59**]. Apparently, the existence problem for curvature measures of $\mathcal{C}_{n-k}$ for general case $k < n$ has not been touched

(see also note 8 on P. 396 in [**110**]). Equation (6.58) was studied in an unpublished notes [**60**] with Yanyan Li. The results in this chapter were obtained in [**63**].

It seems that the estimates in [**26**] and [**62**] can not be obtained through similar way. The uniqueness and C^1 estimates were established for admissible solutions in [**60**]. But C^2 estimates for admissible solutions of equation (6.58) are not known (except for $k = 1$ and $k = n$, the first case follows from the theory of quasilinear equations, and later case was dealt in [**100, 59**]). Since the Alexandrov problem (Gauss curvature measure problem) has already been solved [**5, 106, 100, 59**], Theorem 6.2 yields solutions to two other important measures, the mean curvature measure and scalar curvature measure under convex condition (6.60).

Large part of the study of curvature measures have been carried on for convex bodies. There are some generalizations of these curvature measures to other class of sets in R^{n+1} (e.g., [**44**]). From differential geometric point of view, the notion of $(n-k)-th$ curvature measure can be easily extended to k-convex bodies. Since for $k < n$, admissible solution of (6.58) is not convex in general. By Lemma 6.2, for $k < n$, the prescribing curvature measure equation (6.58) has no convex solution for most of f. This means some condition must be imposed on f for the existence of convex solutions. We believe that for any smooth positive function f, equation (6.58) always has an admissible solution.

Part 2

Fully nonlinear equations in confformal geometry

CHAPTER 7

Some properties of the Schouten tensor in conformal geometry

We now switch our attention to conformal geometry. Let (M, g) be an oriented, compact and manifold of dimension $n > 2$. And let S_g denote the Schouten tensor of the metric g, i.e.,

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right),$$

where Ric_g and R_g are the Ricci tensor and scalar curvature of g respectively. The following decomposition formula reveals why the Schouten tensor is the main object of study in conformal geometry:

$$\text{Riem} = A_g \odot g + \mathcal{W}_g, \tag{7.1}$$

where $\mathcal{W}_g$ is the Weyl tensor of g (which is conformally invariant), and $\odot$ denotes the Kulkarni-Nomizu product (see [**15**]).

We define *σ_k-scalar curvature* of g by

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot S_g),$$

where $g^{-1} \cdot S_g$ is defined, locally by $(g^{-1} \cdot S_g)^i_j = g^{ik}(S_g)_{kj}$. When $k = 1$, σ_1-scalar curvature is just the scalar curvature R (up to a constant multiple). It is natural to consider manifolds with metric of positive k-scalar curvature. However, the surgery might be not preserve this positivity. In fact, we consider a stronger positivity. Define

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \,|\, \sigma_j(\Lambda) > 0, \forall j \leq k\}.$$

A metric g is said to be in Γ_k^+ if $\sigma_j(g)(x) > 0$ for $j \leq k$ and $x \in M$. Such a metric is called a metric of *positive Γ_k-curvature*, or a Γ_k-positive metric. When $k = 1$, it is just the metric of positive curvature. From now on, we only consider the case $k \geq 2$.

We want to analyze the Schouten tensor and derive some of geometric and topological applications.

We note that positive Γ_1-curvature is equivalent to positive scalar curvature, and the condition of positive Γ_1-curvature has some geometric and topological consequences for the manifold M. For example, when (M, g) is locally conformally flat with positive Γ_1-curvature, then $\pi_i(M) = 0, \forall 1 < i \leq \frac{n}{2}$ by a result of Schoen-Yau [**115**]. We will first prove that positive Γ_k-curvature for any $k \geq \frac{n}{2}$ implies positive Ricci curvature.

THEOREM 7.1. *Let (M, g) be a Riemannian manifold and $x \in M$, if g has positive (nonnegative resp.) Γ_k-curvature at x for some $k \geq n/2$. Then its Ricci curvature is positive (nonnegative*

resp.) at x. Moreover, if Γ_k-curvature is nonnegative for some $k > 1$, then

$$Ric_g \geq \frac{2k-n}{2n(k-1)} R_g \cdot g.$$

In particular if $k \geq \frac{n}{2}$,

$$Ric_g \geq \frac{(2k-n)(n-1)}{(k-1)} \binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}(S_g) \cdot g.$$

A direct consequence of Theorem 7.1 is

COROLLARY 7.1. *Let (M^n, g) be a compact, locally conformally flat manifold with nonnegative Γ_k-curvature everywhere for some $k \geq n/2$. Then (M, g) is conformally equivalent to either a space form or a finite quotient of a Riemannian $\mathbb{S}^{n-1}(c) \times \mathbb{S}^1$ for some constant $c > 0$ and $k = n/2$. Especially, if $g \in \Gamma_k^+$, then (M, g) is conformally equivalent to a spherical space form.*

When $k \leq n/2$, we have the following vanishing theorem.

THEOREM 7.2. *Let (M, g) be a compact, oriented and connected locally conformally flat manifold. If g is a metric of positive Γ_k-curvature with $2 \leq k < n/2$, then for any $\left[\frac{n+1}{2}\right] + 1 - k \leq p \leq n - \left(\left[\frac{n+1}{2}\right] + 1 - k\right)$*

$$H^p(M) = 0.$$

We first prove two lemmas. Here, we assume that $k > 1$.

LEMMA 7.1. *Let $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n) \in \mathbb{R}^n$, and define*

$$S_\Lambda = \Lambda - \frac{\sum_{i=1}^n \lambda_i}{2(n-1)}(1, 1, \cdots, 1).$$

If $S_\Lambda \in \bar{\Gamma}_k^+$, then

$$\min_{i=1,\cdots,n} \lambda_i \geq \frac{(2k-n)}{2n(k-1)} \sum_{i=1}^n \lambda_i. \tag{7.2}$$

In particular if $k \geq \frac{n}{2}$,

$$\min_{i=1,\cdots,n} \lambda_i \geq \frac{(2k-n)(n-1)}{(n-2)(k-1)} \binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}(S_\Lambda).$$

Proof: We first note that, for any non-zero vector $A = (a_1, \cdots, a_n) \in \bar{\Gamma}_2^+$ implies $\sigma_1(A) > 0$. This can be proved as follow. As $A \in \bar{\Gamma}_2^+$, $\sigma_1(A) \geq 0$. If $\sigma_1(A) = 0$, there must be $a_i > 0$ for some i since A is a non-zero vector. We may assume $a_n > 0$. Let $(A|n) = (a_1, \cdots, a_{n-1})$, we have $\sigma_1(A|n) \geq 0$. This would give $\sigma_1(A) = \sigma_1(A|n) + a_n > 0$, a contradiction.

Now without loss of generality, we may assume that Λ is not a zero vector. By the assumption $S_\Lambda \in \bar{\Gamma}_k^+$ for $k \geq 2$, so we have $\sum_{i=1}^n \lambda_i > 0$.

Define

$$\Lambda_0 = (1, 1, \cdots, 1, \delta_k) \in \mathbb{R}^{n-1} \times \mathbb{R}$$

and we have $S_{\Lambda_0} = (a, \cdots, a, b)$, where

$$\delta_k = \frac{(2k-n)(n-1)}{2nk-2k-n},$$

$$a = 1 - \frac{n-1+\delta_k}{2(n-1)}, \quad b = \delta_k - \frac{n-1+\delta_k}{2(n-1)}$$

so that

$$\sigma_k(S_{\Lambda_0}) = 0 \quad \text{and } \sigma_j(S_{\Lambda_0}) > 0 \text{ for } j \le k-1. \tag{7.3}$$

It is clear that $\delta_k < 1$ and so that $a > b$. Since (7.2) is invariant under the transformation Λ to $s\Lambda$ for $s > 0$, we may assume that $\sum_{i=1}^n \lambda_i = \mathrm{tr}(\Lambda_0) = n-1+\delta_k$ and $\lambda_n = \min_{i=1,\cdots,n} \lambda_i$. We write

$$S_\Lambda = (a_1, \cdots, a_n).$$

We claim that

$$\lambda_n \ge \delta_k. \tag{7.4}$$

This is equivalent to show

$$a_n \ge b. \tag{7.5}$$

Assume by contradiction that $a_n < b$. We consider $\Lambda_t = t\Lambda_0 + (1-t)\Lambda$ and

$$S_t := S_{\Lambda_t} = tS_{\Lambda_0} + (1-t)S_\Lambda = ((1-t)a + ta_1, \cdots, (1-t)a + ta_{n-1}, (1-t)b + ta_n).$$

By the convexity of the cone Γ_k^+ (see Proposition 1), we know

$$S_t \in \bar{\Gamma}_k^+, \quad \text{for any } t \in (0,1].$$

Especially, $f(t) := \sigma_k(S_t) \ge 0$ for any $t \in [0,1]$. By the definition of δ_k, $f(0) = 0$.

For any vector $V = (v_1, \cdots, v_n)$, let $(V|i) = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$ be the vector with the i-th component removed. Now we compute the derivative of f at 0

$$f'(0) = \sum_{i=1}^{n-1} (a_i - a)\sigma_{k-1}(S_0|i) + (a_n - b)\sigma_{k-1}(S_0|n).$$

Since $(S_0|i) = (S_0|1)$ for $i \le n-1$ and $\sum_{i=1}^n a_i = (n-1)a + b$, we have

$$f'(0) = (a_n - b)(\sigma_{k-1}(S_0|n) - \sigma_{k-1}(S_0|1)) < 0,$$

for $\sigma_{k-1}(S_0|n) - \sigma_{k-1}(S_0|1) > 0$. (Recall that $b < a$.) This is a contradiction, hence $\lambda_n \ge \delta_k$. It follows that

$$\min_{i=1,\cdots,n} \lambda_i \ge \delta_k = \frac{2k-n}{2n(k-1)} \sum_{i=1}^n \lambda_i.$$

Finally, the last inequality in the lemma follows from the Newton-MacLaurin inequality. ■

Remark. It is clear from the above proof that the constant in Lemma 7.1 is optimal.
We next consider the case $S_\Lambda \in \bar{\Gamma}^+_{\frac{n}{2}}$.

LEMMA 7.2. *Let $k = n/2$ and $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ with $S_\Lambda \in \bar{\Gamma}^+_k$. Then either $\lambda_i > 0$ for any i or*

$$\Lambda = (\lambda, \lambda, \cdots, \lambda, 0)$$

up to a permutation. If the second case is true, then we must have $\sigma_{\frac{n}{2}}(S_\Lambda) = 0$.

Proof: By Lemma 7.1, we only need to check that for $\Lambda = (\lambda_1, \cdots, \lambda_{n-1}, 0)$ with $S_\Lambda \in \bar{\Gamma}^+_k$,

$$\lambda_i = \lambda_j, \quad \forall i, j = 1, 2, \cdots, 2k-1.$$

We use the same idea as in the proof of the previous Lemma. Without loss of generality, we may assume that Λ is not a zero vector. By the assumption $S_\Lambda \in \bar{\Gamma}^+_k$ for $k \geq 2$, we have $\sum_{i=1}^{n-1} \lambda_i > 0$. Hence we may assume that $\sum_{i=1}^{n-1} \lambda_i = n-1$. Define

$$\Lambda_0 = (1, 1, \cdots, 1, 0) \in \mathbb{R}^n$$

It is easy to check that

$$S_{\Lambda_0} \in \Gamma^+_{k-1} \quad \text{and} \quad \sigma_k(S_{\Lambda_0}) = 0. \tag{7.6}$$

That is, $S_{\Lambda_0} \in \bar{\Gamma}^+_k$. If λ's are not all the same, we have

$$\sum_{i=1}^{n-1} (\lambda_i - 1) = 0,$$

and

$$\sum_{i=1}^{n-1} (\lambda_i - 1)^2 > 0.$$

Now consider $\Lambda_t = t\Lambda_0 + (1-t)\Lambda$ and

$$S_t := S_{\Lambda_t} = tS_{\Lambda_0} + (1-t)S_\Lambda = (\frac{1}{2} + t(\lambda_1 - 1), \cdots, \frac{1}{2} + t(\lambda_{n-1} - 1), -\frac{1}{2}).$$

From the assumption that $A \in \bar{\Gamma}^+_k$, (7.6) and the convexity of $\bar{\Gamma}^+_k$, we have

$$S_t \in \bar{\Gamma}^+_k \quad \text{for } t > 0. \tag{7.7}$$

For any $i \neq j$ and any vector A, we denote $(A|ij)$ be the vector with the i-th and j-th components removed. Let $\widetilde{\Lambda} = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ be $n-1$-vector, $\Lambda^* = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ be $n-2$-vector. It is clear that $\forall i \neq j, \quad i, j \leq n-1$,

$$\sigma_{k-1}(S_0|i) = \sigma_{k-1}(\widetilde{\Lambda}) > 0,$$

$$\sigma_{k-2}(S_0|ij) = \sigma_{k-2}(\Lambda^*) > 0.$$

Now we expand $f(t) = \sigma_k(S_t)$ at $t = 0$. By (7.6), $f(0) = 0$. We compute

$$\begin{aligned} f'(0) &= \sum_{i=1}^{n-1}(\lambda_i - 1)\sigma_{k-1}(S_0|i) \\ &= \sigma_{k-1}(\widetilde{\Lambda})\sum_{i=1}^{n-1}(\lambda_i - 1) = 0 \end{aligned}$$

and

$$\begin{aligned} f''(0) &= \sum_{i\neq j}(\lambda_i - 1)(\lambda_j - 1)\sigma_{k-2}(S_0|ij) \\ &= \sigma_{k-2}(\Lambda^*)\sum_{i\neq j}(\lambda_i - 1)(\lambda_j - 1) \\ &= -\sigma_{k-2}(\Lambda^*)\sum_{i=1}^{n-1}(\lambda_i - 1)^2 < 0, \end{aligned}$$

for $\sigma_{k-2}(S_0|ij) = \sigma_{k-2}(\Lambda^*) > 0$ for any $i \neq j$ and $\sum_{i\neq j}(\lambda_j - 1) = (1 - \lambda_i)$. Hence $f(t) < 0$ for small $t > 0$, which contradicts (7.7). ■

Remark. From the proof of Lemma 7.2, there is a constant $C > 0$ depending only on n and $\frac{\sigma_{\frac{n}{2}}^{\frac{2}{n}}(S_\Lambda)}{\sigma_1(S_\Lambda)}$ such that

$$\min_i \lambda_i \geq C\sigma_{\frac{n}{2}}^{\frac{2}{n}}(S_\Lambda).$$

Proof of Theorem 7.1. Theorem 7.1 follows directly from Lemmas 7.1 and 7.2. ■

COROLLARY 7.2. *Let (M, g) is a n-dimensional Riemannian manifold and $k \geq n/2$, and let $N = M \times \mathbb{S}^1$ be the product manifold. Then N does not have positive Γ_k-curvature. If N has nonnegative Γ_k-curvature, then (M, g) is an Einstein manifold, and there are two cases: either $k = n/2$ or $k > n/2$ and (M, g) is a torus.*

Proof: This follows from Lemmas 7.1 and 7.2. ■

Proof of Corollary 7.1. From Theorem 7.1, we know that the Ricci curvature Ric_g is nonnegative. Now we deform it by the Yamabe flow considered by Hamilton, Ye [**132**] and Chow [**38**] to obtain a conformal metric $\widetilde{g}$ of constant scalar curvature. The Ricci curvature $Ric_{\widetilde{g}}$ is nonnegative, for the Yamabe flow preserves the non-negativity of Ricci curvature, see [**38**]. Now by a classification result given in [**119, 34**], we have $(M, \widetilde{g})$ is isometric to either a space form or a finite quotient of a Riemannian $\mathbb{S}^{n-1}(c) \times \mathbb{S}^1$ for some constant $c > 0$. In the latter case, it is clear that $k = n/2$, otherwise it can not have nonnegative Γ_k-curvature. ■

Let U be a coordinate chart around a point $x \in M$ and consider the space of $l-jets$ of metrics with respect to the chart U. Let $J^l(p) = (\{g_{ij}\}, \{\partial^1 g_{ij}\}, \cdots, \{\partial^\alpha g_{ij}\}_{|\alpha|=l})$, where α is a multi-index. Let $J^l_{k,+}(p)$ be the subset of $J^l(p)$ consisting of elements with positive σ_j-scalar curvature for all $j \leq k$. It is clear that $J^\infty_{k,+} = J^2_{k,+} \times \mathbb{R}^\infty$, for the curvature tensor depends only on 2-jets. Now we have

PROPOSITION 7.1. *The set $J^2_{k,+}(p)$ is contractible.*

Proof: From the proof of Theorem 1 in [**48**], we only need to check that the set

$$\Delta := \{v : (\delta_{ij}, 0, v) \in J^2_{k,+}(p)\}$$

is contractible. The Christoffel symbols and their derivatives for any element in Δ are $\Gamma^k_{ij} = 0$ and

$$\partial_l \Gamma^k_{ij} = \frac{1}{2}(\partial_l \partial_i g_{jk} \partial_l \partial_j g_{ik} - \partial_l \partial_k g_{ij}).$$

And the Ricci curvature and the scalar curvature are

$$R_{ij} = \frac{1}{2}\sum_l (\partial_l \partial_j g_{il} + \partial_i \partial_l g_{lj} - \partial_l \partial_l g_{ij} - \partial_i \partial_j g_{ll})$$

and

$$R = \sum_{i \neq j} \partial_j \partial_i g_{ij} - \partial_j \partial_j g_{ii}.$$

Hence the Schouten tensor is

$$\begin{aligned} S_{ij} &= \frac{1}{2}\sum_l (\partial_l \partial_j g_{il} + \partial_i \partial_l g_{lj} - \partial_l \partial_l g_{ij} - \partial_i \partial_j g_{ll}) \\ &\quad - \frac{1}{2(n-1)} \sum_{i \neq j} \partial_j \partial_i g_{ij} - \partial_j \partial_j g_{ii}. \end{aligned}$$

By Proposition 13.4, the set Δ is convex, hence contractible. ■

We now prove Theorem 7.2. The proof here follows similar arguments as in the proof of Theorem 7.1. This type argument gives a general condition under which $\Lambda \in \Gamma^+_k$ implies $G_{n,p}(\Lambda) > 0$ is reduced to a combinatoric problem.

Let $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n$ be an n-tuple. For any $j = 1, 2, \cdots, n$, we set $\Lambda|j = (\lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_n)$. Assume that $2 \leq k < n/2$, $1 \leq p \leq n/2$. Define a function $G_{n,p} : \mathbb{R}^n \to \mathbb{R}$ by

$$G_{n,p}(\Lambda) = \min_{(i_1, \cdots, i_n)} \{(n-p)\sum_{j=1}^{p} \lambda_{i_j} + p \sum_{j=p+1}^{n} \lambda_{i_j}\},$$

where $(i_1, \cdots, i_n)$ is a permutation of $(1, 2, \cdots, n)$ and the minimum is taken over all permutations. $G_{n,p}$ is related to a geometric quantity arising in the Weitzenböck form for p-forms (see (7.16)).

We define some special n-tuples, which will be used crucially. Let $I_p = (1, 1, \cdots, 1) \in \mathbb{R}^p$ and $s > 0$. Define n-tuples by

$$E_{n,p} = (I_{n-p}, -I_p) \text{ and } E^s_{n,p} = (I_{n-p}, -sI_p).$$

It is trivial to see that $G_{n,p}(E_{n,p}) = 0$.

LEMMA 7.3. *For any $s > 0$, if $E^s_{n,p} \in \overline{\Gamma}^+_k$, then $E^s_{n-1,p} \in \Gamma^+_{k-1}$ and $E^s_{n-2,p-1} \in \Gamma^+_{k-1}$. If $E_{n,p} \in \overline{\Gamma}^+_k$, then $E_{n-2,p-1} \in \Gamma^+_k$.*

Proof: First, it is easy to check that $E^s_{n-1,p} \in \Gamma^+_{k-1}$ implies $E^s_{n-2,p-1} \in \Gamma^+_{k-1}$. If $E^s_{n,p} \in \Gamma^+_k$ (resp. $\overline{\Gamma}^+_k$), then $E^s_{n-1,p} \in \Gamma^+_{k-1}$ (resp. $\overline{\Gamma}^+_{k-1}$). Hence, we only need to deal with the case that $\sigma_k(E^s_{n,p}) = 0$. Assume by contradiction that $\sigma_{k-1}(E^s_{n-1,p}) = 0$. Since $\sigma_k(E^s_{n,p}) = \sigma_{k-1}(E^s_{n-1,p}) + \sigma_k(E^s_{n-1,p})$, we have $\sigma_k(E^s_{n-1,p}) = 0$. Together with $E^s_{n-1,p} \in \overline{\Gamma}^+_{k-1}$, it implies $E^s_{n-1,p} \in \overline{\Gamma}^+_k$. We may repeat this argument to produce a sequence of integers m such that $E^s_{m,p} \in \overline{\Gamma}^+_k$ and $\sigma_k(E^s_{m,p}) = \sigma_k(E^s_{m+1,p}) = 0$. This process must be stopped somewhere since $-sI_p$ is not in $\overline{\Gamma}^+_k$. We then obtain an integer m such that $\sigma_k(E^s_{m,p}) = \sigma_k(E^s_{m+1,p}) = 0$ and $E^s_{m,p} \in \Gamma^+_{k-1}$. Now

$$0 = \sigma_k(E^s_{m+1,p}) = \sigma_{k-1}(E^s_{m,p}) + \sigma_k(E^s_{m,p}) > 0,$$

this is a contradiction.

To prove the last assertion in the lemma, note that we already have $E_{n-2,p-1} \in \overline{\Gamma}^+_{k-1}$. Now,

$$0 \le \sigma_k(E_{n,p}) = \sigma_k(E_{n-2,p-1}) - \sigma_{k-2}(E_{n-2,p-1}).$$

It follows that

$$\sigma_k(E_{n-2,p-1}) \ge \sigma_{k-2}(E_{n-2,p-1}) > 0.$$

■

LEMMA 7.4. *For $3 \le p \le n/2$, if*

$$k \ge \frac{n - 2p + 4 - \sqrt{n - 2p + 4}}{2}, \tag{7.8}$$

then $E_{n,p} \notin \overline{\Gamma}^+_k$. For $p = 2$,

$$k \ge \frac{n - \sqrt{n}}{2},$$

then $E_{n,2} \notin \Gamma^+_k$. In particular, if $k = [\frac{n+1}{2}] + 1 - p$, then $E_{n,p} \notin \overline{\Gamma}^+_k$.

Proof: If $p = 2$, it is easy to compute

$$\begin{aligned} \sigma_k(E_{n,2}) &= \sigma_k(I_{n-2}) - 2\sigma_{k-1}(I_{n-2}) + \sigma_{k-2}(I_{n-2}) \\ &= = \frac{(n-2)!}{k!(n-k)!}\{(n-2k)^2 - n\} \le 0, \end{aligned}$$

if $k \ge \frac{n-\sqrt{n}}{2}$.

If $p > 2$ and $E_{n,p} \in \overline{\Gamma}_k$, applying Lemma 7.3 (the last assertion) repeatedly, we have $E_{n-2p+4,2} \in \Gamma^+_k$. However, one can compute

$$\begin{aligned} \sigma_k(E_{n-2p+4,2}) &= \sigma_k(I_{n-2p+2}) + \sigma_{k-2}(I_{n-2p+2}) - 2\sigma_{k-1}(I_{n-2p+2}) \\ &= \frac{(n-2p+2)!}{k!(n-2p+2-k)!}\{(n-2p+4-2k)^2 - (n-2p+4) \le 0, \end{aligned}$$

for k satisfies (7.8). A contradiction. ■

LEMMA 7.5. *Let $0 < s < 1$ and $p \le n/2$. If $E^s_{n,p} \in \overline{\Gamma}^+_k$ with $\sigma_k(E^s_{n,p}) = 0$, then for any $\Lambda \in \overline{\Gamma}^+_k$ with $\sigma_1(\Lambda) > 0$*

$$G_{n,p}(\Lambda) > 0.$$

Proof: First note that $\sigma_1(E^s_{n,p}) > 0$. By Lemma 7.3, we have $\sigma_{k-1}(E^s_{n-1,p-1}) > 0$. Using the identity $\sum_{j=1}^n \sigma_{k-1}(\Lambda|j)\lambda_j = k\sigma_k(\Lambda)$ we have

$$0 = k\sigma_k(E^s_{n,p}) = (n-p)\sigma_{k-1}(E^s_{n-1,p}) - sp\sigma_{k-1}(E^s_{n-1,p-1}). \tag{7.9}$$

Now rearrange $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. It is obvious that

$$G_{n,p}(\Lambda) = p\sum_{j=1}^{n-p} \lambda_j + (n-p) \sum_{j=n-p+1}^{n} \lambda_j.$$

We want to show that it is positive for $\Lambda \in \overline{\Gamma^+_k}$ with $\sigma_1(\Lambda) > 0$. Consider a function $f(t) = \sigma_k((1-t)E^s_{n,p} + t\Lambda)$. Denote $E^s_{n,p} = (e_1, e_2, \cdots, e_n)$. By the convexity of $\overline{\Gamma}_k$, we know $f(t) \ge 0$. Since $f(0) = 0$, we have $f'(0) \ge 0$ which implies

$$\begin{aligned} 0 \;\le\; & f'(0) = \sum_{j=1}^n \sigma_{k-1}(E^s_{n,p}|j)(\lambda_j - e_j) = \sum_{j=1}^n \sigma_{k-1}(E^s_{n,p}|j)\lambda_j - \sigma_k(E^s_{n,p}) \\ =\; & \sigma_{k-1}(E^s_{n-1,p})\sum_{j=1}^{n-p}\lambda_j + \sigma_{k-1}(E^s_{n-1,p-1}) \sum_{j=n-p+1}^{n} \lambda_j \\ =\; & \sigma_{k-1}(E^s_{n-1,p})\{\sum_{j=1}^{n-p}\lambda_j + \frac{n-p}{sp}\sum_{j=n-p+1}^{n}\lambda_j\} \qquad \text{(by (7.9))} \\ =\; & \frac{\sigma_{k-1}(E^s_{n-1,p})}{sp}\{sp\sum_{j=1}^{n-p}\lambda_j + (n-p)\sum_{j=n-p+1}^{n}\lambda\}. \end{aligned} \tag{7.10}$$

From Lemma 7.3 we have $\sigma_{k-1}(E^s_{n-1,p}) > 0$. Hence, (7.10) implies that

$$sp\sum_{j=1}^{n-p}\lambda_j + (n-p)\sum_{j=n-p+1}^{n}\lambda \ge 0. \tag{7.11}$$

From assumption that $\sigma_1(\Lambda) = \sum_{j=1}^n \lambda_j > 0$, we have $\sum_{j=1}^{n-p}\lambda_j > 0$. Therefore, (7.11) implies that $G_{n,p}(\Lambda) > 0$. ■

LEMMA 7.6. *Assume that for some $1 \le p < \frac{n}{2}$ and $2 \le k \le n/2$, $E_{n,p} \in \overline{\Gamma}^+_k$ with $\sigma_k(E_{n,p}) = 0$. If $\Lambda \in \Gamma^+_k$, then $G_{n,p}(\Lambda) > 0$.*

Proof: Since the positivity of $G(\Lambda)$ does not change under a rescaling $\Lambda \to \mu\Lambda$, we may assume that $\sigma_1(\Lambda) = \sigma_1(E_{n,p})$. As in the previous lemma, we consider the function $f(t) = \sigma_k((1-t)E_{n,p} + t\Lambda)$. We have $f'(0) \ge 0$. The argument given in the previous Lemma implies that $G_{n,p}(\Lambda) > 0$ or $G_{n,p}(\Lambda) = 0$. Hence, we only need to exclude the latter case. Assume by

contradiction that $G_{n,p}(\Lambda) = 0$. We have $f'(0) = 0$. Since $f(0) = 0$ and $f(t) \geq 0$ for any $t \in [0,1]$, we have $f''(0) \geq 0$. By our choice of $E_{n,p}$, it is clear that $G_{n,p}(E_{n,p}) = 0$. This, together with $G_{n,p}(\Lambda) = 0$, gives

$$p\sum_{i=1}^{n-p}(e_i - \lambda_i) + (n-p)\sum_{i=n-p+1}^{n}(e_i - \lambda_i) = 0. \tag{7.12}$$

Here we denote $E_{n,p}$ by $(e_1, e_2, \cdots, e_n)$. The normalization $\sigma_1(\Lambda) = \sigma_1(E_{n,p})$ gives

$$\sum_{i=1}^{n-p}(e_i - \lambda_i) + \sum_{i=n-p+1}^{n}(e_i - \lambda_i) = 0. \tag{7.13}$$

(7.12) and (7.13) imply

$$\sum_{i=1}^{n-p}(e_i - \lambda_i) = \sum_{i=n-p+1}^{n}(e_i - \lambda_i) = 0. \tag{7.14}$$

Let $\tilde{\Lambda}_1 = (e_1 - \lambda_1, \cdots, e_{n-p} - \lambda_{n-p})$ and $\tilde{\Lambda}_2 = (e_{n-p+1} - \lambda_{n-p+1}, \cdots, e_n - \lambda_n)$. (7.14) means that $\sigma_1(\tilde{\Lambda}_1) = \sigma_1(\tilde{\Lambda}_2) = 0$. Now we compute $f''(0)$

(7.15)

$$\begin{aligned}
0 \leq f''(0) &= \sum_{i\neq j}\sigma_{k-2}(E_{n,p}|ij)(\lambda_i - e_i)(\lambda_j - e_j) \\
&= 2\{\sigma_{k-2}(E_{n-2,p-1})\sigma_1(\tilde{\Lambda}_1)\sigma_1(\tilde{\Lambda}_2) + \sigma_{k-2}(E_{n-2,p-2})\sigma_2(\tilde{\Lambda}_1) + \sigma_{k-2}(E_{n-2,p})\sigma_2(\tilde{\Lambda}_2)\} \\
&= \sigma_{k-2}(E_{n-2,p-2})\{\sigma_1^2(\tilde{\Lambda}_1) - \sum_{i=1}^{n-p}(e_i - \lambda_i)^2\} + \sigma_{k-2}(E_{n-2,p})\{\sigma_1^2(\tilde{\Lambda}_2) - \sum_{n-p+1}^{n}(e_i - \lambda_i)^2\} \\
&= -\sigma_{k-2}(E_{n-2,p-2})\sum_{i=1}^{n-p}(e_i - \lambda_i)^2 - \sigma_{k-2}(E_{n-2,p})\sum_{i=n-p+1}^{n}(e_i - \lambda_i)^2.
\end{aligned}$$

By Lemma 1, we know that $\sigma_{k-2}(E_{n-2,p-2}) > 0$ and $\sigma_{k-2}(E_{n-2,p}) > 0$. Hence, (7.15) implies that

$$e_i = a_i, \quad \text{for any } i.$$

This is a contradiction, since $\Lambda \in \Gamma_k^+$ and $E_{n,p} \notin \Gamma_k^+$ by assumption. ■

PROPOSITION 7.2. (**i**). *Suppose that $\sigma_k(E_{n,p}) < 0$ for some $2 \leq k < n/2$ and $2 \leq p < n/2$. If $\Lambda \in \overline{\Gamma}_k^+$ with $\sigma_1(\Lambda) > 0$, then $G_{n,q}(\Lambda) > 0$ for any $p \leq q \leq n/2$.*

(**ii**). *Suppose that $\sigma_k(E_{n,p}) = 0$ and $E_{n,p} \in \overline{\Gamma}_k^+$ for some $2 \leq k < n/2$ and $2 \leq p < n/2$. If $\Lambda \in \Gamma_k^+$, then $G_{n,q}(\Lambda) > 0$ for any $p \leq q \leq n/2$.*

Proof: It is easy to see that $\sigma_k(E_{n,p}^s)$, as a function of s, is decreasing. Hence there is a $s \in (0,1)$ such that $\sigma_k(E_{n,p}^s) = 0$ and (**i** follows from Lemma 7.5. (**ii**) has be proven in Lemma 7.6. ■

Combine with Lemma 7.4 and Proposition 7.2, we have

COROLLARY 7.3. (**i**) *Let p and k satisfy*

$$k \geq \frac{n-2p+4-\sqrt{n-2p+4}}{2}.$$

Then $G_{n,p}(\Lambda) > 0$, if either $\Lambda \in \Gamma_k^+$ or, $\Lambda \in \overline{\Gamma}_k^+$ with $\sigma_1(\Lambda) > 0$. In particular, if $\Lambda \in \Gamma_2^+$, then $G_{n,q}(\Lambda) > 0$ for any $\frac{n-\sqrt{n}}{2} \leq q \leq n/2$.

(**ii**) *If $\Lambda \in \Gamma_k^+$, then $G_{n,p}(\Lambda) > 0$ for any $\left[\frac{n+1}{2}\right] + 1 - k \leq p \leq n/2$.*

We now prove a vanishing theorem of cohomology group of locally conformally flat manifolds of positive Γ_k curvature.

PROPOSITION 7.3. *Let (M^n, g) be a compact, locally conformally flat manifold with $g \in \Gamma_k^+$. Then*

(a). *the qth Betti number $b_q = 0$ for*

$$\left[\frac{n+1}{2}\right] + 1 - k \leq q \leq n - \left(\left[\frac{n+1}{2}\right] + 1 - k\right).$$

(b). *if $k > \frac{n-\sqrt{n}}{2}$ then $b_q = 0$ for any $2 \leq q \leq n-2$.*

(c). *If $k = \frac{n-\sqrt{n}}{2}$ and $b_2 \neq 0$, then (M, g) is a quotient of $\mathcal{S}^{n-2} \times H^2$. Here H^2 is a hyperbolic plane of sectional curvature -1 and $\mathcal{S}^{n-2}$ is the standard sphere of sectional curvature 1.*

Proof: Recall the Weiztenböck formula for p-forms ω

$$\Delta\omega = \mathrm{tr}\nabla^2\omega + \mathcal{R}\omega,$$

where

$$\mathcal{R}\omega = \sum_{j,l=1} \omega_j \wedge i(e_l)R(e_j, e_l)\omega.$$

Here e_j is a local basis and $i(\cdot)$ denotes the interior product $\Delta = dd^* + d^*d$ is the Hodge-de Rham Laplacian. In local coordinates, let $\omega = \omega_1 \wedge \cdots \wedge \omega_p$. Then

$$\mathcal{R}\omega = \left((n-p)\sum_{i=1}^{p} \lambda_i + p\sum_{i=p+1}^{n} \lambda_i\right)\omega, \tag{7.16}$$

where λ's are eigenvalues of the Schouten tenser S_g. Under the conditions given in (a) or (b) in the proposition, Corollary 2 implies that $\mathcal{R}$ is a positive operator. It is clear from the Weiztenböck formula that $H^q(M) = \{0\}$ for such q considered in (a) and (b) in the proposition. Hence (a) and (b) follow.

Now we prove (c). By assumption, there is a non-zero harmonic 2-form ω. In this case, $\mathcal{R}$ is non-negative. From the Weiztenböck formula, one can prove that ω is parallel. Now one can follows the argument given in [**90**] to prove that the universal cover $\tilde{M}$ of M is $\mathcal{S}^{n-2} \times H^2$. ■

Notes

Theorem 7.1 was proved in [**70**]. Theorem 7.2 was proved in [**63**] when g is a metric of positive Γ_k-curvature with $k < n/2$. When $k = 1$, the above was proved by Bourguignon [**19**] (see also [**96**]). The condition in Theorem 7.2 is optimal. For example, the Hopf manifold $\mathcal{S}^{2m-1} \times \mathcal{S}^1$ is in Γ_{m-1} and has non-vanishing H^1. In the case of positive scalar curvature, there is a developing map of M to $\Omega \subset \mathbb{S}^n$ by Schoen-Yau [**115**]. A substantially deep results regarding the Hausdorff dimension of $\mathbb{S}^n \setminus \Omega$ was proved in [**115**]. If $g \in \Gamma_2^+$, see a recent result of Chang-Hang-Yang [**28**] on improved estimate on the Hausdorff dimension of $\mathbb{S}^n \setminus \Omega$.

CHAPTER 8

Local estimates for elliptic conformal equations

In this chapter, we are interested in the following conformally invariant fully nonlinear equation for $g \in [g_0]$,

$$\frac{\sigma_k(g)}{\sigma_l(g)} = f, \quad 0 \le l < k \le n. \tag{8.1}$$

Equation (8.1) is related to the deformation of conformal metrics. If $g = e^{-2u} g_0$, the Schouten tensor of g can be computed as

$$\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}.$$

Equation (8.1) has the following form:

$$\frac{\sigma_k}{\sigma_l}\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right) = f e^{-2(k-l)u}, \quad 0 \le l < k \le n, \tag{8.2}$$

where f is a nonnegative function.

When of $k = 1$, $l = 0$, equation (8.1) is the Yamabe equation. Equation (8.1) is a type of fully nonlinear equation when $k \ge 2$. To solve the problem, one needs to establish a priori estimates for the solutions of these equations. It is known that such a priori estimates do not exist in general. On the standard sphere there is a non-compact family of solutions to equation (8.1). As in the Yamabe problem, the blow-up analysis is important to rule out the exceptional case. In order to carry on the blow-up analysis, the crucial step is to establish some appropriate **local estimates** for solutions of equation (8.1).

The main objective of this chapter is to prove **local gradient estimates** for the conformal quotient equation (8.1). We will also deduce local C^2 estimates from the local gradient estimates.

A metric g is said to be admissible if $g^{-1} \cdot S_g \in \Gamma_k^+$ for every point $x \in M$. If $g = e^{-2u} g_0$, we say u is admissible if g is admissible.

THEOREM 8.1. *Suppose f is a positive function on M. Let $u \in C^3(B_r)$ be an admissible solution of (8.2) in B_r, the geodesic ball of radius r in a Riemannian manifold (M, g_0). Then, there exists a constant $c_1 > 0$ depending only on r, $\|g_0\|_{C^3(B_r)}$ and $\|f\|_{C^1(B_r)}$ (independent of* $\inf f$*), such that*

$$\sup_{B_{r/2}} \{|\nabla u|^2\} \le c_1 (1 + e^{-2 \inf_{B_r} u}). \tag{8.3}$$

From Theorem 8.1, the "blow-up" analysis usually for semilinear equations, for example, harmonic map equation, Yang-Mills equation and the Yamabe equation, works for (8.2). It is

an interesting phenomenon, since typical fully nonlinear equations do not admit such blow-up analysis.

COROLLARY 8.1. *There exists a constant $\varepsilon_0 > 0$ such that for any sequence of solutions u_i of (8.2) in B_1 with*

$$\int_{B_1} e^{-nu} dvol(g_0) \leq \varepsilon_0, \tag{8.4}$$

either

(1) *There is a subsequence u_{i_l} uniformly converges to $+\infty$ in any compact subset in B_1, or*
(2) *There is a subsequence u_{i_l} converges strongly in $C^{1,\alpha}_{loc}(B_1)$, $\forall 0 < \alpha < 1$. If f is smooth and strictly positive in B_1, then u_{i_l} converges strongly in $C^m_{loc}(B_1)$, $\forall m$.*

Local gradient estimates

We devote the proof of **local gradient estimates** (8.3). The local C^2 estimates has already been proved in Lemma 8.3.

We recall some notations. Let $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$. The k-th elementary symmetric functions is defined as

$$\sigma_k(\Lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Set $\sigma_0 = 1$ and $\sigma_q = 0$ for $q > n$. σ_k can be extended as function on real symmetric $n \times n$ matrices. A real symmetric matrix A is said to lie in Γ_k^+ if its eigenvalues lie in Γ_k^+.

Let $\Lambda_i = (\lambda_1, \cdots, \check{\lambda}_i, \cdots, \lambda_n) = (\lambda_1, \lambda_2, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n)$ and $\Lambda_{ij} = (\lambda_1, \cdots, \check{\lambda}_i, \cdots, \check{\lambda}_j, \cdots, \lambda_n)$ for $i \neq j$. Therefore, $\sigma_q(\Lambda_i)$ ($\sigma_q(\Lambda_{ij})$ resp.) means the sum of the terms of $\sigma_q(\Lambda)$ not containing the factor λ_i (λ_i and λ_j resp.).

Proof of local gradient estimates in Theorem 8.1. We first reduce the proof of the local gradient estimates to **Claim** (8.14) below. This is an easy part of proof, which works for more general form of F. The difficult part is the verification of **Claim** (8.14), which will be carried out in the next section.

We may assume $r = 1$. Let ρ be a test function $\rho \in C_0^\infty(B_1)$ such that

$$\begin{aligned} \rho &\geq 0, \quad \text{in } B_1, \quad \rho = 1, \quad \text{in } B_{1/2}, \\ |\nabla \rho(x)| &\leq 200\rho^{1/2}(x), \quad |\nabla^2 \rho| \leq 100, \quad \text{in } B_1. \end{aligned} \tag{8.5}$$

Set $H = \rho|\nabla u|^2$, we estimate the maximum of H. Assume that H achieves its maximum at x_0. After an appropriate choice of the normal frame at x_0, we may assume that $W = (u_{ij} + u_i u_j - \frac{|\nabla u|^2}{2}\delta_{ij} + S_{ij})$ is diagonal at the point, where u_i and u_{ij} are the first order and second order covariant derivatives respectively. Let w_{ij} be the entries of W, we have at x_0,

$$w_{ii} = u_{ii} + u_i^2 - \tfrac{1}{2}|\nabla u|^2 + S_{ii}, \quad u_{ij} = -u_i u_j - S_{ij}, \quad \forall i \neq j, \tag{8.6}$$

where S_{ij} are entries of S_{g_0} and $u_i = \nabla_i u = \frac{\partial u}{\partial x_i}$.

By the choice of the test function ρ, we have at x_0

$$|\sum_{l=1}^{n} u_{il}u_l| \leq 100\rho^{-1/2}|\nabla u|^2. \tag{8.7}$$

We may assume that $H(x_0) \geq 10^4 A_0^2$, that is $\rho^{-1/2} \leq \frac{1}{100A_0}|\nabla u|$, and $|S_{g_0}| \leq A_0^{-1}|\nabla u|^2$ for some constant A_0 to be chosen later, otherwise we are done. Thus, from (8.7) we have

$$|\sum_{l=1}^{n} u_{il}u_l| \leq \frac{|\nabla u|^3}{A_0}(x_0). \tag{8.8}$$

We denote $\lambda_i = w_{ii}$ and $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. In what follows, we denote C (which may vary from line to line) as a constant depending only on $\|f\|_{C^1(B_1)}$, k, n, and $\|g_0\|_{C^3(B_1)}$ ($\|f\|_{C^2(B_1)}$ and $\|g_0\|_{C^4(B_1)}$ in the next section). By Proposition 13.4 and (8.8),

$$0 \geq F^{ij}H_{ij} = F^{ij}\left\{\left(-2\frac{\rho_i\rho_j}{\rho} + \rho_{ij}\right)|\nabla u|^2 + 2\rho u_{lij}u_l + 2\rho u_{il}u_{jl}\right\}. \tag{8.9}$$

The first term in (8.9) is bounded from below by $10^5 \sum_{i\geq 1} F^{ii}|\nabla u|^2$.

By interchanging covariant derivatives, the second term in (8.9) can be estimated as follows,

$$\begin{aligned}
&\sum_{i,j,l} F^{ij}u_{ijl}u_l \geq \sum_{i,j,l} F^{ij}u_{ijl}u_l - C|\nabla u|^2\sum_i F^{ii} \\
= &\sum_{i,j,l}\{F^{ij}(w_{ij})_l u_l - F^{ij}(u_iu_j - \frac{|\nabla u|^2}{2}\delta_{ij})_l u_l\} - C|\nabla u|^2\sum_i F^{ii} \\
= &\sum_l F_l u_l - 2\sum_{i,j,l} F^{ij}u_{il}u_ju_l + \sum_{i,k,l} F^{ii}u_{kl}u_ku_l - C|\nabla u|^2\sum_i F^{ii} \\
= &\sum_l e^{-2u}(f_lu_l - 2f|\nabla u|^2) - 2\sum_{i,l} F^{ii}u_{il}u_lu_i + \sum_{i,l} F^{ii}u_{il}u_lu_i - C|\nabla u|^2\sum_i F^{ii} \\
\geq &-C(1+e^{-2u})|\nabla u|^2 - \sum_i F^{ii}\frac{|\nabla u|^4}{A_0}.
\end{aligned} \tag{8.10}$$

To obtain the local estimates, we need the following Lemma.

LEMMA 8.1. *There is constant A_0 depending only on k, n, and $\|g_0\|_{C^3(B_1)}$, such that,*

$$\sum_{i,j,l} F^{ij}u_{il}u_{jl} \geq A_0^{-\frac{3}{4}}|\nabla u|^4\sum_{i\geq 1} F^{ii}. \tag{8.11}$$

Assuming the lemma, **local gradient estimate** (8.3) can be proved as follows.

As $\sum_i F^{ii} \geq 1$, inequalities (8.9), (8.10) and (8.11) yield

$$\begin{aligned} 0 &\geq -10^5|\nabla u|^2 \sum_j F^{jj} - Ce^{-2u}\rho|\nabla u|^2 + \left(-\frac{(n+2)^2}{A_0} + A_0^{-\frac{3}{4}}\right)\rho|\nabla u|^4 \sum_j F^{jj} \\ &\geq \sum_j F^{jj}\left\{-10^5 n|\nabla u|^2 - Ce^{-2\inf u}|\nabla u|^2 + \left(-\frac{(n+2)^2}{A_0} + A_0^{-\frac{3}{4}}\right)\rho|\nabla u|^4\right\}. \end{aligned} \tag{8.12}$$

Choosing A_0 large enough so that $A_0 > 2((n+2)^2)^4$ and multiplying (8.12) by ρ, we get

$$H^2 \leq C(1 + e^{-2\inf u})H,$$

thus

$$|\nabla u(x)|^2 \leq C(1 + e^{-2\inf_{x\in B_1} u}) \quad \text{for } x \in B_{1/2}.$$

Therefore (8.3) of Theorem 8.1 is proved, assuming Lemma 8.1.

Proof of Lemma 8.1. Set $\tilde{u}_{ij} = u_{ij} + S_{ij}$, we estimate that,

$$\sum_{i,j,l} F^{ij} u_{il} u_{jl} \geq \frac{1}{2}\sum_{i,l} F^{ii}\tilde{u}_{il}^2 - C\frac{1}{A_0^2}|\nabla u|^4 \sum_i F^{ii}. \tag{8.13}$$

Hence, to prove the Lemma we only need to check the following

Claim: *There is a constant A_0 depending only on k, n, and $\|g_0\|_{C^3(B_1)}$, such that,*

$$\sum_{i,l} F^{ii}\tilde{u}_{il}^2 \geq A_0^{-\frac{5}{8}} \sum_i F^{ii}|\nabla u|^4. \tag{8.14}$$

From (8.6), the left hand side can be expressed as

$$\begin{aligned} &\textstyle\sum_{i,l} F^{ii}\tilde{u}_{il}^2 = \sum_i F^{ii}\tilde{u}_{ii}^2 + \sum_{i\neq l} F^{ii}u_i^2 u_l^2 \\ &= \sum_i F^{ii}\left\{\tilde{u}_{ii}^2 + u_i^2(|\nabla u|^2 - u_i^2)\right\} = \sum_i F^{ii}(\lambda_i^2 - 2u_i^2\lambda_i + \lambda_i|\nabla u|^2 + \frac{|\nabla u|^4}{4}). \end{aligned} \tag{8.15}$$

The **Claim** (8.14) and (8.15) yield

Lemma 8.2. *There is a constant A_0 depending only on k, n, and $\|g_0\|_{C^3(B_1)}$, such that,*

$$\sum_{i,l} F^{ii}\tilde{u}_{il}^2 \geq A_0^{-\frac{5}{8}} \sum_i F^{ii}(|\nabla u|^4 + \lambda_i^2). \tag{8.16}$$

We note by the Newton-MacLaurin inequality, it then follows that

$$\sum_{i,l} F^{ii}\tilde{u}_{il}^2 \geq A_0^{-\frac{5}{8}}\left(\frac{\sigma_k(W)\sigma_{l+1}(W)}{\sigma_l^2(W)} + |\nabla u|^4 \sum_i F^{ii}\right). \tag{8.17}$$

.

Proof of Lemma 8.2. Let $\tilde{I} = \{i| \quad \lambda_i^2 \geq 9|\nabla u|^4\}$. It follows from (8.15) that

$$\sum_{i,l} F^{ii}\tilde{u}_{il}^2 \geq \frac{1}{2}\sum_{i\in\tilde{I}} F^{ii}\lambda_i^2. \tag{8.18}$$

Note that for i *not* in $\tilde{I}$, $\lambda_i^2 \leq 9|\nabla u|^4$. Therefore, (8.16) follows from (8.18) and **Claim** (8.14). ■

We verify the **Claim** (8.14).

Proof of Claim. Set $I = \{1, 2, \cdots, n\}$. Recall that at x_0, by (8.8), we have for any $i \in I$,

$$|u_i\left(u_{ii} - (|\nabla u|^2 - u_i^2)\right) - \sum_l S_{il}u_l| = |\sum_l u_{il}u_l| \leq \frac{1}{A_0}|\nabla u|^3.$$

This implies

$$|u_i\left(u_{ii} - (|\nabla u|^2 - u_i^2)\right)| \leq \frac{2}{A_0}|\nabla u|^3. \tag{8.19}$$

Set $\delta_0 = A_0^{-1/4}$. Sometimes, for simplicity of notation, we denote W_{ii} by λ_i. We divide I into three subsets I_1, I_2 and I_3 by

$$I_1 = \{i \in I \,|\, u_i^2 \geq \delta_0|\nabla u|^2\}, \qquad I_2 = \{i \in I \,|\, u_i^2 < \delta_0|\nabla u|^2 \ \&\ \tilde{u}_{ii}^2 \geq \delta_0^2|\nabla u|^4\}$$

and

$$I_3 = \{i \in I \,|\, u_i^2 < \delta_0|\nabla u|^2 \ \&\ \tilde{u}_{ii}^2 < \delta_0^2|\nabla u|^4\}$$

For any $i \in I_1$, by (8.19) we can deduce that

$$\left|\lambda_i - \frac{|\nabla u|^2}{2}\right| < 2\delta_0^3|\nabla u|^2 < 2\delta_0^2|\nabla u|^2. \tag{8.20}$$

For any $j \in I_3$, since $\lambda_j = \tilde{u}_{jj} + u_j^2 - |\nabla u|^2/2$, we have

$$\left|\lambda_j + \frac{|\nabla u|^2}{2}\right| < 2\delta_0|\nabla u|^2 = 2A_0^{-\frac{1}{4}}|\nabla u|^2. \tag{8.21}$$

In particular, $\lambda_i > 0$ if $i \in I_1$ and $\lambda_j < 0$ if $j \in I_3$, for large small δ_0.

We verify the **Claim** (8.14) by dividing into two cases.

Case 1. $|I_3| = 0$.

First we note that this case includes the case $k = n$. If $\tilde{u}_{ii}^2 + u_i^2(|\nabla u|^2 - u_i^2) \geq \delta_0^2|\nabla u|^4$ for all $i \in I$, the **Claim** follows from (8.15) easily. Therefore we may assume that there is i_0 such that $\tilde{u}_{i_0i_0}^2 \leq \delta_0^2|\nabla u|^4$. Recall that $\tilde{u}_{ii} = u_{ii} + S_{ii}$. Since $I_3 = 0$, we have $i_0 \in I_1$. Thus,

$$\tilde{u}_{i_0i_0}^2 \leq \delta_0^2|\nabla u|^4 \quad \text{and} \quad u_{i_0}^2 \geq \delta_0|\nabla u|^2. \tag{8.22}$$

Assume that $i_0 = 1$. By (8.19) we have $u_1^2 \geq (1 - 2\delta_0)|\nabla u|^2$ and $\lambda_1 > 0$. Now it is clear that $(|\nabla u|^2 - u_j^2) \geq (1 - 2\delta_0)|\nabla u|^2$ for all $j > 1$, and there is no other $j \in I, j \neq 1$ satisfying (8.22) if A_0 is large enough. Hence, for any $j > 1$, $\tilde{u}_{jj} \geq \delta_0^2|\nabla u|^4$ Hence, we have

$$\tilde{u}_{jj}^2 + u_j^2(|\nabla u|^2 - u_j^2) \geq \delta_0^2|\nabla u|^4 \quad \text{for any } j > 1. \tag{8.23}$$

If there is $j_0 \geq 2$ such that $\lambda_{j_0} \leq \lambda_1$, by Lemma 13.3 we have $F^{j_0 j_0} \geq F^{11}$. By (8.23)

$$\sum_{i,l} F^{ii} \tilde{u}_{il}^2 \geq \delta_0^2 |\nabla u|^4 \sum_{i=2}^n F^{ii} \geq \frac{1}{2} \delta_0^2 |\nabla u|^4 \sum_{i=1}^n F^{ii}.$$

Hence, we may assume that $\lambda_j \geq \lambda_1$ for any $j \geq 2$. It follows that $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Gamma_n^+$. By Lemma 13.3 we have $F^{jj} \lambda_j^2 \geq F^{11} \lambda_i^2$ for any $j \geq 2$. And we have $|\nabla u|^2 - 2u_j^2 \geq 0$ for any $j \geq 2$. Note that $\lambda_1 \geq (\frac{1}{2} - 2\delta_0^2)|\nabla u|^2$ by (8.20), altogether we have

$$\sum_{i,l} F^{ii} \tilde{u}_{il}^2 \geq \sum_{j=2}^n F^{jj} (\lambda_j^2 + \frac{|\nabla u|^2}{4}) \geq \sum_{j=2}^n F^{jj} \lambda_j^2 \geq \sum_{j=1}^n F^{jj} \frac{\lambda_j^2}{2} \geq (\frac{1}{4} - \delta_0^2)^2 \sum_{j=1}^n F^{jj} |\nabla u|^2.$$

Case 2. $|I_3| \neq 0$.

By (8.21), for $j \in I_3$ we have

$$\lambda_j^2 - 2u_j \lambda_j \geq (\frac{1}{4} - 2\delta_0) |\nabla u|^4. \tag{8.24}$$

For $j \in I_2$, it is clear that $\lambda_j^2 - 2u_j^2 \lambda_j = (\lambda_j - u_j^2)^2 - u_j^4 \geq -\delta_0^2 |\nabla u|^4$. Set $\tilde{F}^1 = \max_{i \in I_1} F^{ii}$, we have

$$\sum_{j \in I_2} F^{jj} (\lambda_j^2 - 2u_j^2 \lambda_j) \geq -\delta_0^2 |\nabla u|^4 \sum_i F^{ii}. \tag{8.25}$$

Observation: *The* **Claim** *is true if* $\sum_{j \in I_2 \cup I_3} F^{jj} \geq (1 + c_0) \tilde{F}^1$ *for some* $c_0 > 0$ *independent of* δ_0.

The **Observation** follows from (8.19)–(8.25), since

$$
\begin{aligned}
\sum_{i,l} F^{ii}\tilde{u}_{il}^2 &= \sum_i F^{ii}(\lambda_i^2 - 2u_i^2\lambda_i + \lambda_i|\nabla u|^2 + \frac{|\nabla u|^4}{4}) \\
&= \sum_i F^{ii}(\lambda_i^2 - 2u_i^2\lambda_i) + \frac{|\nabla u|^4}{4}\sum_i F^{ii} + F|\nabla u|^2 \\
&\geq (\sum_{i\in I_1} + \sum_{i\in I_2} + \sum_{i\in I_3})F^{ii}(\lambda_i^2 - 2u_i^2\lambda_i) + \frac{|\nabla u|^4}{4}\sum_i F^{ii} \\
&\geq \sum_{i\in I_1} F^{ii}\left(\frac{|\nabla u|^4}{4} - 2u_i^2\frac{|\nabla u|^2}{2}\right) + \sum_{j\in I_3} F^{jj}\frac{|\nabla u|^4}{4} + (1-32\delta_0^2)\frac{|\nabla u|^4}{4}\sum_i F^{ii} \qquad (8.26) \\
&\geq \tilde{F}^1\frac{|\nabla u|^4}{4} - \tilde{F}^1|\nabla u|^4 + \sum_{j\in I_3} F^{jj}\frac{|\nabla u|^4}{4} + (1-32\delta_0^2)\frac{|\nabla u|^4}{4}\sum_i F^{ii}, \\
&\geq -\tilde{F}^1\frac{|\nabla u|^4}{2} + (1-32\delta_0^2)\frac{|\nabla u|^4}{4}\sum_i F^{ii} \\
&\geq (\frac{1}{2}c_0 - 32\delta_0^2)\frac{|\nabla u|^4}{4}\sum_i F^{ii}.
\end{aligned}
$$

We note that, if $|I_3| \geq 2$, (8.20) and (8.21) implies that for any $i \in I_1$ and $j \in I_3$ we have $\lambda_i > \lambda_j$. So $F^{ii} \leq F^{jj}$ by Lemma 13.3. Hence $\sum_{j\in I_3} F^{jj} \geq |I_3|\tilde{F}^1 \geq 2\tilde{F}^1$ and the **Claim** follows from the **Observation**. Therefore in the rest of proof, we may assume $|I_3| = 1$ and may take $I_3 = \{n\}$. We divide it into three subcases.

Subcase 2.1. $|I_3| = 1$, $|I_1| \geq 2$.

Since $\tilde{F}^1 \leq F^{nn}$, we may assume that $F^{jj} \leq \frac{1}{2}\tilde{F}^1$ for any $j \in I_2$. Otherwise, $\sum_{j\in I_2\cup I_3} F^{jj} \geq \frac{3}{2}\tilde{F}^1$ and the **Claim** is true by the **Observation**. From Lemma 13.3 and (8.20), $F^{jj} \leq \tilde{F}^1$ implies that $\lambda_j \geq \inf_{i\in I_1}\lambda_i \geq (\frac{1}{2} - 2\delta_0^2)|\nabla u|^2$. It is clear to see that $u_i^2 \leq (1-\delta_0)|\nabla u|^2$, for $|I_1| \geq 2$. By the **Observation** we may assume $F^{nn} \leq 2\tilde{F}^1$. From these facts, together with

(8.20) and (8.25), we estimate

$$\begin{aligned}
\sum_{i,l} F^{ii}\tilde{u}_{il}^2 &\geq \sum_{i=1}^{n-1} F^{ii}(\lambda_i^2 - 2u_i^2\lambda_i + \lambda_i|\nabla u|^2 + \frac{|\nabla u|^4}{4}) \\
&\geq \sum_{i\in I_1} F^{ii}(\lambda_i^2 - 2u_i^2\lambda_i + \lambda_i|\nabla u|^2) + \frac{|\nabla u|^4}{4}\sum_{i=1}^{n-1} F^{ii} \\
&\geq |\nabla u|^4\{\frac{3}{4}\sum_{i\in I_1} F^{ii} - \sum_{i\in I_1} F^{ii}(1-\delta_0) + \frac{1}{4}\sum_{i=1}^{n-1} F^{ii} - \frac{32\delta_0^2}{4}\sum_i F^{ii}\} \\
&\geq \frac{1}{2}\delta_0\frac{|\nabla u|^4}{4}\sum_{i=1}^{n-1} F^{ii} - 32\delta_0^2\frac{|\nabla u|^4}{4}\sum_i F^{ii} \\
&\geq \frac{1}{4}\delta_0\frac{|\nabla u|^4}{4}\sum_{i=1}^{n-1} F^{ii} \geq \frac{1}{8}\delta_0\frac{|\nabla u|^4}{4}\sum_{i=1}^{n} F^{ii}.
\end{aligned}$$

Subcase 2.2. $|I_3| = 1$, $|I_1| = 1$ and $k \leq n-2$.

In this subcase, $I_2 = \{2, 3, \cdots, n-1\}$. As in Subcase 2.1, we may assume that $\lambda_j \geq \lambda_1$ for any $j \in I_2$. First we assume that there is a $j_0 \in I_2$ such that $\Lambda_{1j_0} \in \Gamma_{k-1}^+$. By Lemma 13.3, we have $F^{j_0j_0}\lambda_{j_0}^2 \geq F^{11}\lambda_1^2$.

Using (8.20) and (8.21), we compute

$$\begin{aligned}
&\sum_{i,l} F^{ii}\tilde{u}_{il}^2 \geq \sum_i F^{ii}(\lambda_i^2 - 2u_i^2\lambda_i + \lambda_i|\nabla u|^2 + \frac{|\nabla u|^4}{4}) \\
\geq\ & F^{11}(\lambda_1^2 - 2|\nabla u|^2\lambda_1) + \frac{1}{2}\sum_{j=2}^{n-1} F^{ii}\lambda_i^2 + F^{nn}\lambda_n^2 + \sum_{i=1}^{n} F^{ii}\frac{|\nabla u|^4}{4} + F|\nabla u|^2 \\
\geq\ & -\frac{3}{4}F^{11}|\nabla u|^4 + \frac{1}{2}F^{j_0j_0}\lambda_{j_0}^2 + F^{nn}\frac{|\nabla u|^4}{4} + \sum_{i=1}^{n} F^{ii}\frac{|\nabla u|^4}{4} - 32\delta_0^2|\nabla u|^4\sum_i F^{ii} \\
\geq\ & \frac{1}{8}|\nabla u|^4\sum_i F^{ii} - 32\delta_0^2|\nabla u|^4\sum_i F^{ii}
\end{aligned}$$

So the **Claim** will follow if we pick A_0 large enough.

Hence, we may assume that for any $j \in I_2$, $\sigma_{k-1}(\Lambda_{1j}) \leq 0$. From this fact, we want to show that

$$\sigma_{k-1}(\Lambda_{1n}) \leq \frac{n-2}{n-k-1}(\lambda_1 + |\lambda_n|)\sigma_{k-2}(\Lambda_{1n}). \tag{8.27}$$

Assume that $\lambda_2 = \min_{j\in I_2}\lambda_j$. From

$$0 \geq \sigma_{k-1}(\Lambda_{12}) = \sigma_{k-1}(\Lambda_{12n}) + \lambda_n\sigma_{k-2}(\Lambda_{12n})$$

we have $\sigma_{k-1}(\Lambda_{12n}) \leq |\lambda_n|\sigma_{k-2}(\Lambda_{12n})$.)Recall that $\lambda_n < 0$.) As $0 < \lambda_2 \leq \lambda_j$ for any $3 \leq j \leq n-1$, by counting the terms, it's easy to see that

$$\sigma_{k-1}(\Lambda_{1n}) \leq \frac{n-2}{n-k-1}\sigma_{k-1}(\Lambda_{12n}).$$

Altogether gives that

$$\begin{aligned}\sigma_{k-1}(\Lambda_{1n}) &\leq \frac{n-2}{n-k-1}\sigma_{k-1}(\Lambda_{1n}) \leq \frac{n-2}{n-k-1}|\lambda_n|\sigma_{k-2}(\Lambda_{1n}) \\ &\leq \frac{n-2}{n-k-1}(\lambda_1 + |\lambda_n|)\sigma_{k-2}(\Lambda_{1n}).\end{aligned}$$

We now want to make use of (8.27). By (8.63) we have

$$\begin{aligned}F^{11} &= F^*\{[\sigma_l(\Lambda_{1n})\sigma_{k-1}(\Lambda_{1n}) - \sigma_{l-1}(\Lambda_{1n})\sigma_k(\Lambda_{1n})] \\ &\quad +\lambda_n[\sigma_l(\Lambda_{1n})\sigma_{k-2}(\Lambda_{1n}) - \sigma_k(\Lambda_{1n})\sigma_{l-2}(\Lambda_{1n})] \\ &\quad +\lambda_n^2[\sigma_{l-1}(\Lambda_{1n})\sigma_{k-2}(\Lambda_{1n}) - \sigma_{k-1}(\Lambda_{1n})\sigma_{l-2}(\Lambda_{1n})]\}.\end{aligned} \tag{8.28}$$

We have a similar expansion for F^{nn}. Hence, we obtain

$$\begin{aligned}F^{nn} - F^{11} &= F^*(\lambda_1 - \lambda_n)\{[\sigma_l(\Lambda_{1n})\sigma_{k-2}(\Lambda_{1n}) - \sigma_k(\Lambda_{1n})\sigma_{l-2}(\Lambda_{1n})] \\ &\quad +(\lambda_1 + \lambda_n)[\sigma_{l-1}(\Lambda_{1n})\sigma_{k-2}(\Lambda_{1n}) - \sigma_{k-1}(\Lambda_{1n})\sigma_{l-2}(\Lambda_{1n})]\}.\end{aligned} \tag{8.29}$$

By the Newton-MacLaurin inequality, there is $C_1 > 0$ depending only on n, k and l, such that

$$\begin{aligned}&\sigma_l(\Lambda_{1n})\sigma_{k-2}(\Lambda_{1n}) - \sigma_k(\Lambda_{1n})\sigma_{l-2}(\Lambda_{1n}) \geq C_1\sigma_l(\Lambda_{1n})\sigma_{k-2}(\Lambda_{1n}), \\ &\sigma_{l-1}(\Lambda_{1n})\sigma_{k-2}(\Lambda_{1n}) \geq \sigma_{k-1}(\Lambda_{1n})\sigma_{l-2}(\Lambda_{1n}).\end{aligned} \tag{8.30}$$

Since $\lambda_1 + \lambda_n \leq 4\delta_0|\nabla u|^2 \leq 2\delta_0\lambda_2$ and $\sigma_{l-1}(\Lambda_{1n})\lambda_2 \leq C_{n-2}^{l-1}\sigma_l(\Lambda_{1n})$, where C_{n-1}^{l-1} is the binomial constant. Combining this fact with (8.30), if $\delta_0 > 0$ small enough, we have

$$(\lambda_1 + \lambda_n)[\sigma_{l-1}(\Lambda_{1n})\sigma_{k-2}(\Lambda_{1n}) - \sigma_{k-1}(\Lambda_{1n})\sigma_{l-2}(\Lambda_{1n})] \geq -\frac{C_1}{2}\sigma_l(\Lambda_{1n})\sigma_{k-2}(\Lambda_{1n}). \tag{8.31}$$

Together with (8.29), (8.27), if $\delta_0 > 0$ small enough, we get

$$\begin{aligned}F^{nn} - F^{11} &\geq \tfrac{C_1}{4}F^*(\lambda_1 - \lambda_n)\sigma_l(\Lambda_{1n})\sigma_{k-2}(\Lambda_{1n}) \\ &\geq \frac{(n-k-1)C_1}{4(n-2)}F^*\sigma_l(\Lambda_{1n})\sigma_{k-1}(\Lambda_{1n}) \geq C_2F^{11},\end{aligned}$$

where the last inequality follows from the expansion (8.28) of F^{11}, the fact that $\lambda_n < 0$ and $\lambda_n^2\sigma_{l-1}(\Lambda_{1n}) \leq 2\lambda_2^2\sigma_{l-1}(\Lambda_{1n}) \leq 2C_{n-2}^{l-1}\sigma_l(\Lambda_{1n})$. Hence, we have $F^{nn} \geq (1 + C_2)F^{11}$ and the **Claim** follows from the **Observation**.

Subcase 2.3 $|I_3| = 1$, $|I_1| = 1$ and $k = n-1$.

Again, we may assume that $\lambda_j \geq \lambda_1$ for any $2 \leq j \leq n-1$. Note that $2u_j^2 \leq |\nabla u|^2$ for any $2 \leq j \leq n-1$. Also as in Subcase 2.2, if $\delta_0 > 0$ is small enough,

$$(l+1)\sigma_{l+1}(\Lambda_{1n}) + (\lambda_1 + \lambda_n)l\sigma_l(\Lambda_{1n}) \geq 0. \tag{8.32}$$

It follows that

$$
\begin{aligned}
&\sum_{j=2}^{n-1}(\sigma_l(\Lambda)\sigma_{k-1}(\Lambda_j)-\sigma_k(\Lambda)\lambda_{l-1}(\Lambda_j))\lambda_j^2\\
=\ &\sum_{j=2}^{n-1}\lambda_j^2\{\sigma_l(\Lambda)(\sigma_{n-2}(\Lambda_{1jn})+(\lambda_1+\lambda_n)\sigma_{n-3}(\Lambda_{1jn})+\lambda_1\lambda_n\sigma_{n-4}(\Lambda_{1jn}))\\
&-\sigma_{n-1}(\Lambda)(\sigma_{l-1}(\Lambda_{1jn})+(\lambda_1+\lambda_n)\sigma_{l-2}(\Lambda_{1jn})+\lambda_1\lambda_n\sigma_{l-3}(\Lambda_{1jn}))\}\\
=\ &\sigma_l(\Lambda)\{[\sigma_k(\Lambda_{1n})\sigma_1(\Lambda_{1n})-(k+1)\sigma_{k+1}(\Lambda_{1n})]+(\lambda_1+\lambda_n)[\sigma_{k-1}(\Lambda_{1n})\sigma_1(\Lambda_{1n})\\
&-k\sigma_k(\Lambda_{1n})]+\lambda_1\lambda_n[\sigma_{k-2}(\Lambda_{1n})\sigma_1(\Lambda_{1n})-(k-1)\sigma_{k-1}(\Lambda_{1n})]\}\\
&-\sigma_k(\Lambda)\{[\sigma_l(\Lambda_{1n})\sigma_1(\Lambda_{1n})-(l+1)\sigma_{l+1}(\Lambda_{1n})]+(\lambda_1+\lambda_n)[\sigma_{l-1}(\Lambda_{1n})\sigma_1(\Lambda_{1n})\\
&-l\sigma_l(\Lambda_{1n})]+\lambda_1\lambda_n[\sigma_{l-2}(\Lambda_{1n})\sigma_1(\Lambda_{1n})-(l-1)\sigma_{l-1}(\Lambda_{1n})]\}\\
=\ &-(n-2)\lambda_1\lambda_n\sigma_{n-2}(\Lambda_{1n})\sigma_l(\Lambda)\\
&+\sigma_{n-1}(\Lambda)[(l+1)\sigma_{l+1}(\Lambda_{1n})+(\lambda_1+\lambda_n)l\sigma_l(\Lambda_{1n})+(l-1)\lambda_1\lambda_n\sigma_{l-1}(\Lambda_{1n})]\\
\geq\ &\lambda_1|\lambda_n|\{(n-2)\sigma_l(\Lambda)\sigma_{n-2}(\Lambda_{1n})-(l-1)\sigma_{n-1}(\Lambda)\sigma_{l-1}(\Lambda_{1n})\}\\
\geq\ &\lambda_1|\lambda_n|(n-l-1)\sigma_l(\Lambda)\sigma_{n-2}(\Lambda_{1n}).
\end{aligned}
\tag{8.33}
$$

From (8.33), we get

$$
\begin{aligned}
&\sum_{i,l}F^{ii}\tilde{u}_{il}^2=\sum_{j=1}^{n}F^{jj}(\lambda_j^2-2u_j^2\lambda_j+\lambda_j|\nabla u|^2+\frac{|\nabla u|^4}{4})\\
\geq\ &\sum_{j=2}^{n-1}F^{jj}(\lambda_j^2-2u_j^2\lambda_j+\lambda_j|\nabla u|^2+\frac{|\nabla u|^4}{4})\\
\geq\ &\sum_{j=2}^{n-1}F^{jj}\lambda_j^2=F^*\sum_{j=2}^{n-1}(\sigma_l(\Lambda)\sigma_{k-1}(\Lambda_j)-\sigma_k(\Lambda)\lambda_{l-1}(\Lambda_j))\lambda_j^2\\
\geq\ &F^*\lambda_1|\lambda_n|(n-l-1)\sigma_l(\Lambda)\sigma_{n-2}(\Lambda_{1n})\\
\geq\ &F^*(\frac{1}{4}-2\delta_0)|\nabla u|^4\sigma_l(\Lambda)\sigma_{n-2}(\Lambda_{1n}).
\end{aligned}
\tag{8.34}
$$

Since $\lambda_j\geq\lambda_1$ for any $j=2,3,\cdots,n-1$, it is easy to see that $\sigma_{n-2}(\Lambda_{1n})\geq\frac{1}{n-1}\sigma_{n-2}(\Lambda_j)$ for any $j=1,2,\cdots,n$. It follows that $F^*\sigma_l(\Lambda)\sigma_{n-2}(\Lambda_{1n})\geq\frac{1}{(n-1)n}\sum_i F^{ii}$. Hence, (8.34) implies

$$
\sum_{j,l}F^{ii}\tilde{u}_{il}^2\geq\frac{1}{(n-1)n}(\frac{1}{4}-2\delta_0)|\nabla u|^4\sum_i F^{ii}.
$$

The proof is complete. ■

REMARK 8.1. *The gradient estimates are also valid for a general equation with term ε^u.*

Finally, Corollary 8.1 follows from Theorem 8.1 and the next Proposition.

PROPOSITION 8.1. *There exist constant $\varepsilon_0 > 0$ and constant $c_{\varepsilon_0} > 0$ (depending only on ε_0) such that any solution u of (8.2) in B_1 with*

$$\int_{B_1} e^{-nu} dvol(g_0) \leq \varepsilon_0$$

satisfies

$$\inf_{B_{1/2}} u \geq -c_{\varepsilon_0}.$$

Proof: We make use of a rescaling argument as in [**112**], together with Theorem 8.1, to prove this Proposition.

Assume by contradiction that there is a sequence of solutions u_i of (8.2) in B_1 such that

$$\int_{B_1} e^{-nu_i} dvol(g_0) \to 0, \quad \text{as } i \to \infty$$

and

$$\inf_{B_{1/2}} u_i \to -\infty, \quad i \to \infty. \tag{8.35}$$

Consider the function $(3/4 - r)^2 \sup_{B_r} e^{-nu_i} : (0, 3/4) \to [0, \infty)$. As the function is continuous, there is $r_0^i \in (0, 3/4)$ such that

$$\left(\frac{3}{4} - r_0^i\right)^2 \sup_{B_{r_0^i}} e^{-nu_i} = \sup_{0<r<3/4} \left(\frac{3}{4} - r\right)^2 \sup_{B_r} e^{-nu_i}.$$

Moreover, there exists $z_0^i \in \overline{B}_{r_0}$ such that $e^{-nu_i(z_0^i)} = \sup_{B_{r_0^i}} e^{-nu_i(z)}$. Let $s_0^i = (3/4 - r_0^i)/2$. From the definition,

$$\sup_{B_{s_0}(z_0^i)} e^{-nu_i} \leq \sup_{B_{s_0+r_0}(z_0^i)} e^{-nu_i} \leq 4e^{-nm_i}, \tag{8.36}$$

where $m_i = u_i(z_0^i)$. Consider the rescaled function $v^i(y) = u_i(\exp_{z_0^i} e^{m_i} y) - m_i$ in $B_{e^{-m_i} s_0^i}$. v^i satisfies equation of type (8.2).

By (8.36), we have,

$$\int_{B_{e^{-m_i} s_0}} e^{-nv^i} = \int_{B_{s_0}(z_0^i)} e^{-nu_i} \to 0, \quad \text{as } i \to \infty$$

and $v^i(0) = 0$, $v^i(x) \geq -\frac{1}{n} \log 4$. From (8.35), one may check that $e^{-m_i} s_0^i \geq a_0 > 0$ for any i. Now by Theorem 8.1, $\sup v^i$ is uniformly bounded in $B_{e^{-m_i} s_0^i/2}$. This is a contradiction. ■

We now treat second derivative estimate for equation

$$f(e^{2v}(\nabla^2 v + dv \otimes dv - \frac{|\nabla v|^2}{2} g + S_g)) = h. \tag{8.37}$$

LEMMA 8.3. *Suppose that f satisfies conditions (13.12), (13.13), and (13.15), and suppose that $v \in C^4$ is an admissible solution of (8.37) with $h = 1$ in B_r. Then, there exists a constant $c > 0$ depending only on r, t_0 and $\|g\|_{C^4(B_r)}$, such that*

$$|\nabla^2 v|(x) < c(1 + \sup_{B_r} |\nabla v|^2), \qquad \text{for } x \in B_{r/2}. \tag{8.38}$$

In general, if $h \in C^{1,1}(B_r)$ (not necessary constant), if $\sum_i \frac{\partial f(\lambda)}{\partial \lambda_i} \geq \delta > 0$ for all $\lambda \in \Gamma$, then there is constant $c > 0$ depending only on r, δ, $\|g\|_{C^4(B_r)}$, and $\|h\|_{C^2(B_r)}$, such that

$$|\nabla^2 v|(x) < c(1 + \sup_{B_r} |\nabla v|^2), \qquad \text{for } x \in B_{r/2}. \tag{8.39}$$

Proof. Choose r' small such that there is a local orthonormal frame in each geodesic ball $B_{r'}(x)$ for all $x \in B_{\frac{2r}{3}}$. We only need to verify (8.38) for such $B_{r'}(x)$, which we will still denote B_r. We may also assume $r = 1$. Let ρ be a smooth nonnegative cut-off function in B_1, $\rho = 1$ in $B_{\frac{1}{2}}$ and $\rho = 0$ in $B_1 \setminus B_{\frac{2}{3}}$. We only need to get an upper bound for $\rho(T^2 v + |Tv|^2)$ for any unit vector field T. Since $\bar{B}_{\frac{2}{3}} \times \mathbb{S}^{n-1}$ is compact, we may assume the maximum attained at some point $y_0 \in B_{\frac{2}{3}}$ and $T = e_1$ for some orthonormal frame $\{e_1, \cdots, e_n\}$ in B_1. Set

$$G = \rho(v_{11} + |v_1|^2).$$

So y_0 is a maximum point of G. By the C^1 bound assumption, we may assume $v_{11} \geq 1 + |v_1|^2$ and $v_{11}(y_0) > \frac{1}{4n}|v_{ij}(y_0)|, \forall i, j$. Now at y_0, we have

$$0 = G_j(y_0) = \frac{\rho_j}{\rho} G + \rho(v_{11j} + 2v_1 v_{1j}) \quad \text{for any } j \tag{8.40}$$

and

$$G_{ij} = \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2} G + \rho(v_{11ij} + 2v_{1i}v_{1j} + 2v_1 v_{1ij}).$$

For any fixed local orthonormal frame, we may view S_g and $S_{\hat{g}}$ as matrices. We denote S_{ij} and U_{ij} the entries of $g^{-1}S_g$ and $\hat{g}^{-1}S_{\hat{g}}$ respectively. By the ellipticity assumption on f, (F^{ij}) is positive definite at $U = \hat{g}^{-1}S_{\hat{g}}$. Since y_0 is a maximum point of G,

$$\begin{aligned} 0 &\geq \sum_{i,j\geq 1} F^{ij} G_{ij} \\ &\geq \sum_{i,j\geq 1} F^{ij} \{\frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2} G + \rho(v_{ij11} + 2v_{1i}v_{1j} + 2v_1 v_{ij1})\} - CG \sum_i F^{ii}, \end{aligned}$$

where the last term comes from the commutators related to the curvature tensor of g and its derivatives.

By (8.40), (13.16) and the concavity of f,

(8.41)

$$\begin{aligned}
0 &\geq \sum_{i,j\geq 1} F^{ij}\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G+\rho\sum_{i,j\geq 1}F^{ij}(v_{ji11}+2v_{1i}v_{1j}+2v_1v_{1ij})-CG\sum_{i\geq 1}F^{ii}\\
&= \sum_{i,j\geq 1} F^{ij}\{\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G+\rho(e^{-2v}U_{ij}-v_iv_j+\frac{1}{2}|\nabla v|^2\delta_{ij}-S_{ij})_{11}\\
&\quad +2\rho v_{1i}v_{1j}+2\rho v_1(e^{-2v}U_{ij}-v_iv_j+\frac{1}{2}|\nabla v|^2\delta_{ij}-S_{ij})_1\}-CG\sum_{i\geq 1}F^{ii}\\
&= \sum_{i,j\geq 1} F^{ij}\{\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G+\rho[e^{-2v}(U_{ij})_{11}-2v_1e^{-2v}(U_{ij})_1+(\frac{1}{2}|\nabla v|^2\delta_{ij}-S_{ij})_{11}\\
&\quad -2v_{11}e^{-2v}U_{ij}-v_iv_{j11}-v_jv_{i11}+2v_1(-v_iv_j+\frac{1}{2}|\nabla v|^2\delta_{ij}-S_{ij})_1]\}-CG\sum_{i\geq 1}F^{ii}\\
&\geq \sum_{i,j\geq 1} F^{ij}\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G+\rho e^{-2v}(h_{11}-2v_1h_1)+\sum_{i\geq 1}F^{ii}[\rho v_{11}^2-C(1+\frac{|\nabla\rho|}{\rho})G]\\
&\quad -2Ct_0\rho v_{11}\sum_i F^{ii},
\end{aligned}$$

where t_0 is the number in (13.15).

From our construction of ρ, $|\nabla\rho(x)|\leq C\rho^{\frac{1}{2}}(x)$ for all $x\in B_1$. We have

$$\sum_{i,j\geq 1} F^{ij}\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G\geq -C\sum_{i\geq 1}F^{ii}\frac{1}{\rho}G.$$

If h is a constant, $h_1=h_{11}=0$. By assumption $v_{11}\geq\frac{1}{2\rho}G$ at y_0. It follows from (8.41) that at y_0, $G\leq C$. So (8.38) follows.

If $h\in C^{1,1}(B_r)$, and $\sum_i\frac{\partial f(\lambda)}{\partial\lambda_i}\geq\delta>0$ for all $\lambda\in\Gamma$, (8.39) also follows from (8.41). ■

1. Conformally invariant uniformly elliptic equations

In this section, we establish a local gradient estimates for solutions $g=e^{-2u}g_0$ of equation (10.103) in Chapter 10.

THEOREM 8.1 (Local gradient estimates). *Suppose F is concave and uniformly elliptic with ellipticity constants λ_0,Λ_0. Let B_1 be a unit disk in a compact Riemannian manifold M and u a C^2 solution of the following equation*

$$F\left(\nabla^2u+du\otimes du-\frac{|\nabla u|^2}{2}g_0\right)=e^{-2u}f(x),\quad x\in B_1 \tag{8.42}$$

for a C^1 function $f : B_1 \to \mathbb{R}$. Then there is a constant $C > 0$ depending only on $\lambda_0, \Lambda_0, g_0$ such that

$$|\nabla u|^2(x) \leq C(1 + \|f\|_{C^1(B_1)} e^{-2\inf_{B_1} u}), \quad \text{for any } x \in B_{1/2}. \tag{8.43}$$

Combining Theorem 8.1 and Theorem 10.4, we deduce the following.

COROLLARY 8.2. *Let B_1 be a unit disk in a compact Riemannian manifold M and $f \in C^1(B_1)$. Suppose u is a solution of equation (8.42), then there is a constant $C > 0$ depending only on $\lambda_0, \Lambda_0, g_0, \|f\|_{C^1 B_1}, \inf_{B_1} u$, such that*

$$\|u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C. \tag{8.44}$$

In particular, (8.44) is true for any solution of (10.103) when $p \leq \frac{n}{2}$.

We remark that the local gradient estimates (8.43) does not true for $p = n-1$. The operator G_p we are considering here is only Lipschitz, which we will deal with by a smoothing argument. Actually we can prove the local gradient estimates for a more general class of uniformly elliptic fully nonlinear conformal equations.

We first prove the local gradient estimates for C^2 uniformly elliptic operator F. Let $F : \mathbb{R}^n \to \mathbb{R}$ is a C^2 symmetric function and consider the following equation

$$F\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0\right) = \tilde{f}, \tag{8.45}$$

for some C^1 function $\tilde{f}$. We denote the left hand side of (8.45) by $F(W)$ and set

$$F^{ij} = \frac{\partial F}{\partial w_{ij}},$$

where w_{ij} is the entry of the matrix W. As mentioned above, $F(W) = F(\Lambda)$, where Λ is the set of eigenvalues of W.

PROPOSITION 8.1. *Let B_1 be a unit disk in a compact Riemannian manifold M and u a C^3 solution of (8.45) for a C^1 function $\tilde{f} : B_1 \to \mathbb{R}$. Let $F : \mathcal{S} \to \mathbb{R}$ be a C^2 function satisfying*

(1) *F is an uniformly elliptic with ellipticity constants λ_0 and Λ_0*
(2) *F is concave.*

Then for any $\rho \in C_0^2(B_1)$ with $0 \leq \rho(x) \leq 1$, there is a constant $C > 0$ depending only on λ_0, Λ_0, $\|\rho\|_C^2(B_1)\|$ and g_0 such that

$$\max_{B_1}\{\rho(x)|\nabla u|^2(x)\} \leq C(1 + \max_{B_1}\{\rho(x)|\nabla \tilde{f}(x)|\}). \tag{8.46}$$

Proof of Proposition 8.1. The Proof follows closely the argument given in [**71**] and [**72**]. As in [**71**], we first reduce the proof of the Lemma to the following claim.

Claim. *There is a constant A_0 depending, such that*

$$\sum_{i,j} F^{ii} \tilde{u}_{ij}^2 \geq A_0^{-\frac{5}{8}} \sum_i F^{ii} |\nabla u|^4, \tag{8.47}$$

where $\tilde{u}_{ij} = u_{ij} + S_{ij}$.

For convenience of the reader, we sketch the reduction. Let ρ be a test function $\rho \in C_0^2(B_1)$. We may assume

$$|\nabla\rho(x)| \le 2b_0\rho^{1/2}(x) \quad \text{and} \quad |\nabla^2\rho| \le b_0, \text{ in } B_1, \tag{8.48}$$

for $b_0 > 1$. Set $H = \rho|\nabla u|^2$. Our aim is to bound $\max_{B_1} H$. Let $x_0 \in B_1$ be a maximum point of H and assume that W is a diagonal matrix at the point x_0 by choosing a suitable normal coordinates around x_0. Set $\lambda_i = w_{ii}$ and $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. Since W is diagonal at x_0, we have at x_0

$$w_{ii} = u_{ii} + u_i^2 - \frac{1}{2}|\nabla u|^2 + S_{ii}, \quad u_{ij} = -u_iu_j - S_{ij}, \quad \forall i \ne j, \tag{8.49}$$

where S_{ij} are entries of S_{g_0}. We may assume that

$$H(x_0) \ge b_0^2A_0^2,$$

for some large, but fixed constant $A_0 > 0$ which will be fixed later. We may also assume that

$$|S_{g_0}|(x_0) \le A_0^{-1}|\nabla u|^2(x_0). \tag{8.50}$$

Otherwise, we are done. The fact that the derivatives of H at x_0 vanish imply

$$\left|\sum_{l=1}^n u_{il}u_l\right|(x_0) \le \frac{|\nabla u|^3}{A_0}(x_0) \quad \text{for any } i. \tag{8.51}$$

Applying the maximum principle to H, we have

$$0 \ge F^{ij}H_{ij} = F^{ij}\left\{\left(-2\frac{\rho_i\rho_j}{\rho} + \rho_{ij}\right)|\nabla u|^2 + 2\rho u_{lij}u_l + 2\rho u_{il}u_{jl}\right\}. \tag{8.52}$$

The first term in the left hand side of (8.52) is bounded from below by $-10nb_0\Lambda_0|\nabla u|^2$. By using equation (8.45) and inequality (8.51), the second term can be bounded by

$$\begin{aligned} \sum_{i,j,l} F^{ij}u_{ijl}u_l &\ge \sum_l F_lu_l - 2\sum_{i,l} F^{ii}u_{il}u_lu_i + \sum_{i,l} F^{ii}u_{jl}u_ju_l - C|\nabla u|^2\sum_i F^{ii} \\ &\ge -|\tilde{f}|^2 - 2n\sum_i F^{ii}\frac{|\nabla u|^4}{A_0} - C|\nabla u|^2, \end{aligned} \tag{8.53}$$

where $C > 0$ depends only on g_0 and Λ_0. See also (2.20) in [**71**]. It is easy to see that the third term is bounded by the Claim. Hence if the Claim is true, from (8.52) we have

$$0 \ge -C|\nabla u|^2 - \rho|\nabla\tilde{f}|^2 - \rho\sum_i F^{ii}\frac{|\nabla u|^4}{A_0} + \rho A_0^{-\frac{5}{8}}\sum_i F^{ii}|\nabla u|^4. \tag{8.54}$$

Multiplying (8.54) by ρ, we have

$$0 \le \sum_i F^{ii}(A_0^{-\frac{5}{8}} - A_0^{-1})H^2 - CH - \rho^2|\nabla\tilde{f}|^2,$$

from which we have (8.46).

Now we prove the Claim. By (8.49), we have

$$\begin{aligned}\sum_{i,l} F^{ii}\tilde{u}_{il}^2 &= \sum_i F^{ii}\tilde{u}_{ii}^2 + \sum_{i\neq l} F^{ii}u_i^2u_l^2 \\ &= \sum_i F^{ii}\left\{\tilde{u}_{ii}^2 + u_i^2(|\nabla u|^2 - u_i^2)\right\} \\ &= \sum_i F^{ii}(w_{ii}^2 - 2u_i^2 w_{ii} + w_{ii}|\nabla u|^2 + \frac{|\nabla u|^4}{4}).\end{aligned} \tag{8.55}$$

Set $\delta_0 = A_0^{-1/4} < 0.1$. We divide the set $I = \{1, 2, \cdots, n\}$ as in [**71**] into two parts:

$$I_1 = \{i \in I \,|\, u_i^2 \geq \delta_0|\nabla u|^2\} \quad \text{and} \quad I_2 = \{i \in I \,|\, u_i^2 < \delta_0|\nabla u|^2\}.$$

It is clear that I_1 is non-empty.

Case 1. There is j_0 satisfying

$$\tilde{u}_{jj}^2 \leq \delta_0^2|\nabla u|^4 \quad \text{and} \quad u_j^2 < \delta_0|\nabla u|^2. \tag{8.56}$$

We may assume that $j_0 = n$. We have $|w_{nn} + \frac{|\nabla u|^2}{2}| = |\tilde{u}_{nn} + u_n^2| < 2\delta_0|\nabla u|^2$ by (8.56). From (8.51) and (8.49), we have

$$\left|w_{ii} - \tfrac{|\nabla u|^2}{2}\right| = \left|u_{ii} + u_i^2 - |\nabla u|^2 + S_{ii}\right| \leq 3\delta_0^2|\nabla u|^2,$$

for any $i \in I_1$.

Using these estimates, we repeat the derivation of equation (2.38) in [**71**] to obtain

$$\sum_{i,l} F^{ii}\tilde{u}_{il}^2 \geq \tilde{F}^1\frac{|\nabla u|^4}{4} - \tilde{F}^1|\nabla u|^4 + F^{nn}\frac{|\nabla u|^4}{4} + (1 - 32\delta_0^2)\frac{|\nabla u|^4}{4}\sum_i F^{ii}, \tag{8.57}$$

where $\tilde{F}^1 := \max_{i\in I_1} F^{ii}$. Recall that I_1 is necessarily non-empty. We assume $1 \in I_1$ with $F^{11} = \tilde{F}^1$. The concavity of F implies that

$$F^{nn} \geq F^{11}, \tag{8.58}$$

for $w_{11} > w_{nn}$. Hence, from (8.57) we have

$$\begin{aligned}\sum_{i,l} F^{ii}\tilde{u}_{il}^2 &\geq -16\delta_0^2|\nabla u|^4 + (1 - 32\delta_0^2)\frac{|\nabla u|^4}{4}\sum_{i=2}^{n-1} F^{ii} \\ &\geq \left(1 - 32\delta_0^2 - 64\delta_0^2\frac{\Lambda_0}{(n-2)\lambda_0}\right)\frac{|\nabla u|^4}{4}\sum_{i=2}^{n-1} F^{ii} \\ &\geq \left(1 - 32\delta_0^2 - 64\delta_0^2\frac{\Lambda_0}{(n-2)\lambda_0}\right)\frac{(n-2)\lambda_0}{n\Lambda_0}\frac{|\nabla u|^4}{4}\sum_{i=2}^{n-1} F^{ii},\end{aligned}$$

Case 2. There is no $j \in I$ satisfying (8.56).

For this case, the proof is the same as in [**72**]. We repeat it here for completeness.

We may assume that there is i_0 such that $\tilde{u}^2_{i_0 i_0} \leq \delta_0^2 |\nabla u|^4$, otherwise the claim is automatically true. Assume $i_0 = 1$. As in Case 4 in [**71**], we have $u_1^2 \geq (1 - 2\delta_0)|\nabla u|^2$ and $\tilde{u}^2_{jj} + u_j^2(|\nabla u|^2 - u_j^2) \geq \delta_0^2 |\nabla u|^4$ for $j > 1$. From equation (8.55), we have

$$\begin{aligned}
\sum_{i,l} F^{ii} \tilde{u}^2_{il} &= \sum_i F^{ii} \left\{ \tilde{u}^2_{ii} + u_i^2(|\nabla u|^2 - u_i^2) \right\} \\
&\geq \sum_{i \geq 2} F^{ii} (\tilde{u}^2_{ii} + u_i^2(|\nabla u|^2 - u_i^2)) \\
&\geq \delta_0^2 |\nabla u|^4 \sum_{i \geq 2} F^{ii} \geq C \delta_0^2 |\nabla u|^4 \sum_{i \geq 1} F^{ii}.
\end{aligned}$$

The latter inequality follows from the uniformly ellipticity of F. This finishes the proof the Claim and hence the Proposition. ■

We have a direct corollary.

Corollary 8.3. *Suppose F is a C^2 concave and uniformly elliptic operator with ellipticity constants λ_0, Λ_0. Let B_1 be a unit disk in a compact Riemannian manifold M and u a C^2 solution of equation*

$$F\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0\right) = e^{-2u} f(x), \quad x \in B_1 \tag{8.59}$$

for a C^1 function $f : B_1 \to \mathbb{R}$. Then there is a constant $C > 0$ depending only on $\lambda_0, \Lambda_0, g_0$ such that

$$|\nabla u|^2(x) \leq C(1 + \|f\|_{C^1(B_1)} e^{-2 \inf_{B_1} u}), \quad \text{for any } x \in B_{1/2}. \tag{8.60}$$

Proof: We pick $\rho \in C_0^2(B_1)$ such that $\rho(x) = 1, \forall x \in B_{\frac{1}{2}}$ and $0 \leq \rho(x) \leq 1, \forall x \in B_1$. (8.60) follows directly from (8.46) with $\tilde{f} = e^{-2u} f$. ■

In what follows in the next sections, we will only need Corollary 8.3 as we will work on smooth operator F. We note that estimates (8.60) and (8.44) are independent of the smoothness of F. Theorem 8.1 can also be proved by certain appropriate approximations.

A sketch proof of Theorem 8.1. Since $u \in C^2$, u is in fact $C^{2,\alpha}$ by the Evans-Krylov theorem. We may find two sequences of smooth functions $\{u_k\}$ and $\{f_k\}$, such that $u_k \to u$ in $C^{2,\alpha}(\bar{B}_1)$, $f_k \to f$ in $C^{0,1}(\bar{B}_1)$, and

$$F\left(\nabla^2 u_k + du_k \otimes du_k - \frac{|\nabla u_k|^2}{2} g_0 + S_{g_0}\right) \geq e^{-2u_k} f_k.$$

We now construct a sequence of smooth concave $F_k : \mathcal{S} \to \mathbb{R}$ such that F_k converges to F uniformly in compacts of $\mathcal{S}$ and F_k is uniformly elliptic with ellipticity constants $\frac{\lambda_0}{2}$ and $2\Lambda_0$. We may assume

$$F_k(\lambda) \geq F(\lambda), \quad \forall |\lambda| \leq \sup_{B_1} \left| \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 \right| + 1.$$

By the symmetry of F_k, $\frac{\partial F_k(1,\cdots,1)}{\partial \lambda_i} = \frac{\partial F_k(1,\cdots,1)}{\partial \lambda_j}$ for all i, j. Set $\frac{\partial F_k(1,\cdots,1)}{\partial \lambda_j} = A$. Let $R_0(x)$ be the scalar curvature of g_0, and let $\tilde{u}$ be the solution of $\Delta \tilde{u}_k = \frac{f_k e^{-2u_k}}{A} - R_0 + n - \frac{F_k(1,\cdots,1)}{A}$ in B_1 with $\tilde{u}_k = u_k$ on ∂B_1.

We consider the following Dirichlet problem

$$\begin{aligned} F_k\left(\nabla^2 v_k + dv_k \otimes dv_k - \frac{|\nabla v_k|^2}{2} g_0 + S_{g_0}\right) &= f_k e^{-2u_k}, \qquad \text{in } B_1, \\ v_k &= u_k, \qquad \text{on } \partial B_1. \end{aligned} \tag{8.61}$$

By the concavity of F_k,

$$F_k(\lambda) \leq A\sigma_1(\lambda) - nA + F_k(1, \cdots, 1).$$

We have

$$\begin{aligned} A\left(\Delta v_k + R_0 - n + \frac{F_k(1,\cdots,1)}{A}\right) &\geq A\left(\Delta v_k - \frac{n-2}{2}|\nabla v_k|^2 + R_0 - n + \frac{F_k(1,\cdots,1)}{A}\right) \\ &\geq f_k e^{-2u_k} \\ &\geq F_k\left(\nabla^2 v_k + dv_k \otimes dv_k - \frac{|\nabla v_k|^2}{2} g_0 + S_{g_0}\right). \end{aligned}$$

In turn, we have $v_k \leq \tilde{u}_k$ in B_1. On the other hand, we have

$$\begin{aligned} F_k\left(\nabla^2 u_k + du_k \otimes du_k - \frac{|\nabla u_k|^2}{2} g_0 + S_{g_0}\right) &\geq F\left(\nabla^2 u_k + du_k \otimes du_k - \frac{|\nabla u_k|^2}{2} g_0 + S_{g_0}\right) \\ &= F_k\left(\nabla^2 v_k + dv_k \otimes dv_k - \frac{|\nabla v_k|^2}{2} g_0 + S_{g_0}\right). \end{aligned}$$

This gives $v_k \geq u_k$ in B_1. From this, we obtain a C^0 bound of v_k and a bound of $|\nabla v_k|$ at the boundary ∂B_1. Using the same proof of Proposition 8.1, we can obtain a bound of $|\nabla v_k|$ on $\bar{B}_1$ (simply let $H(x) = |\nabla v_k|^2$ and estimate at the maximum point if it is not on the boundary). At this end, we obtain

$$\|v_k\|_{C^1(\bar{B})} \leq C, \tag{8.62}$$

for some constant C depending only on $\|f\|_{C^1(\bar{B})}, \|u\|_{C^1(\bar{B})}, \lambda, \Lambda$ and independent of k.

The standard barrier construction $\omega_\pm$ similar to the one in Step 3 in Chapter 9 of [**23**] (page 91), with the modified operator $\tilde{F}(\omega_\pm) = F_k(\nabla^2 \omega_\pm + dv_k \otimes dv_k - \frac{|\nabla v_k|^2}{2} g_0 + S_{g_0})$, will give a C^2 bound near boundary. The global C^2 estimate follows easily along the lines of proof in Proposition 3.1 in [**71**] (see also proof of Lemma 3 in [**64**]). Higher regularity estimates follow from the Krylov Theorem [**88**]. We note that global C^2 and higher regularity bounds of v_k on $\bar{B}$ may depend on higher smoothness assumptions on u_k and f_k. But by Theorem 10.4 and (8.62), we have the following the interior $C^{2,\alpha}$ estimates of v_k,

$$\|v_k\|_{C^{2,\alpha}(\Omega)} \leq C,$$

for all compact subset Ω in B, where C depending only on $dist(\Omega, \partial B)$, $\|u_k\|_{C^{2,\alpha}(\bar{B}_1)}$ and $\|f_k\|_{C^1(\bar{B}_1)}$. In any case, we can establish the existence of the Dirichlet problem (8.61) by using the method

of continuity for equation

$$F_t\left(\nabla^2 v_k + dv_k \otimes dv_k - \frac{|\nabla v_k|^2}{2} g_0 + S_{g_0}\right) = t f_k e^{-2u_k} + (1-t) f_k^*,$$

where $F_t(\lambda) = tF_k(\lambda) + (1-t)\sigma_1(\lambda)$ and $f_k^* = \sigma_1\left(\nabla^2 u_k + du_k \otimes du_k - \frac{|\nabla u_k|^2}{2} g_0 + S_{g_0}\right)$. Now by Proposition 8.1, we have for any $\rho \in C_0^2(B_1)$ with $0 \le \rho \le 1$, there exists C independent of k and u_k such that

(8.63)

$$\max_{B_1}\{|\rho(x)\nabla v_k|^2(x)\} \le C(1 + (\max_{B_1}\{|\rho(x) f_k(x)||\nabla u_k(x)|\} + \max_{B_1}\{\rho(x)|\nabla f_k(x)|\})e^{-2u_k(x)}).$$

By (8.62) and Theorem 10.4, $v_k \to v_0$ (after passing a subsequence) in $C^{2,\alpha}(B_1)$, v_k converges to $v_0 = u$ by the uniqueness. Therefore, there is a constant $C > 0$ depending only on λ_0, Λ_0, and the geometry of B_1 such that

$$\max_{B_{\frac{1}{2}}} |\nabla u|^2(x) \le C(1 + \|f\|_{C^1(B_1)} e^{-2\inf_{B_1} u(x)}).$$

This finishes the proof of Theorem 8.1. ■

As for Corollary 8.1, the following is a direct consequence of Theorem 8.1.

Corollary 8.4. *Let B_1 be a unit disk in a Riemannian manifold (M, g_0) and $p < n/2$. There exists a small constant $\varepsilon_0 > 0$ depending only on (B_1, g_0) such that for any sequence of solutions u_i of (10.103) in B_1 with*

$$\int_{B_1} e^{-nu_i} dvol(g_0) \le \varepsilon_0,$$

either

(1) *there is a subsequence u_{i_l} uniformly converging to $+\infty$ in any compact subset of B_1, or*
(2) *there is a subsequence u_{i_l} converging strongly in $C^{2,\alpha}_{loc}(B_1)$.*

Notes

The equation we treat in this chapter is a fully nonlinear version of the Yamabe problem. We refer to the works of Trudinger [**123**], Aubin [**11**] and Schoen [**111**] on the Yamabe problem. Equation (8.1) was introduced by Viacolovsky in [**126**] for $2 \le k \le n$, $l = 0$. When $l = 0$, these local estimates were proved in [**71**]. For general $l < k \le n$, the estimates were obtained in [**66**]. **Claim** (8.14) in [**71**] was renamed as H_α condition in [**91**], where it was used to get local gradient estimates for conformal invariant equations in a general form. It is obvious in [**71**] that local estimates follows from **Claim** (8.14) for general conformally invariant equations. In [**91**], it was proved that if $F(g^{-1}S_g)$ satisfies **Claim** (8.14), $F(tg^{-1}S_g + (1-t)R_g g)$ also satisfies **Claim** (8.14 for $0 \le t \le 1$. This is a useful fact in a deformation process.

We note that local estimates are a special feature of conformally invariant equations (which is generally not true for elliptic fully nonlinear equations). The negative sign in front of $|\nabla u|^2$ in equation (8.1) plays an important role. The equation is similar to the Monge-Ampère type

equation arising from reflector antenna, local second derivative estimates were proved for reflector antenna equation in [**130**] for $n = 2$ and in [**75**] for general dimensions.

CHAPTER 9

Method of moving planes and conformal equations

The main theme of this chapter is the application of the method of moving planes to conformally invariant fully nonlinear elliptic equations. We want to investigate the following conformally invariant equation:

$$f(\lambda(S_{\hat{g}})) = 1. \tag{9.64}$$

where $\hat{g} \in [g]$, $S_{\hat{g}}$ is the Schouten tensor of $\hat{g}$, $\lambda(S_{\hat{g}})$ is the set of the eigenvalues of $S_{\hat{g}}$ with respect to $\hat{g}$, and f is a certain function on symmetric matrices we will specify. If we write $\hat{g} = u^{\frac{4}{n-2}} g$ for some positive smooth function u, the Schouten tensor $S_{\hat{g}}$ can be computed as

$$S_{\hat{g}} = -\frac{2}{n-2} u^{-1} \nabla_g^2 u + \frac{2n}{(n-2)^2} u^{-2} \nabla_g u \otimes \nabla_g u - \frac{2}{(n-2)^2} u^{-2} |\nabla_g u|^2 g + S_g. \tag{9.65}$$

Equation (9.64) is indeed a second order nonlinear differential equation on u.

We now specify conditions on f so that (9.64) is elliptic. Let $\mathcal{G}$ be an open symmetric convex cone in $\mathbf{R}^n$, that is, for $\lambda \in \mathcal{G}$ and any permutation σ, $\sigma \cdot \lambda = (\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(n)}) \in \mathcal{G}$. It is clear that $(1, 1, \cdots, 1) \in \Gamma$. Set $\tilde{\mathcal{G}} = \{S \mid S$ is a symmetric matrix whose eigenvalues $(\lambda_1, \cdots, \lambda_n) \in \mathcal{G}\}$. We assume condition (13.11). Since the regularity of f is not an issue here, we assume that f is a smooth function defined in $\mathcal{G} \subseteq \mathcal{G}_1^+$, and satisfies condition (13.12). Condition (13.12) implies that f is elliptic in $\tilde{\mathcal{G}}$. A metric $\hat{g}$ is called *admissible* if $\hat{g}^{-1} S_{\hat{g}} \in \tilde{\mathcal{G}}$ for every point in M. This is equivalent to say that $\lambda(S_{\hat{g}}) \in \mathcal{G}$ for every point in M. We further assume a concavity condition (13.13) on f. Since we are concerned with equation (9.64), it is necessary that there is $\gamma \in \mathcal{G}$ such that $f(\gamma) = 1$. The symmetry and the concavity of f imply $f(t, \cdots, t) \geq 1$ for some $t > 0$. Therefore, we assume condition (13.15) on f.

Our first result is concerned with a Harnack type inequality.

THEOREM 9.1. *Suppose that f satisfies (13.12), (13.13) and (13.15). Then there exists a constant $C > 0$ such that for any admissible solution $u^{\frac{4}{n-2}} |dx|^2$ of (9.64) in a open ball B_{3R}, we have*

$$\max_{B_R} u(x) \cdot \min_{B_{2R}} u(x) \leq \frac{C}{R^{n-2}}. \tag{9.66}$$

As an application, the following global regularity and existence for equation (9.65) on a general compact locally conformally flat manifold (M, g) will be proved via fundamental work of Schoen-Yau on developing maps in [**115**]. Here, we need an additional condition (13.18). We note that (13.18) implies (13.15).

THEOREM 9.2. *Let (M, g) be an n-dimensional smooth compact locally conformally flat manifold with g admissible. Suppose that f satisfies (13.12), (13.13) and (13.18), and (M, g) is not conformally diffeomorphic to the standard n-sphere. Then there exists a positive constant $C > 0$, such that*

$$\|u\|_{C^3} + \|u^{-1}\|_{C^3} \leq C. \tag{9.67}$$

Furthermore, there is a smooth admissible solution $u^{\frac{4}{n-2}} g$ satisfying equation (9.64).

Theorem 9.1 will be proved by contradiction. Before going to the proof, we want to give a sketch of our idea first. Suppose that the inequality does not hold. Then there exists a sequence of blowup solutions for equation (9.64). We then rescale the solutions. The main step is to give C^1 estimates for these rescaled solutions. Actually, the C^1-norm of the rescaled solution will be proved to be uniformly small, and then the C^2 estimates or higher-order derivatives follows by the concave assumption accordingly. Therefore, the rescaled solutions converges to a constant in $C^{2,\alpha}$ and that will yield a contradiction to assumptions (13.12) and (13.15).

Obviously, the crucial step is the C^1 estimate of those rescaled solutions. Here, as in section 2, the method of moving planes will be employed to obtain a local gradient estimates. As in previous works, we first extend our rescaled solutions to the whole space $\mathbf{R}^n$, and obtain a viscosity super-solution. Then, we apply the Kelvin transformation twice on those extended super-solutions. Finally the local gradient estimates follows from the application of the method of moving planes.

It seems a new idea to obtain the local gradient estimates via the method of moving planes for the fully nonlinear elliptic equation. For geometric fully nonlinear elliptic equation with the concave assumption, the local gradient estimate is generally the crucial step to obtain the a priori bound for solutions. Here, our proof relies on the conformal invariance of the equation. This leads us to suspect that for conformally invariant fully nonlinear elliptic equation, the concave assumption alone should be enough for the a priori bound. This is partially confirmed in our proof of Theorem 9.1 here. We shall study this problem for general manifolds later.

Since we use of Kelvin transformations repeatedly in our proof, we shall keep our notations as clean as possible.

Suppose u is a C^2 function. Recall that the Schouten tensor $S(u)$ related to the metric $u^{\frac{4}{n-2}}|dx|^2$ is the matrix whose (i, j)-th component is defined by

$$S_{ij}(x) = u^{-\frac{4}{n-2}}(-\frac{2}{n-2}u^{-1}u_{x_ix_j} + \frac{2n}{(n-2)^2}u^{-2}u_iu_j - \frac{2}{(n-2)^2}u^{-2}|\bigtriangledown u|^2\delta_{ij}).$$

Let $\lambda(S(u))(x) = (\lambda_1, \ldots, \lambda_n)$ denote the eigenvalues of $(S_{ij}(x))$. Assume that u satisfies

$$\begin{cases} f(\lambda(S(u)))(x) = 1 \\ \lambda(S(u))(x) \in \mathcal{G} \quad \text{for} \quad x \in B_{3R}(0), \end{cases} \tag{9.68}$$

where $B_r(p)$ is the open ball with center p and radicals $r > 0$.

Proof of Theorem 9.1. By scaling invariance of the equation, we may assume $R = 1$. The inequality (9.66) will be proved by contradiction. Suppose it does not hold. Then there exists a

sequence of solutions u_i such that

$$\max_{B_1} u_i \cdot \min_{B_2} u_i \geq i. \tag{9.69}$$

Let

$$M_i = \max_{\bar{B}_1} u_i = u_i(\bar{x}_i),$$

and $x_i \in B_1$ with $\bar{B}(x_i, r_i) \subset \bar{B}_1$ and $|x_i - \bar{x}_i| = r_i$, where $r_i = M_i^{-\frac{2}{n-2}}$. By (9.69), $M_i \to +\infty$ as $i \to +\infty$. Set

$$\begin{cases} v_i(y) = M_i^{-1} u_i(x_i + M_i^{-\frac{2}{n-2}} y) \\ \bar{x}_i = x_i + M_i^{\frac{-2}{n-2}} \bar{y}_i. \end{cases} \tag{9.70}$$

Then $|\bar{y}_i| = 1$ and v_i satisfies

$$\begin{cases} f(\lambda(S(v_i))(x)) = 1, \\ \lambda(S(v_i))(x) \in \mathcal{G}, \end{cases} \quad \text{for} \quad |x| < M_i^{\frac{2}{n-2}}. \tag{9.71}$$

For simplicity, we let $L_i = M_i^{\frac{2}{n-2}}$ and choose $l_i \to +\infty$ as $i \to +\infty$ such that

$$l_i^2 < L_i, \tag{9.72}$$

and

$$l_i^{n-2} < i. \tag{9.73}$$

We extend v_i to $\mathbf{R}^n$ via the Kelvin transformation, i.e., $\tilde{v}_i(y)$ is defined by

$$\tilde{v}_i(y) = \left(\frac{l_i}{|y|}\right)^{n-2} v_i\left(\frac{l_i^2 y}{|y|^2}\right) \quad \text{for} \quad |y| \geq l_i.$$

Then $\tilde{v}_i(y)$ also satisfies (9.71) for $|y| \geq l_i$.

For $|y| = L_i$, we have $|\frac{l_i^2 y}{|y|^2}| \leq 1$ and then,

$$\tilde{v}_i(y) \leq \left(\frac{l_i}{L_i}\right)^{n-2}.$$

On the other hand, by (9.73), $v_i(y)$ satisfies,

$$v_i(y) \quad \geq \quad M_i^{-1} \inf_{B_2(0)} u_i \geq \frac{i}{M_i^2} = \frac{i}{L_i^{n-2}} > \left(\frac{l_i}{L_i}\right)^{n-2}.$$

Therefore,

$$v_i(y) > \tilde{v}_i(y) \quad \text{for} \quad |y| = L_i. \tag{9.74}$$

Set

$$\tilde{\tilde{v}}_i(y) = \begin{cases} v_i(y) & |y| \leq l_i, \\ \min(v_i(y), \tilde{v}_i(y)) & l_i \leq |y| \leq L_i, \\ \tilde{v}_i(y) & |y| \geq L_i. \end{cases} \tag{9.75}$$

By (9.74), $\tilde{\tilde{v}}_i$ is a continuous function defined in the whole space $\mathbf{R}^n$ and one may try to prove that $\tilde{\tilde{v}}_i$ is a viscosity super-solution. But, we will not pursue this fact in our proof. We will rather keep both v_i and $\tilde{v}_i(y)$ as solutions of (9.71) in the regions $\{y \mid |y| < L_i\}$ and $\{y \mid |y| \geq l_i\}$ respectively.

In what follows, we want to prove the first derivatives of v_i are uniformly small in the ball $B(\bar{y}_i, \frac{1}{2})$. In fact, we will prove for each $j = 1, 2, \ldots, n$ and $\delta > 0$,

$$\left|\frac{\partial v_i(y)}{\partial y_j}\right| \leq \delta \left\{1 + \sum_{k \neq j} \sup_{y \in B(\bar{y}_j, \frac{1}{2})} \left|\frac{\partial v_i}{\partial y_k}(y)\right|\right\} \tag{9.76}$$

for all $i \geq i_0 = i_0(\delta)$, and $|y - \bar{y}_i| \leq \frac{1}{2}$. Without loss of generality, we may assume $j = 1$, and $\bar{y}_i = (-1, 0, \ldots, 0)$. To obtain (9.76), we apply the Kelvin transformation twice on $\tilde{\tilde{v}}_i$. In the rest of the proof, in order to keep the simplicity, we will abuse some notations if there is no confusion. For any small δ, we first make the inversion T_1 with respect to the ball $B(e_\delta, 1)$ and denote the Kelvin transformation of $\tilde{\tilde{v}}_i$ by u_i, that is,

$$u_i(x) = |x - e_\delta|^{2-n} \tilde{\tilde{v}}_i \left(\frac{x - e_\delta}{|x - e_\delta|^2} + e_\delta\right), \tag{9.77}$$

where $e_\delta = (\delta^2, 0, \ldots, 0)$. From now on, u_i will be the one defined in (9.77). So $u_i(x)$ satisfies (9.71) except the small ball $\{x \mid |x - e_\delta| < 2l_i^{-1}\}$. We choose i large so that the small ball is contained in the ball $B(e_\delta, \frac{1}{2}\delta^2)$. We also denote Y and $\tilde{Y}$ as the image of $\{y \mid l_i \leq |y| \leq L_i\}$ and $\{y \mid |y| \geq L_i\}$ under the inversion T_1. Next, we denote T_2 to be the inversion $x \to \frac{x}{|x|^2}$, and $u_i^*(y)$ to be the corresponding Kelvin transform, that is,

$$u_i^*(y) = |y|^{2-n} \left|\frac{y}{|y|^2} - e_\delta\right|^{2-n} \tilde{\tilde{v}}_i \left(\frac{\frac{y}{|y|^2} - e_\delta}{|\frac{y}{|y|^2} - e_\delta|^2} + e_\delta\right). \tag{9.78}$$

We also denote Z and $\tilde{Z}$ to be the image of Y and $\tilde{Y}$ under T_2 respectively. Clearly, Z and $\tilde{Z}$ lie in a small ball with center $(\frac{1}{\delta^2}, 0, \ldots, 0)$. Note that the composition $T_2 \circ T_1(y) \to y$ in C^2 for $\bar{B}(\bar{y}_i, \frac{1}{2})$ as $\delta \to 0$. Hence

$$\frac{\partial}{\partial y_1}\left(\frac{\frac{y}{|y|^2} - e_\delta}{|\frac{y}{|y|^2} - e_\delta|^2} + e_\delta\right) = (1, 0, \cdots, 0) + O(\delta^2), \tag{9.79}$$

and

$$\frac{\partial}{\partial y_1}\left(|y|^{2-n}\left|\frac{y}{|y|^2} - e_\delta\right|^{2-n}\right) = O(\delta^2) \tag{9.80}$$

for $y \in B(\bar{y}_i, \frac{1}{2})$. Both (9.79) and (9.80) can be computed by straightforward way.

Now we fix i and δ and apply the method of moving planes to u_i^*. We use the same notations as in section 1, for any $\lambda \in \mathbf{R}$ we set $\Sigma_\lambda = \{y \mid y_1 > \lambda\}$ and y^λ to denote the reflection of $y \in \Sigma_\lambda$ with respect to the hyperplane $y_1 = \lambda$. Since $u_i^*(y)$ has a harmonic expansion at ∞, we list here for the convenience of reference (see [**50**]).

LEMMA 9.1. *For any* $\lambda < \frac{a_1}{(n-2)a_0}$, *there exists* $R = R(\lambda)$ *depending only on* $\min(1 + |a_1|, \lambda)$ *such that for* $x = (x_1, y')$ *and* $y = (y_1, y')$ *satisfying*

$$x_1 < y_1, x_1 + y_1 \leq 2\lambda, |y| \geq R$$

we have

$$u_i^*(x) < u_i^*(y).$$

Before we start the process of moving planes by using Lemma 3.1, we note that a_0, a_j and R in Lemma 9.1 could be large, because it also depends on i and δ. By our construction, $u_i^*(y)$ is a positive C^2 function except at $Z \cup \tilde{Z}$. But $u_i^*(y)$ is a super-harmonic function in the distribution sense. Therefore, for any small neighborhood N of $Z \cup \tilde{Z}$,

$$u_i^*(y) \geq \inf_{\partial N} u_i^* \geq c_0 = c_0(i, \delta) > 0 \tag{9.81}$$

for $y \in \bar{N}$. Thus, by Lemma 3.1, λ can be chosen negatively large so that

$$u_i^*(y^\lambda) < u_i^*(y) \quad \text{for} \quad y \in \Sigma_\lambda. \tag{9.82}$$

As usual, we set

$$\lambda_0 = \sup\{\lambda \mid u_i^*(y^{\lambda'}) < u_i^*(y) \ \text{ for } y \in \Sigma_{\lambda'} \text{ and } \lambda' < \lambda\}.$$

We claim if δ is small enough, then

$$\lambda_0 \geq \min\left(-\frac{1}{4}, \frac{a_1}{(n-2)a_0}\right).$$

Clearly, by the continuity, we have

$$w_{\lambda_0}(y) := u_i^*(y) - u_i^*(y^{\lambda_0}) \geq 0 \quad \text{for} \quad y \in \Sigma_{\lambda_0}. \tag{9.83}$$

We claim

$$w_{\lambda_0}(y) > 0 \quad \text{for} \quad y \in \Sigma_{\lambda_0}. \tag{9.84}$$

Recall that $w_{\lambda_0}(y)$ is continuous in $\bar{\Sigma}_{\lambda_0}$ and is C^2 in $\Sigma_{\lambda_0} \backslash Z$. Now suppose $y_0 \in \Sigma_{\lambda_0}$ such that

$$w_{\lambda_0}(y_0) = 0. \tag{9.85}$$

If $y_0 \notin Z \cup \tilde{Z}$, by the strong maximum principle $w_{\lambda_0}(y) \equiv 0$ for $y \notin Z \cup \tilde{Z}$. Let $v_i^*(y)$ denote the double Kelvin transformation of $v_i(y)$ through the conformal mapping $T_2 \circ T_1$. Note that

$$v_i^*(y) = u_i^*(y) \quad \text{for} \quad y \in \mathbf{R}^n \backslash Z \cup \tilde{Z},$$

where $\mathbf{R}^n \backslash \tilde{Z}$ is *connected*. Since $w_{\lambda_0}(y) \equiv 0$ for $y \notin Z \cup \tilde{Z}$, by the unique continuation, we have

$$v_i^*(y^{\lambda_0}) = v_i^*(y) \quad \text{for} \quad y \in \Sigma_{\lambda_0} \backslash \tilde{Z}. \tag{9.86}$$

For $y \in Z$, by (9.86) and (9.83),

$$v_i^*(y^{\lambda_0}) = v_i^*(y) \geq u_i^*(y) \geq u_i^*(y^{\lambda_0}) = v_i^*(y^{\lambda_0}). \tag{9.87}$$

Thus, $v_i^*(y) = u_i^*(y)$ for $y \in Z$, which implies

$$v_i(y) \leq \tilde{v}_i(y) \quad \text{for} \quad l_i \leq |y| \leq L_i. \tag{9.88}$$

By (9.74), this is a contradiction. Thus, $y_0 \in Z \cup \tilde{Z}$.

If $y_0 \in Z$ and $v_i(y_0) \le \tilde{v}_i(y_0)$, then $v_i(y_0) = v_i(y_0^{\lambda_0})$ and by (9.83), $v_i^*(y) \ge u_i^*(y) \ge u_i^*(y^{\lambda_0}) = v_i^*(y^{\lambda_0})$ for $y \in \Sigma_{\lambda_0} \backslash \tilde{Z}$. Thus, the strong maximum principle again yields

$$v_i^*(y) = v_i^*(y^{\lambda_0}) \quad \text{for} \quad y \in \Sigma_{\lambda_0} \backslash \tilde{Z}.$$

And it is reduced to the previous case. Thus, $v_i(y_0) > \tilde{v}_i(y_0)$. Set $\tilde{v}_i^*(y)$ be the corresponding double Kelvin transformation of $\tilde{v}_i$. Clearly, $\tilde{v}_i^*(y)$ is defined only on $Z \cup \tilde{Z}$. By (9.83), $\tilde{v}_i^*(y) \ge u_i^*(y^{\lambda_0})$ for $y \in \tilde{Z}$ and the equality holds at y_0, which implies

$$\tilde{v}_i^*(y) = u_i^*(y^{\lambda_0}) \quad \text{in} \quad \tilde{Z}. \tag{9.89}$$

Therefore

$$\tilde{v}_i(y) \le v_i(y) \quad \text{for} \quad l_i \le |y| \le L_i.$$

But $\tilde{v}_i(y) = v_i(y)$ for $|y| = l_i$. Hence (9.89) yields $u_i^*(y) = u_i^*(y^{\lambda_0})$ for $y \in \partial(\tilde{Z} \cup Z)$, which is reduced to the previous case. Therefore $y_0 \notin Z$. But $y_0 \in \tilde{Z}$ also leads to (9.89) by the strong maximum principle, which in turn yields a contradiction again. Hence the claim (9.84) is proved.

Once (9.84) is established, it is easy to see $\lambda_0 \ge \min(-\frac{1}{4}, \frac{a_1}{(n-2)a_0})$ follows from Lemma 9.1 by the standard argument of the method of moving planes. We omit the details here.

By the Hopf boundary lemma, we have

$$\frac{\partial}{\partial y_1} u_i^*(y) \ge 0 \quad \text{for} \quad y_1 \le \min(-\frac{1}{4}, \frac{a_1}{(n-2)a_0}).$$

We want to prove $\frac{\partial}{\partial y_1} u_i^*(y) \ge 0$ for $y_1 \le -\frac{1}{4}$. If not, then there exists $y_0 = (y_{0,1}, y_0')$ such that $y_{0,1} \le -\frac{1}{4}$ and $\frac{\partial}{\partial y_1} u_i^*(y_0) = 0$. Then we do the Kelvin transformation u_i^{**} as,

$$u_i^{**}(y) = \left(\frac{r_0}{|y|}\right)^{n-2} v\left(\frac{r_0^2 y}{|y|^2} + y_0\right), \tag{9.90}$$

where $r_0 = \frac{1}{2}|y_0|$. Obviously, the singular set of u_i^{**} is in the half-space $\{y \mid y_1 > 0\}$. Then we can apply the method of moving planes to show

$$u_i^{**}(y^\lambda) < u_i^{**}(y) \quad \text{for } y \in \Sigma_\lambda \text{ and } \lambda < 0, \tag{9.91}$$

by Lemma 3.1 and by the fact $\frac{\partial u_i^*}{\partial y_1}(y_0) = 0$. The same argument as the proof of (9.84) yields that (9.91) holds for $\lambda = 0$ too. This implies

$$u_i^*(y^\lambda) < u_i^*(y) \quad \text{for } y \in \Sigma_\lambda \text{ and } \lambda = y_{0,1}.$$

But it yields a contradiction to $\frac{\partial}{\partial y_1} u_i^*(y_0) = 0$. Hence $\frac{\partial}{\partial y_1} u_i^*(y) > 0$ for $y_1 \le -\frac{1}{4}$.

By the expression of (9.78), using (9.79) and (9.80), we then have

$$-\frac{\partial}{\partial y_1} \tilde{\tilde{v}}_i(y) \le O(\delta^2) \tilde{\tilde{v}}_i(y) + O(\delta^2) \sum_{k=2}^{n} \left| \frac{\partial}{\partial y_k} \tilde{\tilde{v}}_i \right| \tag{9.92}$$

for $|y-\bar{y}_i| \le \frac{1}{2}$. We can repeat the process by taking $e_\delta = (-\delta^2, 0, \dots, 0)$. In this case, u_i^* has singularity near $(-\frac{1}{\delta^2}, 0, \dots, 0)$. So, we can move the plane from the right-hand side and obtain the following inequality,

$$\frac{\partial}{\partial y_1}\tilde{\tilde{v}}_i(y) \le O(\delta^2)\tilde{\tilde{v}}_i(y) + O(\delta^2)\sum_{k=2}^{n}\left|\frac{\partial}{\partial y_k}\tilde{\tilde{v}}_i\right| \tag{9.93}$$

for $|y-\bar{y}_i| \le \frac{1}{2}$. Note that $v_i(\bar{y}_i) = \max_{|y|\le 1} v_i(y) = 1$. Since u_i^* is increasing in y_1, we obtain

$$v_i(y) \le 2 \quad \text{for} \quad |y-\bar{y}_i| \le \frac{1}{2}. \tag{9.94}$$

Thus, together with (9.92) and (9.93), (9.94) yields

$$\left|\frac{\partial}{\partial y_1}v_i(y)\right| \le O(\delta^2)\left(1 + \sum_{k=2}^{n}\frac{\partial}{\partial y_k}v_i(y)\right)$$

for $|y-\bar{y}_i| \le \frac{1}{2}$. Therefore (9.76) is proved.

After (9.76) is established, we have $v_i(y)$ uniformly converges to the constant 1 in C^1 for $|y-\bar{y}_i| \le \frac{1}{2}$. This gives $\sigma_1(S(v_i))$ convergent weakly to 0 in $|y-\bar{y}_i| \le \frac{1}{2}$. On the other hand, by (13.17) in Lemma 13.6, $\sigma_1(S(v_i)) \ge C > 0$ in $|y-\bar{y}_i| \le \frac{1}{2}$ as $f(S(v_i)) = 1$. This yields a contradiction. The proof of Theorem 9.1 is complete.

We note that we only used (13.17) in our proof, not the full concavity condition (13.13). Though (13.13) implies (13.17) by Lemma 13.6. Q.E.D.

Now we establish the global gradient estimate of $\log u$ via the method of moving planes. It is well known that once gradient estimates are available, C^2 estimates of $\log u$ will follow easily. Then higher-order derivatives follow readily the Krylov-Evans theory.

PROPOSITION 9.1. *Let (M, g) be an n-dimensional smooth compact locally conformally flat manifold. Suppose that f satisfies (13.12), (13.13) and (13.18), and (M, g) is not conformally diffeomorphic to the standard n-sphere. Then there exists a positive constant $C > 0$, such that*

$$\max_M u \le C, \quad \|\nabla \log u\|_{L^\infty} + \|\nabla^2 \log u\|_{L^\infty} \le C. \tag{9.95}$$

Theorem 9.2 is a consequence of the proposition.

Proof of Theorem 9.2. First we prove the C^2 bound of the solutions. We by Proposition 9.1 we only need o prove u has a positive lower bound. It is sufficient to prove $\max_M u$ has a positive lower bound. We now use an observation from Viaclovsky [**127**]. We would like to note that this is the only place where the admissible condition of S_g is used. At any maximum point x_0 of u_k, $u_k^{-\frac{4}{n-2}} S_{\hat{g}}(x_0) \ge u_k^{-\frac{4}{n-2}} S_g(x_0)$. Therefore,

$$1 = f(u_k^{-\frac{4}{n-2}}(x_0) g^{-1}(x_0) S_{\hat{g}}(x_0)) \ge f(u_k^{-\frac{4}{n-2}}(x_0) g^{-1}(x_0) S_g(x_0)).$$

Since $g^{-1}\cdot S_g(x_0)$ is admissible, and $K = \{g^{-1}\cdot S_g(x) | x \in M\}$ is compact, by (13.19), $u_i^{-\frac{4}{n-2}}(x_0) \le C_0$ for some constant C_0. Therefore, the C^0 and C^1 estimates are proved. By Lemma 8.3, we have C^2 estimates. Then it follows from the second condition in (13.18) that f is uniformly

elliptic. The higher-derivatives follow from the Krylov-Evans Theorem and standard elliptic theory. So, the a priori estimates (9.67) is proved for the case when $\Omega \neq \mathbb{S}^n$.

The existence of solutions can be obtained by using the degree theory following the argument of Li-Li in [**91**]. We define a deformation

$$f_t(\lambda) = \begin{cases} f((1-t)\lambda + t\sigma_1(\lambda)e), & \text{for } t \in [0,1], \\ (2-t)f(\sigma_1(\lambda)e) + \frac{t-1}{nt_0}\sigma_1(\lambda), & \text{for } t \in [1,2] \end{cases}$$

with the corresponding cone

$$\mathcal{G}_t = \begin{cases} \{\lambda \in \Gamma_1^+ \mid (1-t)\lambda + t\sigma_1(\lambda)e \in \mathcal{G}\}, & \text{for } t \in [0,1], \\ \Gamma_1^+, & \text{for } t \in [1,2], \end{cases}$$

where $e = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ in $\mathcal{G}$. Obviously, f_t in the deformation satisfies the assumptions of Theorem 1.3 and $f_t(t_0, \ldots, t_0) = 1$, where t_0 as in (13.15). By a priori estimates (9.67), the C^3-norms of solutions are uniformly bounded. Therefore, the degree remains the same during the deformation. Since the degree for the Yamabe problem (i.e. for f_2) is -1 (see [**113**]), the degree for our equation is -1. The existence of solutions follows. ■

Proof of Proposition 9.1. We should first use the theory of Schoen-Yau in [**115**] to set up the situation where the method of moving planes can work. Let $(\tilde{M}, \tilde{g})$ be the universal cover of M with $\tau : \tilde{M} \to M$ be a covering and $\tilde{g} = \tau^*(g)$ is the pull-back metric of g. By applying the theory of Schoen-Yau on locally conformally flat manifold, there exists a developing map $\Phi : (\tilde{M}, \tilde{g}) \to (\mathbb{S}^n, \sigma)$ where σ is the standard metric on $\mathbb{S}^n$. The map Φ is conformal and one to one. Let

$$\Omega = \Phi(\tilde{M}). \tag{9.96}$$

Then Ω is an open set of $\mathbb{S}^n$. In our case, the scalar curvature of g is positive. Then Schoen-Yau's Theorem tells us that the Hausdorff-dimension of $\partial\Omega$ is at most $\frac{n-2}{2}$.

If $\Omega = \mathbb{S}^n$, then M has an unique conformal structure, and solution always exists, which can be derived from the solutions on $\mathbb{S}^n$. Hence we consider $\partial\Omega$ is not empty. Now fix a point $p \in M$ and choose $\tilde{p} = \tau^{-1}(p)$ such that $\text{dist}(\tilde{p}, \partial\Omega) \geq \delta_0 > 0$. By composing a conformal transformation on $\mathbb{S}^n$ and identifying $\mathbf{R}^n = \mathbb{S}^n \backslash \{\text{North pole}\}$ through the stereographic projection, we may assume $\tilde{p} = (-1, 0, \ldots, 0)$ and $\partial\Omega \subset \{x \mid |x| \geq \frac{1}{\delta}\}$ for some $\delta > 0$. For the simplicity, we assume $\infty \notin \partial\Omega$. We still denote the conformal map: $(\tilde{M}, \tilde{g}) \to (\mathbf{R}^n, |dx|^2)$ by Φ. Set $v(x)$ to be the conformal factor:

$$\Phi^*(|dx|^2) = v(\Phi^{-1}(x))^{\frac{4}{n-2}}\tilde{g}.$$

Then $\tilde{u}(x) = v(x)u(\tau\Phi^{-1}(x))$ for $x \in \Omega$ is a solution of

$$\begin{cases} f(\lambda(S(\tilde{u}))(x)) = 1 \ \text{ and } \ \lambda(S(\tilde{u}))(x) \in \mathcal{G} \ \text{ for } \ x \in \Omega, \\ \lim\limits_{x \to \partial\Omega} \tilde{u}(x) = +\infty. \end{cases} \tag{9.97}$$

Note that the boundary condition of (9.97) follows from [**115**], because M is compact. By composition with a rotation, we may assume

$$\frac{\partial \tilde{u}}{\partial x_i}(-1,0,\ldots,0) = 0 \quad \text{if} \quad i \neq 1 \tag{9.98}$$
$$\frac{\partial \tilde{u}}{\partial x_1}(-1,0,\ldots,0) > 0$$

Let u^* be the Kelvin transformation with respect to the unit ball, that is,

$$u^*(y) = |y|^{2-n}\tilde{u}\left(\frac{y}{|y|^2}\right).$$

Then $u^*(y)$ satisfies equation (9.97) in Ω^*, where Ω^* is the image of Ω under the inversion $y \to \frac{y}{|y|^2}$, and $\partial\Omega^* \subset B(0,\delta)$. Since $\infty \notin \partial\Omega$, $u^*(x)$ is C^2 at the origin and $\lim_{x\to\partial\Omega^*} u^*(x) = +\infty$. Because $u^*(x)$ has a harmonic expansion at ∞, we can employ the method of moving planes as before (by Lemma 9.1). Hence, we conclude that $u^*(y)$ is increasing in y_1 as long as $y_1 \leq -\frac{1}{2}$. Thus,

$$\frac{\partial u^*}{\partial y_1}(-1,0,0,\ldots,0) > 0,$$

which by (9.98) implies

$$|\bigtriangledown \tilde{u}(-1,0,\ldots,0)| = \frac{\partial \tilde{u}}{\partial y_1}(-1,0,\ldots,0) < (n-2)\tilde{u}(-1,0,\ldots,0). \tag{9.99}$$

By noting $\tilde{u}(x) = v(x)u(\tau \circ \Phi^{-1}(x))$, we then obtain

$$|\bigtriangledown \log u(p)| \leq c \quad \text{for} \quad p \in M. \tag{9.100}$$

Clearly, the gradient estimate (9.100) yields

$$\frac{\max_M u}{\min_M u} \leq C. \tag{9.101}$$

Together with Theorem 9.1, we get

$$\max_M u \leq C.$$

Then C^2 estimates follows Lemma 8.38. ■

Notes

The type of inequality in Theorem 9.1 was initially established by Schoen for the Yamabe problem. A different proof was given by Chen and Lin [**32**]. In the fully nonlinear setting, the inequality was first proved for $f = \sigma_k$ by Li-Li in [**91**]. The proof given in [**91**] relies on the local estimates, or the "H_α condition". Theorem 9.1 and Theorem 9.2 for general f were proved in [**64**] and [**92**] independently. The proof here is from [**64**], where the main argument follows from [**32, 33**] by employing the method of moving planes. It is clear that key ingredients of the arguments in the proof of these results are the work of Schoen-Yau [**115**] on developing maps for locally conformally flat manifolds and Alexandrov's moving plane method.

CHAPTER 10

Deformation of the smallest eigenvalue of Ricci tensor

We now switch our attention to the smallest eigenvalue of Ricci tensor. The Ricci curvature tensor of a Riemannian metric plays an important role in comparison geometry for Riemannian manifolds, in particular the lower bounds of Ricci curvature. We are interested in conformal deformations of the smallest eigenvalue of the Ricci tensor. Let (M^n, g_0) be an n-dimensional compact Riemannian manifold and $[g_0]$ its conformal class. And let Ric_g and R_g be the Ricci curvature tenser and the scalar tensor of a metric g respectively. Define $\min Ric_g(x)$ the smallest eigenvalue of $g^{-1} \cdot Ric_g$ at $x \in M$. Our problem is to find a conformal metric $g = e^{-2u} g_0$ such that

$$\min Ric_g(x) = constant. \tag{10.102}$$

The problem is equivalent to solving an interesting fully nonlinear uniform elliptic equation. First we recall that the Schouten tensor of the metric g is defined by

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} g\right).$$

Let $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n$. Assume that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. For an integer $1 \leq p \leq n-1$, define a function $G_p : \mathbb{R}^n \to \mathbb{R}$ by

$$G_p(\Lambda) = (n-p) \sum_{i \leq p} \lambda_i + p \sum_{i > p} \lambda_i.$$

For a symmetric matrix A, $G_p(A) = G_p(\Lambda)$, where Λ is the set of eigenvalues of A. It is easy to check $\min Ric = G_1(g^{-1} \cdot S_g)$.

We may also ask if there is a conformal metric with a constant $W_p(g) := G_p(g^{-1} \cdot S_g)$. $W_p(g)$ is also an interesting geometric object, which will be called *p-Weitzenböck curvature*, for it arises from the Weitzenböck formula for p-forms in a locally conformally flat manifold as we have seen in Chapter 7. We will consider the following general equation

$$W_p(g)(x) = f(x), \quad \forall x \in M. \tag{10.103}$$

We first treat the case when the background metric has negative curvature, i.e., $W_p(g_0) < 0$.

THEOREM 10.1. *Let (M, g_0) be a compact Riemannian manifold and $1 \leq p \leq n$. Suppose that $W_p(g_0)(x) < 0$ for any $x \in M$, then there is a unique $C^{2,\alpha}$ metric $g \in [g_0]$ for some $\alpha > 0$ such that $W_p(g)(x) = -1, \quad \forall x \in M$.*

A geometric consequence of Theorem 10.1 is the existence of an extremal metric in the conformal class with minimal volume. Although (10.103) has no variational structure in general,

a solution of (10.103) does achieve the minimum of the minimal volume in a conformal class in this case. The following is a simple consequence, see section 4 for other related results.

COROLLARY 10.1. *Suppose that* $\min Ric_{g_0}(x) < 0$ *for any* $x \in M$. *Then there is a unique conformal metric* $g^* \in [g_0]$ *such that* $vol(g^*) = \min vol(g)$, *where minimum is taken over all* $g \in [g_0]$ *with* $\min Ric_g(x) \geq -1$. *The extremal metric* g^* *is characterized by a unique solution to equation* $\min Ric_{g^*}(x) = -1, \forall x \in M$.

We now turn to the case of the positive Ricci curvature.

THEOREM 10.2. *Let* (M, g_0) *be a compact Riemannian manifold with* $Ric_{g_0} > 0$. *Then there is a conformal metric* $g \in [g_0], g \in C^{2,\alpha}(M)$ *for some* $\alpha > 0$ *such that* $\min Ric_g(x) = n-1$ *for all* $x \in M$.

The positivity of p-Weitzenböck curvature for $1 < p \leq n/2$ plays an important role in the investigation of the topological structure of locally conformally flat manifolds in [**64**].

THEOREM 10.3. *Let* (M, g_0) *be an* n*-dimensional smooth compact locally conformally flat manifold with* $W_p(g_0) > 0$ *and* $p \leq n/2$. *If* (M, g_0) *is not conformally equivalent to the standard* n*-sphere, then there exists* $g \in [g_0], g \in C^{2,\alpha}(M)$ *for some* $\alpha > 0$ *such that*

$$W_p(g)(x) = 1, \quad \forall x \in M. \tag{10.104}$$

Furthermore, the solution space is compact. That is, there is positive constant $C > 0$, *such that*

$$\|u\|_{C^{2,\alpha}} \leq C \tag{10.105}$$

for any $C^{2,\alpha}$ *solution* $g = e^{-2u} g_0$ *of (10.104).*

1. Fully nonlinear uniformly elliptic equations and Caffarelli's estimates

In this section, we will deduce our problem to fully nonlinear uniformly elliptic elliptic equations. We will make use of Caffarelli's fundamental $W^{2,p}$ and $C^{2,\alpha}$ estimates in [**22**] (see also Safonov's work on $C^{2,\alpha}$ estimates in [**109**]). Some of these results have been subsequently generalized to certain type of equations of form $F(\nabla^2 u, \nabla u, x) = f(x)$ by L. Wang in [**129**] as well as for the case of parabolic equations. As we will see that equation (10.103) involves ∇u in a delicate way, we need certain appropriate a priori estimates depending only on one side bound of u (to be more explicit, $e^{-2\inf u}$). This type of local gradient estimates have been established in Chapter 8.

Let (M^n, g_0) be a compact, oriented Riemannian manifold of dimension $n > 2$. Let $[g_0]$ be the conformal class of g_0. For any $g \in [g_0]$, we denote Ric_g, S_g the Ricci tensor and the Schouten tensor of the metric g respectively. We write

$$\begin{aligned}
\mathcal{R}^s_g(x) &= \text{smallest eigenvalue of} \quad g^{-1} Ric_g(x), \\
\mathcal{R}^l_g(x) &= \text{largest eigenvalue of} \quad g^{-1} Ric_g(x), \\
\mathcal{S}^s_g(x) &= \text{smallest eigenvalue of} \quad g^{-1} S_g(x), \\
\mathcal{S}^l_g(x) &= \text{largest eigenvalue of} \quad g^{-1} S_g(x).
\end{aligned}$$

It is clear that $\mathcal{R}_g^s$ and $\mathcal{R}_g^l$ do not depend on the choice of coordinates. From the definition of the Schouten tensor

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)}\right),$$

we have the following relations

$$\begin{aligned} \mathcal{R}_g^s(x) &= (n-2)\mathcal{S}_g^s(x) + \mathrm{tr}(S_g(x)), \\ \mathcal{R}_g^l(x) &= (n-2)\mathcal{S}_g^l(x) + \mathrm{tr}(S_g(x)). \end{aligned} \tag{10.106}$$

If $g = e^{-2u}g_0$, there is a transformation formula between two Schouten tensors

$$S_g = \nabla^2 u + \nabla u \otimes \nabla u - \frac{|\nabla u|^2 g_0}{2} + S_{g_0}, \tag{10.107}$$

where all covariant derivatives are with respect to g_0. (The same convention will be used in the rest of this paper, unless it is stated otherwise). Locally, denoting S_{ij} the Schouten tensor of g_0 under the frame, we have

$$(S_g)_{ij} = u_{ij} + u_i u_j - \frac{|\nabla u|^2}{2}\delta_{ij} + S_{ij}.$$

For any symmetric matrix A, we denote $\lambda_s(A)$ and $\lambda_l(A)$ to be the smallest and largest eigenvalues of A respectively. And we denote $\sigma_1(A) = \mathrm{tr}(A)$. We obtain the equations for constant $\mathcal{R}_g^s$ and $\mathcal{R}_g^l$ respectively:

$$F_*(u) =: ((n-2)\lambda_s + \sigma_1)(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0) = e^{-2u}\mathcal{R}_g^s, \tag{10.108}$$

$$F^*(u) =: ((n-2)\lambda_l + \sigma_1)(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0) = e^{-2u}\mathcal{R}_g^l. \tag{10.109}$$

Hence F_* and F^* are uniformly elliptic with ellipticity constants 1 and $n-1$. It is clear that F_* is concave and F^* is convex. We also note that F^* and F_* are homogeneous of degree 1.

There are other similar fully nonlinear equations arising in the Weitzenböck formula for p-form on local conformally flat manifolds. Let $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n$ be the set of eigenvalues of a symmetric matrix A. For an integer $1 \le p \le n/2$, define a function $G_p : \mathbb{R}^n \to \mathbb{R}$ by

$$G_p(A) = G_p(\Lambda) = \min\{(n-p)\sum_{k\le p}\lambda_{i_k} + p\sum_{k>p}\lambda_{i_k}\},$$

where min is over all permutations of $1, 2, \cdots, n$. We define $W_p(g)$ the p-Weitzenböck curvature of g by

$$W_p(g) = G_p(g^{-1}\cdot S_g).$$

It is easy to check $\mathcal{R}_g^s = W_1(g)$. The p-Weitzenböck curvature is as much interest as the scalar curvture, at least for locally conformally flat manifold. For example, from the Weitzenböck formula one can easily show that a locally conformally flat manifold with positive p-Weitzenböck curvature has vanishing q-cohomology group for $n/2 - p \le q \le n/2 + p$ (for $p \le n/2$).

We may also ask whether we can find a conformal metric with constant W_p. The corresponding equation is

$$G_p\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0\right) = constant \cdot e^{-2u}. \tag{10.110}$$

Equation (10.110) is also uniformly elliptic. It is easy to see that (10.110) is concave for $p \leq \frac{n}{2}$. From the transformation formula (10.107), one can check that Equation (10.110) is also conformally invariant. Hence, in general there is no compactness for equation (10.110).

We also define for $p \geq n/2$ a function $G_p : \mathbb{R}^n \to \mathbb{R}$ by

$$G_p(A) = G_p(\Lambda) = \max\{(n-p)\sum_{k \leq p} \lambda_{i_k} + p \sum_{k > p} \lambda_{i_k}\},$$

where max is over all permutations of $1, 2, \cdots, n$. We still define $W_p(g) = G_p(g^{-1} \cdot S_g)$. It is clear that $\mathcal{R}_g^l = W_{n-1}(g)$. We can also consider equation (10.110) for $p \geq n/2$. For $p \geq n/2$ equation (10.110) is still uniformly elliptic, but it is convex. Since G_p is homogeneous of order 1, we may rewrite equation (10.103) as the following equivalent equation by setting $v = e^u$

$$G_p(\nabla^2 v + S_{g_0} v) = p(n-p)\frac{|\nabla v|^2}{v} + \frac{f}{v}. \tag{10.111}$$

The function G_p is only Lipschitz, when $p \neq n/2$. (Remark that when $p = n/2$, equation (10.110) is equivalent to the Yamabe equation.) One can find a sequence of smooth functions $\{F_k\}$ such that F_k uniformly converges to G_p in any compact domain of $\mathbb{R}^n$ and homogeneous 1 outside the unit ball in $\mathbb{R}^n$, i.e., $F_k(x) = |x| F(\frac{x}{|x|})$ for $|x| \geq 1$. Furthermore, for $p < n/2$ ($p > n/2$) F_k is concave (convex).

One may consider a more general class of conformal equations. Let $\mathcal{S}$ be the space of symmetric 2-tensors on M. Let $F : \mathcal{S} \to \mathbb{R}$ a real continuous function. We consider the following general equation

$$F\left(e^{-2u}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0)\right) = f(x), \tag{10.112}$$

for some function $f : M \to \mathbb{R}$. F is *uniformly elliptic* with constants λ_0 and Λ_0 if there exists two positive constants λ_0 and Λ_0 such that for any $W \in \mathcal{S}$

$$\lambda_0 \|N\| \leq F(W + N) - F(W) \leq \Lambda_0 \|N\| \quad \forall N \geq 0,$$

here by $N \geq 0$ means that N is nonnegative definite and $\|N\| = \sup_{|v|=1} |Nv|$. If F is uniformly elliptic, we call equation (10.112) a uniformly elliptic fully nonlinear equation with ellipticity constants λ_0 and Λ_0. There are many typical uniformly elliptic fully nonlinear equations. Our equation (10.110) is similar to the Pucci equation, see [**22**]. Let $\mathcal{M}^{\pm}$ be the Pucci's extremal operators, namely for two given constant $0 < \lambda_0 < \Lambda_0$ and $W \in \mathcal{S}$

$$\begin{aligned} \mathcal{M}^-(W) &= \lambda_0 \sum_{e_i > 0} e_i + \Lambda_0 \sum_{e_i < 0} e_i, \\ \mathcal{M}^+(W) &= \Lambda_0 \sum_{e_i > 0} e_i + \lambda_0 \sum_{e_i < 0} e_i, \end{aligned}$$

where $e_i = e_i(W)$ are the eigenvalues of W. One can also consider

$$\begin{aligned} \mathcal{M}^-(e^{-2u}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0)) &= 1, \\ \mathcal{M}^+(e^{-2u}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0)) &= 1. \end{aligned} \tag{10.113}$$

$\mathcal{M}^-$ is concave, while $\mathcal{M}^+$ is convex.

We now deduce $C^{2,\alpha}$ estimates for equation (10.103) from the work of Caffarelli [**22**] assuming the gradient bound and a lower bound of u. The crucial part in Caffarelli's perturbation theory for uniformly elliptic fully nonlinear equation $F(\nabla^2 u, x) = f(x)$ is the $C^{1,1}$ interior estimates for $F(\nabla^2 u, x_0) = constant$. He obtained such fundamental estimates for concave or convex operator F (note that here concavity and convexity of F can always be switched by $\tilde{F}(\Lambda) = -F(-\Lambda)$). Though Caffarelli proved these estimates in [**22**] for equations with flat metric, his arguments work under general Riemannian metrics. And the generalization of Caffarelli's estimates by L. Wang [**129**] to uniformly elliptic equations of form $F(\nabla^2 u, \nabla u, u, x)$ gives the following $C^{2,\alpha}$ estimates for equations of type (10.110).

THEOREM 10.4 ($C^{2,\alpha}$-estimates). *Suppose F is a uniformly elliptic concave operator with elliptic constants λ_0, Λ_0. Let B_1 be a unit disk in a compact Riemannian manifold M and $f, h \in C^1(B_1)$. Suppose $g = e^{-2u} g_0$ with $|\nabla u|_{B_1} \leq A$ is a solution of equation*

$$F\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0\right) = e^{-2u} f(x), \quad x \in B_1, \tag{10.114}$$

then there exist $\alpha > 0$ and $C > 0$ depending only on $\lambda_0, \Lambda_0, A, \|u\|_{C^0(B_1)}$ and g_0 such that

$$\|u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C. \tag{10.115}$$

In fact, we may directly apply Caffarelli's estimates [**22**] to obtain $C^{2,\alpha}$ estimates for equations of type (10.111). Let B_1 be a unit disk in a compact Riemannian manifold M and $f, h \in C^1(B_1)$. Suppose $g = e^{-2u} g_0 = v^{-2} g_0$ with $|\frac{\nabla v}{v}|_{B_1} \leq A$ is a solution of equation

$$F(\nabla^2 v(x) + v(x) S_{g_0}(x)) = h(x) \frac{|\nabla v(x)|^2}{v(x)} + \frac{f(x)}{v(x)}, \quad x \in B_1. \tag{10.116}$$

Since F is concave, by Theorem 6.6 in [**23**], the equation

$$F(\nabla^2 v + S_{g_0}(x_0) v) = constant,$$

has $C^{1,1}$ interior estimates for any $x_0 \in B_1$. It follows from Theorem 7.1 in [**23**] that equation (10.116) has interior $W^{2,p}$ estimate for any $n < p < \infty$ since $\frac{|\nabla v|}{v} \leq A$. This in turn gives $C^{1,\beta}$ a priori bound for the solution v of equation (10.116) for all $0 < \beta < 1$. Finally estimate (10.115) for $v = e^u$ follows from Theorem 8.1 in [**23**] since the right hand side of equation (10.116) is a C^β function now.

2. negative curvature case

In this section, we discuss the negative curvature case, where the geometry is rich. By [**49**] ($n = 3$) and [**94**] (general dimension $n \geq 3$), every higher dimensional manifold has a metric with negative Ricci tensor. It is clear that such a metric also has negative W_p for any $1 \leq p \leq n-1$. Hence every higher dimensional manifold has a metric with negative W_p. The conformal deformation will yield interesting geometric information about the extremal metrics in a given conformal class.

Proof of Theorem 10.1. First we take the sequence of smooth F_k considered in the previous section. For each F_k we consider the following equation

$$F_k\left((e^{2u}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0)\right) = -1. \tag{10.117}$$

For large k, from the condition of the Theorem, we have

$$F_k(S_{g_0})(x) < 0, \quad \forall x \in M.$$

We first prove the existence of solutions to equation (10.117). Here we use the method of continuity. Let us consider the following equation

$$F_t(u) := F_k\left(e^{2u}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0)\right) + t - (1-t)F_k(S_{g_0}) = 0 \tag{10.118}$$

and define $J = \{t \to [0,1] \,|\, (10.118) \text{ has a solution for } t\}$. It is clear that $0 \in J$. First, we prove the openness of J. Let $t_0 \in J$. By the maximum principle, we know that there is only one solution u of (10.118) for $t = t_0$. Let L be its linearization. We want to show that L is invertible. By the maximum principle again, we know that the kernel of L is trivial. Note that L might be not self-adjoint. To show the invertibility of L, we need to show that the cokernel of L is also trivial. However, one can readily check that the Fredholm index of L is zero, and hence the cokernel of L is trivial. Now the openness follows from the implicit function theorem.

Then we show the closeness. Let x_0 and x_1 be the minimum and maximum of u respectively. By the maximum principle, we have

$$e^{2u(x_0)} \geq \frac{t - (1-t)F_k(S_{g_0})(x_0)}{-F_k(S_{g_0})(x_0)} \tag{10.119}$$

and

$$e^{2u(x_1)} \leq \frac{t - (1-t)F_k(S_{g_0})(x_1)}{-F_k(S_{g_0})(x_1)}. \tag{10.120}$$

Hence, we have C^0 bound of u independent of t.

By a global estimates proven in Proposition 10.1 below and Theorem 10.4, we have the closeness. Hence we have a solution u_k of (10.117) with the bound

$$\frac{1}{-\min F_k(S_{g_0})(x_0)} \leq e^{2u_k} \leq \frac{1}{-\max F_k(S_{g_0})(x_0)}. \tag{10.121}$$

In viewing of (10.121), we use again the global estimates and Theorem 10.4 to obtain a $C^{2,\alpha}$ uniform bound of u_k for some $\alpha > 0$. Hence u_k converges (by taking a subsequence) to u. It is clear that $e^{-2u} g_0$ satisfies (10.103). ■

PROPOSITION 10.1. *Let u be a solution of*

$$F(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0) = f e^{-2u} \tag{10.122}$$

with C^1 function f. Suppose that F is uniformly elliptic and is homogeneous of degree 1. Assume that u has C^0 bound. Then u has a C^1 bound and a $C^{2,\alpha}$ bound.

Proof: Since we already have full C^0 bound, the proposition can be proved using standard Pogorelov type of trick, for example, as in [**78**]. Let $v = e^u$ and consider the following equivalent form of (10.122)

$$F\left(\frac{\nabla^2 v}{v} - \frac{|\nabla v|^2}{v^2} g_0 + S_{g_0}\right) = f v^{-2}. \tag{10.123}$$

Without loss of generality, we may assume that $v \leq 1$. Set $H = e^{2\phi(v)}|\nabla v|^2$. Here ϕ will be fixed later. Let x_0 be a maximum point of H. At x_0, we have

$$\sum_k (2 v_k v_{ki} + 2\phi'(v) v_i |\nabla v|^2) = 0, \qquad \text{for any } i. \tag{10.124}$$

Without loss of generality, we may assume that $v_1 = |\nabla v|$, $v_i = 0$ for any other i and that v_{ij} is diagonal at x_0. Hence, (10.124) is equivalent to $v_{11} = -\phi'(v)|\nabla v|^2$. Set $w_{ij} = v_{ij} - \frac{|\nabla v|^2}{2v}\delta_{ij} + vS_{ij}$. The maximum principle, together with (10.124), implies

$$0 \geq \sum F^{ij}(v_{kj}v_{ki} + v_k v_{kij} + \phi'(v) v_{ij}|\nabla v|^2 + v_i v_j \phi''(v)|\nabla v|^2 + 2\phi'(v) v_i v_{ki} v_k) \tag{10.125}$$

By (10.124), it is easy to check that

$$\sum F^{ij} v_{ki} v_{kj} \geq F^{11} v_{11} v_{11} = F^{11}(\phi'(v))^2 \frac{|\nabla v|^4}{v},$$

$$\begin{aligned}
\sum F^{ij} v_k v_{kij} &= \sum F^{ij} v_k v_{ijk} + \sum F^{ij} v_k v_m R^m_{ijk} \\
&\geq \sum F^{ij} v_k (w_{ij} + \frac{|\nabla v|^2}{2v}\delta_{ij} + vS_{ij})_k - C \max v |\nabla v|^2 \\
&\geq 2 v_k (f v^{-1})_k + \sum F^{ii} v_1 (-\frac{v_1^3}{2v^2} - \phi'(v)\frac{|\nabla v|^3}{v}) - C \max v |\nabla v|^2 \\
&\geq 2 f_k v_k - 2 v^{-1}|\nabla v|^2 - (\frac{1}{2v} + \phi'(v)) \sum F^{ii} \frac{|\nabla v|^4}{v} - C \max v |\nabla v|^2,
\end{aligned}$$

$$\begin{aligned}
\phi'(v) \sum F^{ij} v_{ij} |\nabla v|^2 &= \phi'(v) \sum F^{ij} (w_{ij} + \frac{|\nabla v|^2}{2v}\delta_{ij} - vS_{ij}) |\nabla v|^2 \\
&\geq \phi'(v) f v^{-1} |\nabla v|^2 + \phi'(v) \sum F^{ii} \frac{|\nabla v|^4}{2v} - C \max v |\nabla v|^2,
\end{aligned}$$

$$\phi''(v) \sum F^{ij} v_i v_j |\nabla v|^2 = \phi''(v) F^{11} |\nabla v|^4,$$

$$2\phi'(v) \sum F^{ij} v_j v_k v_{ki} = 2\phi'(v) F^{11} v_1^2 v_{11} = -2(\phi'(v))^2 F^{11} |\nabla v|^4.$$

Here C is a positive constant depending only on the Riemannian curvature of the background metric and it varies from line to line. Therefore, we have

$$\begin{aligned} 0 \geq\ & F^{11}(\phi''(v)-(\phi'(v))^2)|\nabla v|^4+\frac{1}{2v}\sum F^{ii}(-\frac{1}{v}+\phi(v))|\nabla v|^4-C\max v|\nabla v|^2 \\ & +2f_kv_k-2v^{-1}|\nabla v|^2+\phi'(v)fv^{-1}|\nabla v|^2. \end{aligned} \tag{10.126}$$

Choose

$$\phi=-\frac{1}{2}\log t(ct-2),$$

for a large constant $c>0$ so that $c\min v>3$. One can easily check that for any $t\in[\min v,\infty)$

$$\phi''(v)-(\phi'(v))^2=\frac{1}{t^2(ct-2)^2}>0$$

and

$$-(\frac{1}{t}+\phi(t))=\frac{1}{t(ct-2)}.$$

In view of (10.126), we have

$$0\geq\frac{1}{2(\max v)^2(c\max v-2)}|\nabla v|^4F^{ii}-c(f,C)|\nabla v|^2.$$

Now we have a global bound of $|\nabla v|$, which depend only on g_0, f, $\min v$ and $\max v$. The $C^{2,\alpha}$ bound follows from Theorem 10.4. ■

Remark 10.1. *The condition $W_p(g_0)(x)<0,\forall x\in M$ in Theorem 10.1 can be weaken to $W_p(g_0)(x)\leq 0,\forall x\in M$ and $W_p(g_0)(x_0)<0$ for some $x_0\in M$. In fact, under the weaker condition, one may produce a metric $g\in[g_0]$ with the stronger condition holds. This can be done using the short time existence of the fully nonlinear flow*

$$u_t=G_p(\nabla^2u+du\otimes du-\frac{|\nabla u|^2}{2}g_0)e^{2u},\quad u|_{t=0}=0.$$

The short time existence follows from standard nonlinear parabolic theory, and the strict negativity of $W_p(g)$ (which is equal to u_t) follows from the strong maximum principle.

Remark 10.2. *It is of interest to characterize when the condition in Theorem 10.1 is true by some conformal geometric quantities. The difficulty here is the lack of variational structure for this type of equations. We note that when $p>\frac{n}{2}$, if the Yamabe constant $Y([g_0])$ of (M,g_0) is non-positive, then the condition in Theorem 10.1 is satisfied unless (M,g_0) is conformally equivalent to a Ricci flat manifold. This simple observation follows from the solution of the Yamabe problem and the fact that if the scalar curvature vanishing identically, for $p>\frac{n}{2}$, $G_p\leq 0$ and G_p vanishes identically if and only if the metric is Ricci flat.*

Remark 10.3. *It is also an interesting problem to consider the equation $W_p(g)=-1$ on a complete, non-compact manifold. The arguments in the proof of the existence of Dirichlet problem (8.61) can be extended to deal with a given boundary condition at the infinity for equation $W_p(g)=-1$ on a complete non-compact negatively curved manifold.*

As a direct consequence of Theorem 10.1, we have

COROLLARY 10.2. *If there is $\tilde{g} \in [g_0]$ with $\mathcal{R}^s_{\tilde{g}}(x) < 0$ for all $x \in M$ then there is a unique $C^{2,\alpha}$ metric $g^* \in [g_0]$ for some $\alpha > 0$ such that $\mathcal{R}^s_{g^*}(x) = -1, \forall x \in M$. Similarly, If there is $\tilde{g} \in [g_0]$ with $\mathcal{R}^l_{\tilde{g}}(x) < 0$ for all $x \in M$, then there is a unique $g_* \in [g_0]$ and $g_* \in C^{2,\alpha}(M)$ for some $\alpha > 0$ such that $\mathcal{R}^l_{g_*}(x) = -1, \forall x \in M$.*

Corollary 10.2 can be applied to consider minimal volumes in conformal classes. Set

$$\mathcal{C}^- = \{g \in [g_0] | \mathcal{R}^s_g(x) \geq -1, \forall x \in M\},$$

$$\mathcal{C}_- = \{g \in [g_0] | \mathcal{R}^l_g(x) \geq -1, \forall x \in M\},$$

$$\tilde{\mathcal{C}}^- = \{g \in [g_0] | \mathcal{R}^s_g(x) \leq -1, \forall x \in M\},$$

$$\tilde{\mathcal{C}}_- = \{g \in [g_0] | \mathcal{R}^l_g(x) \leq -1, \forall x \in M\}.$$

Define

$$\mathcal{V}^s([g_0]) = \inf_{g \in \mathcal{C}^-} vol(g), \qquad \mathcal{V}^l([g_0]) = \inf_{g \in \mathcal{C}_-} vol(g),$$

$$\mathbb{V}^s([g_0]) = \sup_{g \in \tilde{\mathcal{C}}^-} vol(g), \qquad \mathbb{V}^l([g_0]) = \sup_{g \in \tilde{\mathcal{C}}_-} vol(g),$$

where $vol(g)$ is the volume of g.

LEMMA 10.1. *Let $g_1, g, g_2 \in [g_0]$. If $\mathcal{R}^s_g(x) < 0$, and $\mathcal{R}^s_{g_1}(x) \leq \mathcal{R}^s_g(x) \leq \mathcal{R}^s_g(x), \forall x \in M$, then $vol(g_1) \leq vol(g) \leq vol(g_2)$, any one of the equalities holds if and only if the metric is equal to g. Similarly, if $\mathcal{R}^l_g(x) < 0$, and $\mathcal{R}^l_{g_1}(x) \leq \mathcal{R}^l_g(x) \leq \mathcal{R}^l_g(x), \forall x \in M$, then $vol(g_1) \leq vol(g) \leq vol(g_2)$, any one of the equalities holds if and only if the metric is equal to g.*

The Lemma is a simple consequence of the maximum principle applied to equations (10.108) or (10.109). From the lemma, we have the following relations

$$\mathcal{V}^s([g_0]) \geq \mathbb{V}^s([g_0]) \geq \mathcal{V}^l([g_0]) \geq \mathbb{V}^l([g_0]).$$

And we can show that the minimal volumes $\mathcal{V}^s([g_0])$ and $\mathcal{V}^l([g_0])$ are achieved.

COROLLARY 10.3. *Suppose that $\mathcal{R}^s_{g_0}(x) < 0$ for any $x \in M$. Then there is a unique conformal metric $g^* \in [g_0]$ such that $vol(g^*) = \mathcal{V}^s([g_0])$ with $\mathcal{R}^s_{g^*}(x) = -1, \forall x \in M$ and*

$$\mathbb{V}^s([g_0]) = \mathcal{V}^s([g_0]) \geq \mathcal{V}^l([g_0]).$$

The equality holds if and only if there is an Einstein metric in $[g_0]$. If $\mathcal{R}^l_{g_0}(x) < 0$ for any $x \in M$, then there is a unique $g^* \in [g_0]$ such that $vol(g_*) = \mathcal{V}^l([g_0])$ with $\mathcal{R}^l_{g_*}(x) = -1, \forall x \in M$. In this case, we have

$$\mathcal{V}^s([g_0]) = \mathbb{V}^s([g_0]) \geq \mathcal{V}^l([g_0]) = \mathbb{V}^l([g_0]).$$

For the study of minimal volumes in general Riemannian manifolds, we refer to [**55**] and [**17**].

3. positive curvature case

Now we consider conformal classes with metrics of positive Ricci curvature. Let $[g_0]$ be such a conformal class. After a suitable scaling, we may assume that $Ric_{g_0} \geq (n-1)g_0$. Define $[g_0]_+ = \{g \in [g_0] \,|\, Ric_g \geq (n-1)g\}$ and $V_{\max}(M, [g_0]) = \sup_{g\in[g_0]_+} vol_g(M)$. This definition is motivated by Gursky and Viaclovsky [**77**]. From the Bishop comparison, we know

$$V_{\max}(M, [g_0]) \leq vol(\mathcal{S}^n), \tag{10.127}$$

the volume of the unit sphere.

Proposition 10.2. *Let (M, g_0) be a compact Riemannian manifold with $Ric_{g_0} \geq (n-1)g_0$. If $V_{\max}(M, [g_0]) < vol(\mathcal{S}^n)$, then there is a conformal metric $g \in [g_0]_+$ with*

$$\mathcal{R}^s_g = n-1.$$

Proof. Consider the sequence of approximating function F_k as in Section 2 with a normalization condition that $F_k(1, 1, \cdot, 1) = n-1$). We first want to find a solution to the following equation

$$F_k\left(e^{2u}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0)\right) = n-1, \tag{10.128}$$

for large k. Define

$$V^k_{\max}([g_0]) = \max\{vol(g) \,|\, g \in [g_0] \text{ with } F_k(g^{-1} \cdot S_g) \geq n-1\}.$$

It is easy to check that $\lim_{k\to\infty} V^k_{\max}([g_0]) = V_{\max}([g_0])$. Hence for large k we have

$$V^k_{\max}([g_0]) < vol(\mathcal{S}^n). \tag{10.129}$$

To show the existence of solution of (10.128), we consider a deformation, which is similar to a deformation considered by Gursky and Viaclovsky in their study of σ_k-Yamabe problem.

$$\begin{aligned} &F_k\left(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + \psi(t) S_g + (1-\psi) g\right) \\ &= (n-1)(1-t)\left(\frac{1}{vol(g_0)}\int_M e^{-(n+1)u}\right)^{\frac{2}{n+1}} + (n-1)\psi(t)e^{-2u}, \end{aligned} \tag{10.130}$$

where $\psi(t) : [0, 1] \to [0, 1]$ is a C^1 function satisfying $\psi(0) = 0$ and $\psi(t) = 1$ for $t \geq 1/2$. We now prove that there is a solution of (10.130) when $t = 1$, provided that $V^k_{\max}(M, [g]) < vol(\mathcal{S}^n)$.

When $t = 0$, it is easy to check that (10.130) has a unique solution $u = 0$ and its corresponding linearization has no nontrivial kernel. Hence its Leray-Schauder degree is non-zero. If the solution space of (10.130) for any $t \in [0, 1]$ is compact, then using degree theoretic argument, we are done. Assume by contradiction that there is no compactness. Assume without loss of generality that there is a sequence of solutions $g_i = e^{-2u_i} g$ of (10.102) with $t = 1$ such that u_i does not converge in $C^{2,\alpha}$. In view of Theorem 8.1 and Corollary 8.2, we have either

(a) $\inf_M u_i \to -\infty$, or
(b) $\inf_M u_i \to +\infty$.

The latter is easy to be excluded as follows. At the minimum point of u_i, we have

$$e^{2\inf u_i} F_k(S_{g_0}) = F_k(e^{2\inf u_i} S_0) \leq (n-1),$$

which certainly implies that $\inf u_i$ is bounded from above uniformly. Hence we are left to exclude (a). Let x_i be the minimum point of u_i and assume that $x_i \to x_0$ as $i \to \infty$. Consider a scaled function

$$\tilde{u}_i = u(\exp_{x_i} \varepsilon_i x) - \log \varepsilon_i,$$

where $\varepsilon_i = \exp u_i(x_i)$. It is clear that $\tilde{u}_i \geq 0$ satisfies a similar equation on $B(0, \varepsilon_i^{-1} r_0/2) \subset \mathbb{R}^n$ with a scaled metric, where r_0 is the injectivity radius of (M, g). The set $B(0, \varepsilon_i^{-1} r_0/2)$ with the scaled metric converges to $\mathbb{R}^n$. By local estimates and local C^2 estimates in [**64**] for a more general concave case one can show that $\tilde{u}_i$ converges to an entire solution v of

$$F_k(\nabla^2 v + dv \otimes dv - \frac{1}{2}|\nabla v|^2 g_{\mathbb{R}^n}) = (n-1)e^{-2v} \tag{10.131}$$

and

$$V^k_{\max}(g) \geq \liminf_i vol(e^{-2u_i} g) \geq vol(\mathbb{R}^n, e^{-2v} g_{\mathbb{R}^n}), \tag{10.132}$$

where $g_{\mathbb{R}^n}$ is the standard Euclidean metric. By a classification result of Li-Li in [**92**], we know that $(\mathbb{R}^n, e^{-2v} g_{\mathbb{R}^n})$ is equivalent to $\mathcal{S}^n$. Hence $V^k_{\max}([g_0]) \geq vol(\mathbb{R}^n, e^{-2v} g_{\mathbb{R}^n}) = vol(\mathcal{S}^n)$. This contradicts (10.129). Therefore, we have a solution u_k of (10.130) for large k.

Now we consider the sequence $\{u_k\}$. As above, we can show first that u_k has a uniform upper bound. If u_k has a uniform lower bound, Corollary 8.2 implies that the sequence $\{u_k\}$ has a uniform $C^{2,\alpha}$ bound. And hence we have a limit u_0 which is a solution we desire. Hence to prove the Proposition, we only need to exclude the case that $\min u_k \to -\infty$. Assume that we are in this case. By a similar argument presented above, after considering a suitable rescaling we have a limit $C^{2,\alpha}$ function v_∞ satisfying

$$G_{n-1}(\nabla^2 v + dv \otimes dv - \frac{1}{2}|\nabla v|^2 g_{\mathbb{R}^n}) = (n-1)e^{-2v} \tag{10.133}$$

and

$$V^k_{\max}([g_0]) \geq \liminf_i vol(e^{-2u_i} g) \geq vol(\mathbb{R}^n, e^{-2v_\infty} g_{\mathbb{R}^n}). \tag{10.134}$$

The contradiction follows from the following Lemma. We finish the proof of the Proposition. ■

PROPOSITION 10.3. *Let $p < n/2$ and $g = e^{-2u} g_{\mathbb{R}^n}$ be a C^2 function on $\mathbb{R}^n$ such that*

$$G_p(g^{-1} S_g)(x) = c, \forall x \in \mathbb{R}^n,$$

for some constant c. Then $u = 0$ if $c \leq 0$ and $u(x) = \log \frac{\lambda^2 + |x - x_0|^2}{2\lambda\sqrt{\frac{(n-p)p}{c}}}$ if $c > 0$. That is, $(\mathbb{R}^n, e^{-2u} g_{\mathbb{R}^n})$ can be compactified as a standard sphere if $c > 0$.

Proof: The proof follows [**92**] closely. The only difference is that the operator there is required to be C^1. Here our operator G_p is Lipschitz only. However G_p is uniformly elliptic and concave. We will show in Lemma 10.2 below that the Hopf lemma holds for our equation. Then the argument in [**92**] can be applied to our equation.

■

LEMMA 10.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $p < n/2$. If (10.110) has two solutions w and v with $w \geq v$ and $w(x_0) = v(x_0)$ for some $x_0 \in \partial\Omega$, then $x_0 \in \partial\Omega$. Furthermore,*

$$\frac{\partial w}{\partial \nu}(x_0) < \frac{\partial v}{\partial \nu}(x_0),$$

unless $w = v$. Here ν is the outer normal of $\partial\Omega$ at x_0.

Proof: For any function, set

$$A^u = \nabla u + du \otimes ds - \frac{1}{2}|\nabla u|^2 g_0 + S_{g_0}.$$

Since G_p is concave and homogeneous one, we have

$$G_p(A^w - A^v) \leq G_p(A^w) - G_p(A^v) = e^{-2w} - e^{-2v} \leq 0.$$

Let $\tilde{w} = w - v$. Now we can write $G_p(A^w - A^v)$ as follows

$$G_p(A^w - A^v) = a_{ij}(x)\tilde{w}_{ij} + b_i(x)\tilde{w}_i$$

with $\lambda_0 Id \leq (a_{ij}(x)) \leq \Lambda_0 Id$ and $b_i(x)$ bounded for any i. Therefore, we can apply Theorem 5 on page 61 and Theorem 7 on page 65 in [**104**] to prove the Lemma. ■

REMARK 10.4. *Proposition 10.3 does not hold for $p = n - 1$. For example $u = kx_1$ for any $k > 0$ is a solution of*

$$F(u) = G_{n-1}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0) = 0.$$

The same example indicates that without the concavity of F, Proposition 8.1 and Theorem 8.1 are not true. It is easy to check that $F(u_k) = 0$. On a domain $\Omega \subset \{x_1 \geq 0\}$, we have $u_k \geq 0$. But $|\nabla u_k| = k \to \infty$.

PROPOSITION 10.4. *Let (M, g_0) be a compact Riemannian manifold with $Ric_{g_0} \geq (n-1)g_0$. If $V_{\max}(M, [g_0]) = vol(\mathcal{S}^n)$, then (M, g) is conformally equivalent to the standard unit sphere.*

This Proposition is a direct consequence of the following

PROPOSITION 10.5. *Let (M, g) be a compact Riemannian manifold with $Ric_g \geq n - 1$. If $vol(M)$ is close to ω_n, the volume of $\mathcal{S}^n$, then, the Yamabe constant of $(M, [g])$, $Y(M, [g])$ is close to $n(n-1)\omega_n^{2/n}$, the Yamabe constant of $\mathcal{S}^n$.*

Proof: Let us first recall the well-known Yamabe constant of (M, g), which is defined by

$$Y(M, [g]) := \inf \left(\int v^{\frac{2n}{n-2}} dvol(g) \right)^{-\frac{n-2}{n}} \left\{ 4\frac{n-1}{n-2} \int |\nabla v|^2 dvol(g) + \int R_g v^2 dvol(g) \right\}$$

By a result of Ilias [**83**], which is based on a result of Gromov (see also [**16**]), we have

$$n(n-1)\omega_n^{2/n} \left(\int v^{\frac{2n}{n-2}} \right)^{(n-2)/2} \leq \left(\frac{\omega_n}{vol(g)} \right)^{2/n} \left\{ 4\frac{n-1}{n-2} \int |\nabla v|^2 + n(n-1) \int v^2 \right\},$$

for any $v \in H_1^2(M)$. Note that $R_g \geq n(n-1)$. Therefore, we have

$$Y(M,[g]) \geq \left(\frac{\omega_n}{vol(g)}\right)^{-2/n} n(n-1)\omega_n^{2/n} \geq n(n-1)\omega_n^{2/n} - \delta,$$

for any small $\delta > 0$, provided that $vol(g)$ is close to ω_n. ■

By the results of Colding (see [**39**],[**40**] and [**103**]), we know that the condition in Proposition 10.5 is equivalent to the one of following three other conditions:

1) rad$M$ is close to π,
2) M is Gromov-Hausdorff close to $\mathcal{S}^n$,
3) the $(n+1)$th eigenvalue of the Laplacian, $\lambda_{n+1}(M)$, is close to n.

Proof of Proposition 10.4. $V_{\max}(M,[g_0]) = vol(\mathcal{S}^n)$ implies by definition that there is a sequence $g_i \in [g_0]_+$ with $\lim_{i\to\infty} vol(g_i) = vol(\mathcal{S}^n)$. Proposition 10.5 implies that $Y(M,[g_0]) = Y(M,[g_i]) \to n(n-1)\omega_n^{2/n}$, the Yamabe constant of $\mathcal{S}^n$. Hence, the Yamabe constant of $(M,[g_0])$ equals to the Yamabe constant of the standard sphere. By the resolution of the Yamabe problem by Aubin [**11**] and Schoen [**111**], $(M,[g_0])$ is conformally equivalent to the standard sphere. ■

Proof of Theorem 10.2. Theorem 10.2 follows from Propositions 10.4 and 10.2. ■

Remark 10.5. *It is interesting to know weather $V_{\max}$ is achieved as in the negative case. One can show that if $V_{\max}$ is achieved by $\tilde{g}$, then $\mathcal{R}^s_{\tilde{g}}$ is constant.*

Now we prove Theorem 10.3.

Proof of Theorem 10.3. It follows the exact same arguments in the proof of Theorem 3 in [**64**], since that proof works for general uniform elliptic concave equations as well, as we note that $W_p(g) > 0$ implies the positivity of the mean curvature when $p \leq \frac{n}{2}$. We only give a sketch here.

Step 1. We define a deformation

$$f_t(g) := tW_p(g) + (1-t)R_g = 1, \tag{10.135}$$

where $g = e^{-2u}g_0$. Equation (10.135) ($\forall t \in [0,1]$) is still uniformly elliptic and concave.

Step 2. (Harnack inequality) There is a constant $C > 0$ such that for a solution u of (10.135) in B_{3R} we have

$$\min_{B_R} u + \max_{B_{2R}} u \geq 2\log R - \log C. \tag{10.136}$$

Here B_R is the ball of radius R in $\mathbb{R}^n$. (10.136) can be proved as in [**64**] using the method of moving planes.

By scaling argument, we may assume that $R = 1$. Assume by contradiction that (10.136) is not true. Then there exists a sequence of solutions of (10.135) in B_3 such that

$$\min_{B_1} u_i + \max_{B_2} u_i < -i. \tag{10.137}$$

Let $m_i = \min_{B_1} u_i = u_i(\bar{x}_i)$, and let $x_i \in B_1$ with $\bar{B}_{r_i}(x_i) \subset \bar{B}_1$ and $|x_i - \bar{x}_i| = r_i$. Here $r_i = e^{m_i}$. In view of (10.137), we know that $r_i \to 0$ as $i \to \infty$. Consider a new sequence of functions v_i defined by

$$v_i(x) \quad = \quad u_i(x_i + r_i x) - m_i$$

and set $\bar{x}_i = x_i + r_i \bar{y}_i$. It is clear that v_i satisfies (10.135) in $\{|x| < r_i^{-1}\}$ and $|\bar{y}_i| = 1$. ¿From (10.137), we extend v_i by the Kelvin transformation to the whole Euclidean space as in (2.8) of [**64**]. Now applying the method of moving planes as in [**64**], which in turn follows closely from [**33**], we can show that v_i converges to 0 in $B_{\frac{1}{2}}(\bar{y}_i)$. This is a contradiction. Note that though we are dealing with the Lipschitz operators, the method of moving planes works by using the fact that f_t in (10.135) is uniformly elliptic and concave.

Step 3. Consider a solution u of equation (10.135). First, it is clear that we have that the scalar curvature of $g = e^{-2u} g_0$ is positive. Hence we can apply the result of Schoen-Yau in [**115**] to embed the universal cover $\widetilde{M}$ of (M, g) into $\mathcal{S}^n$ by a map Φ conformally. Therefore we can use the method of moving planes (again make use of uniformly ellipticity and concavity of f_t) to obtain as in [**64**] that

$$|\nabla u|(x) \leq C, \qquad \text{for any } x \in M,$$

for some constant independent of u, provided that (M, g_0) is not equivalent to $\mathcal{S}^n$. It follows that

$$\max u - \min u \leq C, \tag{10.138}$$

for some constant independent of u. (10.138), together with the Harnack inequality (10.136), implies that

$$\min u \geq C,$$

for some constant independent of u. Hence by Theorem 10.4, we know that the solution space of equation (10.135) is compact.

Step 4. From Step 3 we can apply the degree theory. We may use a result of Li in [**93**], a variation of the original Leray-Schauder theorem. We also refer to Nirenberg's lecture notes [**98**] on the exposition of the degree theory in nonlinear differential equations. When $t = 0$, the topological degree for equation (10.135) is -1, which was proved by Schoen [**113**]. Since the solution space is compact, the topological degree for equation (10.135) with $t = 1$ is also -1. This finishes the proof of the Theorem. ■

Remark 10.6. *As in Remark 10.1, the conditions in Theorem 10.2 and Theorem 10.3 can be weakened to the assumption that the corresponding curvature of the background metric is nonnegative and positive at some point. The same argument using the short time existence of the corresponding curvature flows as in Remark 10.1 can produce a metric $g \in [g_0]$ with the positive curvatures.*

Notes

The results in this chapter were obtained in [**74**]. One of the interesting aspect of the equations treated here is that our geometric problem is deduced to a Pucci type fully nonlinear uniformly elliptic equations with respect to the Schouten tensor. These equations have been studied extensively in Euclidean domains, in particular in connection to stochastic optimization. With the breakthrough of the Krylov-Safonov's Harnack estimate [**89**] for non-divergent elliptic equations, it followed the fundamental Evans-Krylov theorem [**42, 88**] on the Hölder regularity of the second derivatives.

CHAPTER 11

Conformal curvature flow

In this chapter, we want to deform the metric in the conformal class $[g_0]$ of a fixed background metric g_0 along some curvature flow to certain extremal metric. The conformal curvature flow equation has some advantage such that it enable us to analyze the extremal metric, in turn to obtain some geometric information (which will be dealt with in the next chapter).

We consider the following general fully nonlinear flow:

$$\begin{cases} \dfrac{d}{dt}g &= -\left(\log\dfrac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}\right)\cdot g, \\ g(0) &= g_0, \end{cases} \tag{11.1}$$

where

$$r_{k,l} = \exp\left(\frac{\int \sigma_l(g)\log(\sigma_k(g)\sigma_l(g)^{-1})dg}{\int \sigma_l(g)dg}\right)$$

is defined so that the flow (11.1) preserves $\int \sigma_l(g)dg$ when $l \neq n/2$ and $\mathcal{E}_{n/2}$ when $l = n/2$. We have the following result for flow (11.1).

THEOREM 11.1. *For any smooth initial metric $g_0 \in \Gamma_k^+$, flow (11.1) has a global solution $g(t)$. Moreover, there is $h \in \mathcal{C}_k$ satisfying equation (12.5) such that for all m,*

$$\lim_{t\to\infty} \|g(t) - h\|_{C^m(M)} = 0.$$

A real symmetric $n \times n$ matrix A is said to lie in Γ_k^+ if its eigenvalues lie in Γ_k^+. Let A_{ij} be the $\{i,j\}$-entry of an $n \times n$ matrix. Then for $0 \leq k \leq n$, the kth *Newton transformation* associated with A is defined to be

$$T_k(A) = \sigma_k(A)I - \sigma_{k-1}(A)A + \cdots + (-1)^k A^k.$$

We have

$$T_k(A)^i_j = \frac{1}{k!}\delta^{i_1\ldots i_k i}_{j_1\ldots j_k j} A_{i_1 j_1}\cdots A_{i_k j_k},$$

where $\delta^{i_1\ldots i_k i}_{j_1\ldots j_k j}$ is the generalized Kronecker delta symbol. Here we use the summation convention. By definition,

$$\sigma_k(A) = \frac{1}{k!}\delta^{i_1\ldots i_k}_{j_1\ldots j_k} A_{i_1 j_1}\cdots A_{i_k j_k}, \quad T_{k-1}(A)^i_j = \frac{\partial \sigma_k(A)}{\partial A_{ij}}.$$

For $0 < l < k \leq n$, let

$$\tilde{T}_{k-1,l-1}(A) = \frac{T_{k-1}(A)}{\sigma_k(A)} - \frac{T_{l-1}(A)}{\sigma_l(A)}.$$

It is important to note that if $A \in \Gamma_k^+$, then $\tilde{T}_{k-1,l-1}(A)$ is positive definite.

The operator $F(A) = \left(\frac{\sigma_k(A)}{\sigma_l(A)}\right)^{\frac{1}{k-l}}$ is elliptic and concave in Γ_k^+. For simplicity of the notation, we will denote $\frac{\sigma_k(A)}{\sigma_l(A)}$ by $\frac{\sigma_k}{\sigma_l}(A)$.

LEMMA 11.1. *A conformal class of metric $[g]$ with $[g] \cap \Gamma_k^+ \neq \emptyset$ does not have a $C^{1,1}$ metric $g_1 \in \overline{\Gamma}_k^+$ with $\sigma_k(g_1) = 0$, where $\overline{\Gamma}_k^+$ is the closure of Γ_k^+.*

Proof: By the assumption, there is a smooth admissible metric g_0 with $\sigma_k(g_0) > 0$. Assume by contradiction that there is a $C^{1,1}$ metric g_1 with $\sigma_k(g_1) = 0$. Write $g_1 = e^{-2u} g_0$, so u satisfies

$$\sigma_k\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g + S_g\right) = 0. \tag{11.2}$$

Let

$$W = (\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0), \quad \text{and} \quad a_{ij}(W) = \frac{\partial \sigma_k(W)}{\partial w_{ij}}.$$

A Set $u_1 = u, u_0 = 1$, we may assume $u_1 \geq u_0 + 1$ since $u_1 + c$ also satisfies (11.2) for any constant c. Let $v = e^{-u_1} - e^{-u_0}$, $h_t = te^{-u_1} + (1-t)e^{-u_0}$, $u_t = -\log h_t$ and $W_t = \nabla^2 u_t + du_t \otimes du_t - \frac{|\nabla u_t|^2}{2} g + S_g$. As in [**127**], one can check that $W_t \in \Gamma_k^+$ and $(a_{ij}(W_t))$ positive definite (nonnegative definite for all $0 \leq t < 1$ ($0 \leq t \leq$!). We have the following

$$0 > \sigma_k(W_1) - \sigma_k(W_0) = \sum_{ij} (\int_0^1 \frac{a_{ij}(W_t)}{h_t^2} dt) \nabla_{ij}^2 v + \sum_l b^l(t,x) \nabla_l v + dv, \tag{11.3}$$

for some bounded functions d and b^l, $l = 1, ..., n$. This is a contradiction to the strong maximum principle. ■

The follow Proposition is a uniqueness result.

PROPOSITION 11.1. *Let (M, g_0) be a spherical space form. If $g \in [g_0] \cap \Gamma_k^+$ is a solution of (12.5), then (M, g) is also a spherical space form.*

Proof: The Proposition is a special case of a Liouville type result in [**92**]. But it can be proved in simpler way, following the similar argument as in [**128**]. After transfer the equation to R^n as in [**128**], the method of moving plane in [**50**] can be used as in [**128**] to show that the solution is symmetric at some point. We may assume the solution is symmetric about the origin and its value and gradient at the origin are same as the standard solution (after a rescaling if necessary). Since both are radial functions, expanding the solution to the power series, if some of the derivatives of the solution does not match the standard solution at the origin, then the difference of two solution is either non-negative or non-positive in a neighborhood of the origin since it is a function in one variable only and analytic (since they satisfy analytic elliptic equation). But, this contradicts the strong minimum principle, as the difference of two solutions satisfies certain elliptic equation. This implies that all the derivatives are the same at the origin, which in turn gives the uniqueness by the analyticity again. ■

For any $0 < k \leq n$, let

$$\tilde{\mathcal{F}}_k(g) = \begin{cases} \dfrac{1}{n-2k}\displaystyle\int_M \sigma_k(g)dg, & k \neq n/2, \\ \mathcal{E}_{n/2}(g), & k = n/2. \end{cases}$$

LEMMA 11.2. *The flow (11.1) preserves $\tilde{\mathcal{F}}_l$. It also decreases the functional $\tilde{\mathcal{F}}_k$. In fact, the evolution equations for $\log \frac{\sigma_k}{\sigma_l}$ and $\tilde{\mathcal{F}}_k$ are*

$$\frac{d}{dt}\log\frac{\sigma_k(g)}{\sigma_l(g)} = \frac{1}{2}\mathrm{tr}\{\tilde{T}_{k-1,l-1}(S_g)\nabla_g^2 \log\frac{\sigma_k(g)}{\sigma_l(g)}\} + (k-l)(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}) \tag{11.4}$$

and

$$\frac{d}{dt}\tilde{\mathcal{F}}_k(g) = -\frac{1}{2}\int_M (\frac{\sigma_k(g)}{\sigma_l(g)} - r_{k,l})\left(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}\right)\sigma_l(g)dg. \tag{11.5}$$

Proof: We prove $\frac{d\tilde{\mathcal{F}}_l}{dt} = 0$ for $l \neq n/2$, the proof for the case $l = n/2$ is the same using $\tilde{\mathcal{F}}_{n/2} = \mathcal{E}_{n/2}$. On any locally conformally flat manifold, from the computation in [**126**],

$$\begin{aligned} \frac{d}{dt}\int_M \sigma_l(g)\, dvol(g) &= \frac{n-2l}{2}\int_M \sigma_l(g)g^{-1}\cdot\frac{d}{dt}g\, dvol(g) \\ &= \frac{2l-n}{2}\int_M \sigma_l(g)g^{-1}\cdot\left(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}\right) dvol(g) = 0. \end{aligned}$$

The first identity follows from simple direct computation, we omit it. We verify the second identity. When $k \neq \frac{n}{2}$,

$$\begin{aligned} \frac{d}{dt}\tilde{\mathcal{F}}_k(g) &= \frac{1}{2}\int_M \sigma_k(g)g^{-1}\frac{d}{dt}gdg \\ &= -\frac{1}{2}\int_M \sigma_k(g)\left(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}\right)dg \\ &= -\frac{1}{2}\int_M \frac{\sigma_k(g)}{\sigma_l(g)}\left(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}\right)\sigma_l(g)dg \\ &= -\frac{1}{2}\int_M (\frac{\sigma_k(g)}{\sigma_l(g)} - r_{k,l})\left(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}\right)\sigma_l(g)dg. \end{aligned}$$

By [**20**], the above also holds for $k = \frac{n}{2}$. ■

If $g = e^{-2u}\cdot g_0$, one may compute that

$$\sigma_k(g) = e^{2ku}\sigma_k\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0\right).$$

Equation (11.1) can be written in the following form

$$\begin{cases} 2\dfrac{du}{dt} &= \log\dfrac{\sigma_k}{\sigma_l}\left(\nabla^2 u + du\otimes du - \dfrac{|\nabla u|^2}{2}g_0\right) + 2(k-l)u - \log r_{k,l}(g) \\ u(0) &= u_0. \end{cases} \tag{11.6}$$

The short time existence of flow (11.1) follows from the standard implicit theorem as $g_0 \in \Gamma_k^+$. We want to prove the long time existence and convergence.

Let

$$T^* = \sup\{T_0 > 0 \,|\, (11.1) \text{ exists in } [0, T_0] \text{ and } g(t) \in \Gamma_k^+ \text{ for } t \in [0.T_0]\}.$$

Proposition 11.2. *There is a constant $C > 0$ independent of T such that*

$$|\nabla u| \le c, \quad \textit{and} \quad |\nabla^2 u| \le c. \tag{11.7}$$

Proof. The gradient estimate follows from Schoen-Yau's theorem on developing maps on locally conformally flat manifolds and the method of moving planes as in the proof of Proposition 9.1 (see also [**132**]), we won't repeat it here. We now prove the second derivative boundedness.

Set

$$F = \log\frac{\sigma_k}{\sigma_l}(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0).$$

By equation (11.6), $F = 2u_t - 2(k-l)u - \log r_{k,l}$. We only need to consider the case $k > 1$, therefore we only need to give a upper bound of Δu which dominates all other second order derivatives. Consider $G = \Delta u + m|\nabla u|^2$ on $M \times [0, T]$, where m is a large constant which will be fixed later. Without loss of generality, we may assume that the maximum of G on $M \times [0, T]$ achieves at a point $(x_0, t_0) \in M \times (0, T]$ and $G(x_0, t_0) \ge 1$. We may assume that at (x_0, t_0)

$$2\sigma_1(W) \ge G \ge \frac{1}{2}\sigma_1(W), \tag{11.8}$$

where $W = \nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0$. Consider everything in a small neighborhood near x_0. We may consider W as a matrix with entry $w_{ij} = u_{ij} + u_i u_j - \frac{1}{2}|\nabla u|^2\delta_{ij} + S(g_0)_{ij}$. In the rest of the proof, c denotes a positive constant independent of T, which may vary from line to line.

Since G achieves its maximum at (x_0, t_0), we have at this point

$$G_t = \sum_l (u_{llt} + 2mu_{lt}u_l) \ge 0, \tag{11.9}$$

and

$$G_i = \sum_l (u_{lli} + 2mu_{li}u_l) = 0, \quad \forall\ i. \tag{11.10}$$

(11.10) and (11.7) imply that at (x_0, t_0)

$$|\sum_l u_{lli}| \le cG. \tag{11.11}$$

By the Harnack inequality (11.7), we may assume that

$$|u_{lij} - u_{ijl}| < c \quad \text{and} \quad |u_{ijkl} - u_{ijlk}| < cG. \tag{11.12}$$

We may assume by choosing coordinates that the matrix (w_{ij}) at (x_0, t_0) is diagonal. At the maximum point, G_{ij} is non-positive definite. Set $F^{ij} = \frac{\partial F}{\partial w_{ij}}$. Since $g(t) = e^{-2u(t)} g_0 \in \Gamma_k^+$, we know that the matrix (F^{ij}) is positive. Hence in view of (11.9)-(11.12) and the concavity of F we have

$$
\begin{aligned}
0 &\geq \sum_{i,j} F^{ij} G_{ij} = \sum F^{ij}(u_{llij} + 2m u_{li} u_{lj} + 2m u_{lij} u_l) \\
&\geq \sum_{i,j,l} F^{ij}(u_{ijll} + 2m u_{li} u_{lj} + 2m u_{ijl} u_l) - c \sum_i F^{ii} G \\
&= -c \sum_i F^{ii} G + \sum_{i,j,l} F^{ij} \{ w_{ijll} - (u_i u_j - \frac{1}{2} |\nabla u|^2 \delta_{ij} + S(g_0)_{ij})_{ll} \\
&\quad + 2m u_{li} u_{lj} + 2m w_{ijl} u_l - 2m u_l (u_i u_j - \frac{1}{2} |\nabla u|^2 \delta_{ij} + S(g_0)_{ij})_l \} \\
&\geq \Delta F + 2m \sum_l F_l u_l + \sum_i F^{ii} u_{jl}^2 + 2(m-1) \sum_{i,l} F^{ii} u_{li}^2 - c \sum_i F^{ii} G \\
&\geq \Delta F + 2m \sum_l F_l u_l + \frac{1}{n} G^2 \sum_i F^{ii} + 2(m-1) \sum_{i,l} F^{ii} u_{li}^2 - c \sum_i F^{ii} G.
\end{aligned} \tag{11.13}
$$

From equation (11.6), $F = 2u_t - 2(k-l)u - \log r(g)$. In view of (11.9) and (11.10), (11.13) yields

$$
\begin{aligned}
0 &\geq -2(k-l)G + \frac{1}{n} \sum_i F^{ii} G^2 + 2(m-1) \sum_i F^{ii} u_{ii} - c \sum_i F^{ii} G \\
&\geq -2(k-l)\Delta u + \sum_i F^{ii} G^2 + 2(m-1) \sum F^{ii} u_{ii}^2 - c \sum F^{ii} G \\
&\geq \{ -2(k-l)G + 2(m-1) \sum F^{ii} u_{ii}^2 \} + \frac{1}{n} \sum_i F^{ii} (G^2 - cG).
\end{aligned} \tag{11.14}
$$

We claim that for large $m > 0$

$$
\frac{1}{2n} G^2 \sum_i F^{ii} + 2(m-1) \sum_i F^{ii} w_{ii}^2 \geq 2(k-l)G. \tag{11.15}
$$

It is easy to check, from the Newton-MacLaurin inequality, that

$$
\begin{aligned}
\sum F^{ii} w_{ii}^2 &= \frac{\sigma_1(W)\sigma_k(W) - (k+1)\sigma_{k+1}(W)}{\sigma_k(W)} - \frac{\sigma_1(W)\sigma_l(W) - (l+1)\sigma_{l+1}(W)}{\sigma_l(W)} \\
&= (l+1)\frac{\sigma_{l+1}}{\sigma_l}(W) - (k+1)\frac{\sigma_{k+1}}{\sigma_k}(W) \geq c_{n,k,l} \frac{\sigma_{l+1}}{\sigma_l}(W),
\end{aligned} \tag{11.16}
$$

and

$$
\sum_i F^{ii} = (n-k+1)\frac{\sigma_{k-1}}{\sigma_k}(W) - (n-l+1)\frac{\sigma_{l-1}}{\sigma_l}(W) \geq \frac{\tilde{c}_{n,k,l}}{\sigma_1(W)}, \tag{11.17}
$$

where $c_{n,k,l}$ and $\tilde{c}_{n,k,l}$ are two positive constant depending only on n, k and l. From these two facts, we can prove the claim as follows. First, if

$$\frac{\tilde{c}_{n,k,l}}{4n}\frac{\sigma_1(W)\sigma_{k-1}(W)}{\sigma_k(W)} \geq 4(k-l),$$

then the claim follows from (11.17) and (11.8). Hence we may assume that

$$\frac{\sigma_1(W)\sigma_{k-1}(W)}{\sigma_k(W)} \leq c^*_{n,k,l},$$

for some positive constant $c^*_{n,k,l}$ depending only on n, k and l. Together with the Newton-MacLaurin inequality, it implies

$$\frac{\sigma_{l+1}(W)}{\sigma_l(W)} \geq \hat{c}_{n,k,l}\frac{\sigma_k(W)}{\sigma_{k-1}(W)} \geq \hat{c}_{n,k,l}c^*_{n,k,l}\sigma_1(w),$$

which, in turn, together with (11.16) implies

$$\sum_i F^{ii}w_{ii}^2 \geq c_{n,k,l}\frac{\sigma_{l+1}(W)}{\sigma_l(W)} \geq c^1_{n,k,l}\sigma_1(W) \geq \frac{1}{2}c^1_{n,k,l}G.$$

Hence, if we choose m large, then the claim is true. The Proposition follows directly from the claim. ■

PROPOSITION 11.3. *Suppose $\|u\|_{C^2(M)}$ is bounded independent of $t \in [0,T]$. Then there is a constant $C_0 > 0$ independent of T such that*

$$\frac{\sigma_k(g)}{\sigma_l(g)}(t) \geq C_0, \qquad \textit{for } t \in [0,\infty).$$

Proof: Here we will make use of Lemma 13.4. We consider $H = \log\frac{\sigma_k(g)}{\sigma_l(g)} - e^{-u}$ on $M \times [0,T]$ for any $T < T^*$. From (11.1) and (11.4) we have

$$\begin{aligned}\frac{dH}{dt} &= \frac{1}{2}\mathrm{tr}\{\tilde{T}_{k-1,l-1}(S_g)\nabla_g^2\log\frac{\sigma_k(g)}{\sigma_l(g)}\} + (k-l+\frac{1}{2}e^{-u})(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}(g)) \\ &= \frac{1}{2}\mathrm{tr}\{\tilde{T}_{k-1,l-1}(S_g)\nabla_g^2(H+e^{-u})\} + (k-l+\frac{1}{2}e^{-u})(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}(g)).\end{aligned}$$

Without loss of generality, we may assume that the minimum of H in $M \times [0,T]$ achieves at $(x_0,t_0) \in M \times (0,T]$. Let H_j and H_{ij} are the first and second derivatives with respect to the back-ground metric g_0. At this point, we have $\frac{dH}{dt} \leq 0$, $0 = H_l = \sum_{ij} F^{ij}w_{ijl} + e^{-u}u_l$ for all l, and (H_{ij}) is non-negative definite. Also we have (F^{ij}) is positive definite and

$$\sum_{i,j} F^{ij}w_{ij} = \frac{1}{\sigma_k(g)}\frac{\partial\sigma_k(g)}{\partial w_{ij}}w_{ij} - \frac{1}{\sigma_l(g)}\frac{\partial\sigma_l(g)}{\partial w_{ij}}w_{ij} = k-l.$$

Recall that in local coordinates $\tilde{T}^{ij}_{k-1,l-1}(S_g) = F^{ij}$ and

$$\sum_{i,j} F^{ij}(\nabla_g^2)_{ij}H = \sum_{i,j} F^{ij}(H_{ij} + u_iH_j + u_jH_j - \sum_l u_lH_l\delta_{ij}).$$

It follows that at the point,

(11.18)

$$
\begin{aligned}
0 &\geq H_t - \frac{1}{2}\sum_{i,j} F^{ij} H_{ij} \\
&= \frac{1}{2}\mathrm{tr}\{\tilde{T}_{k-1,l-1}(S_g)\nabla_g^2 e^{-u}\} + (k-l+\frac{1}{2}e^{-u})(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}(g)) \\
&= \frac{1}{2}\sum_{i,j} F^{ij}\{(e^{-u})_{ij} + u_i(e^{-u})_j + u_j(e^{-u})_i - u_l(e^{-u})_l\delta_{ij}\} \\
&\quad +(k-l+\frac{1}{2}e^{-u})(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}(g)) \\
&= \frac{e^{-u}}{2}\sum_{i,j} F^{ij}\{-u_{ij} - u_i u_j + |\nabla u|^2\delta_{ij}\} + (k-l+\frac{1}{2}e^{-u})(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}(g)) \\
&= \frac{e^{-u}}{2}\sum_{i,j} F^{ij}\{-w_{ij} + S_{ij} + \frac{1}{2}|\nabla u|^2\delta_{ij}\} + (k-l+\frac{e^{-u}}{2})(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}(g)) \\
&\geq \frac{e^{-u}}{2}\sum_{i,j} F^{ij}\{-w_{ij} + S_{ij}\} + (k-l+\frac{1}{2}e^{-u})(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}(g)) \\
&= \frac{e^{-u}}{2}\sum_{i,j} F^{ij} S_{ij} + (k-l+\frac{1}{2}e^{-u})(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}(g)) - \frac{k-l}{2}e^{-u},
\end{aligned}
$$

where S_{ij} are the entries of $S(g_0)$. Since $S(g_0) \in \Gamma_k^+$, by Lemma 13.4,

$$
F^{ij}S_{ij} = \{\frac{1}{\sigma_k(g)}\frac{\partial\sigma_k(g)}{\partial w_{ij}} - \frac{1}{\sigma_l(g)}\frac{\partial\sigma_l(g)}{\partial w_{ij}}\}S_{ij} \geq (k-l)e^{2u}\left(\frac{\sigma_k(g_0)}{\sigma_l(g_0)}\right)^{\frac{1}{k-l}}\left(\frac{\sigma_k(g)}{\sigma_l(g)}\right)^{-\frac{1}{k-l}}. \tag{11.19}
$$

By C^2 estimates, $\log r_{k,l}(g)$ is bounded from above, we have

$$
\begin{aligned}
0 &\geq \frac{(k-l)e^u}{2}\left(\frac{\sigma_k(g_0)}{\sigma_l(g_0)}\right)^{\frac{1}{k-l}}\left(\frac{\sigma_k(g)}{\sigma_l(g)}\right)^{-\frac{1}{k-l}} \\
&\quad +(k-l+\frac{1}{2}e^{-u})(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}(g)) - \frac{k-l}{2}e^{-u} \\
&\geq c_1\left(\frac{\sigma_k(g)}{\sigma_l(g)}\right)^{-\frac{1}{k-l}} + c_2\log\frac{\sigma_k(g)}{\sigma_l(g)} - c_3
\end{aligned}
$$

for positive constants c_1, c_2 and c_3 independent of T. It follows that there is a positive constant c_4 independent of T such that

$$
\frac{\sigma_k(g)}{\sigma_l(g)} \geq c_4,
$$

at point (x_0, t_0). Then the Proposition follows, as $|u|$ is bounded by Proposition 11.7. ■

PROPOSITION 11.4. *If there is C independent of t such that $\|u\|_{C^2(M)} \leq C$ for all $t \in [0, T^*)$, then $T^* = \infty$, and all the results in Theorem 11.1 are true.*

Proof of Proposition 11.4. First, by Krylov's theorem, the flow has $C^{2,\alpha}$ estimates. The standard parabolic theory gives the longtime existence of the flow. Lemma 11.2 implies that

$$\int_0^\infty \int_M (\sigma_k(g) - r_{k,l}\sigma_l(g))^2 dgdt < \infty,$$

which, in turn, implies that there is a sequence $\{t_l\}$ such that

$$\int_M (\sigma_k(g) - r_{k,l}\sigma_l(g))^2(t_l)dg \to 0$$

as $t_l \to \infty$. The above estimates imply that $g(t_l)$ converges in $C^{2,\alpha}$ to a conformal metric h, which is a solution of (12.5).

Now we want to use Simon's argument [**117**] to prove that h is the unique limit of flow (11.1)(see also [**8**]). Since the arguments are essentially the same, here we only give a sketch. First, with the regularity estimates established for flow (11.1), one can show that, for all m,

$$\lim_{t\to\infty} \|\frac{\sigma_k(g(t))}{\sigma_l(g(t))} - \beta\|_{C^m(M)} = 0,$$

for some positive constant β. It is clear that $\frac{\sigma_k(h)}{\sigma_l(h)} = \beta$. By Proposition 11.3 and the Newton-MacLaurin inequality, there is a constant $c > 1$ such that $c^{-1} \leq \sigma_l(g(t)) \leq c$. We want to show that flow (11.1) is a pseudo-gradient flow, though it is not a gradient flow. The crucial step is to establish the angle estimate (11.21) for the L^2 gradient of some proper functionals. We may now switch the back-ground metric to h and all derivatives and norms are taken with respect to the metric h.

The following is the version of Theorem 3 in Simon [**117**] for our flow (11.1) (which is a infinite dimensional generalization of Lojasiewicz' result).

PROPOSITION 11.5. *There exist $\theta \in (0, 1/2]$ and $r_0 > 0$ such that for any $\|g - h\|_{C^{2,\alpha}} \leq r_0$*

$$\left(\int_M |\nabla \mathcal{F}_k|^2(g)dvol(h)\right)^{1/2} \geq |\mathcal{F}_k(g) - \mathcal{F}_k(h)|^{1-\theta}. \tag{11.20}$$

Proof: Simon [**117**] proved such inequality for gradients of functionals. Our flow (11.1) is different in the fact that the gradient is a fully nonlinear operator rather than a quasilinear one. But as Andrews [**8**] observed, Simon's argument can be carried through for $\mathcal{F}_k$. The details otherwise are identical, we refer to the proof of Theorem 3 in [**117**]. ∎

Here we only give a proof for $l < k < n/2$. The proof for the other cases is similar by taking consideration of the corresponding functionals.

Consider a functional defined by

$$\mathcal{F}_{k,l}(g) = \left(\int \sigma_l(g)dg\right)^{-\frac{n-2k}{n-2l}} \int_M \sigma_k(g)dg.$$

Its L^2-gradient is

$$\nabla \mathcal{F}_{k,l} = -c_0((\sigma_k(g) - \tilde{r}_{k,l}(g)\sigma_l(g))e^{-nu},$$

where c_0 is a non-zero constant and $\tilde{r}_{k,l}(g)$ is given by

$$\tilde{r}_{k,l}(g) := \frac{\int_M \sigma_k(g)dg}{\int_M \sigma_l(g)dg},$$

which is different from $r_{k,l}$. But it is easy to check that $r_{k,l}(t) - \tilde{r}_{k,l}(t) \to 0$ as $t \to \infty$. Since $\frac{\sigma_k(g(t))}{\sigma_l(g(t))}$ is very close to a constant for large t, from (11.5) we have

$$\begin{aligned}
\tfrac{d}{dt}\mathcal{F}_{k,l}(g) &\leq -c\int_M (\frac{\sigma_k(g)}{\sigma_l(g)} - r_{k,l})\left(\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}\right)\sigma_l(g)dg \\
&\leq -c\left(\int_M \left|\frac{\sigma_k(g)}{\sigma_l(g)} - r_{k,l}\right|^2 \sigma_l(g)dg \int_M \left|\log\frac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}\right|^2 \sigma_l(g)dg\right)^{1/2} \\
&\leq -c\left(\int_M |\frac{\sigma_k(g)}{\sigma_l(g)} - r_{k,l}|^2\sigma_l(g)dg\right)^{1/2}\left(\int_M |\frac{dg}{dt}|^2\sigma_l(g)dg\right)^{1/2} \\
&\leq -c\left(\int_M |\frac{\sigma_k(g)}{\sigma_l(g)} - \tilde{r}_{k,l}|^2\sigma_l(g)dg\right)^{1/2}\left(\int_M |\frac{dg}{dt}|^2\sigma_l(g)dg\right)^{1/2} \\
&\leq -c\left(\int_M |\sigma_k(g) - \tilde{r}_{k,l}\sigma_l(g)|^2\sigma_l(g)dg\right)^{1/2}\left(\int_M |\frac{dg}{dt}|^2\sigma_l(g)dg\right)^{1/2} \\
&\leq -c\left(\int_M |\nabla\mathcal{F}_{k,l}|^2 dh\right)^{1/2}\left(\int_M |\frac{dg}{dt}|^2 dh\right)^{1/2},
\end{aligned} \tag{11.21}$$

where $c > 0$ is a constant varying from line to line. The angle estimate (11.21) means that flow (11.1) is a pseudo-gradient flow.

Step 1. For a fixed constant $t_0 > 0$, by Propositions 11.2, 11.3, Lemma 11.2 and the Krylov theorem, we have that for any small $\varepsilon > 0$ there is a constant $\delta_1 > 0$ such that

$$\|g(t) - h\|_{C^{2,\alpha}} < \varepsilon, \qquad \text{for } t \in [t, t + t_0]$$

if $\|g(t) - h\|_{L^2} < \delta_1$.
Step 2. Since there exist $T_0 > 0$, $c_1 > 0$ and $r_0 > 0$ such that (11.21) and (11.20) hold. By the continuity of $\mathcal{F}_k$ in $C^{2,\alpha}$, there exists $r_1 > 0$ such that

$$\|g - h\|_{C^{2,\alpha}} < r_1 \quad \text{implies} \quad |\mathcal{F}_k(g) - \mathcal{F}_k(h)| < \left(\frac{c_0\theta\delta_1}{2}\right)^{1/\theta}. \tag{11.22}$$

We **claim** that for any $[a, b] \subset [T_0, \infty)$,

$$\begin{aligned}
&\|g - h\|_{C^{2,\alpha}} < \min\{r_0, r_1\} \quad \forall t \in [a, b] \quad \text{implies} \\
&\|g(t_1) - g(t_2)\|_{L^2} < \frac{\delta_1}{2} \quad \forall a \leq t_1 < t_2 \leq b.
\end{aligned}$$

From (11.21) and (11.20), we have for ant $t \in [a, b]$

$$\begin{aligned}\left(\int_M \left|\frac{dg}{dt}\right|^2 dvol(h)\right)^2 &\leq \frac{1}{c_1}\frac{d\mathcal{F}_k}{dt}(g)\|\nabla \mathcal{F}_k(g(t))\|_{L^2} \\ &\leq \frac{1}{c_1\theta}\left|\frac{d}{dt}\left((\mathcal{F}_k(g(t)-\mathcal{F}_k(h))^\theta\right)\right|.\end{aligned}$$

Integrating the previous inequality over $[t_1, t_2] \subset [a, b]$ and by the monotonicity of $\mathcal{F}_k$, we have

$$\|g(t_1) - g(t_2)\|_{L^2} \leq \frac{1}{c_1\theta}|\mathcal{F}_k(g(t_1) - \mathcal{F}_k(h)|^\theta < \frac{1}{2}\delta_1.$$

The **claim** is proved.
Step 3. Now recall that there is a sequence $t_l \to \infty$ such that $g(t_l)$ converges to h in $C^{2,\alpha}$. Hence for any $\delta_1 > 0$ there is t_{l_0} such that $\|g(t_{l_0}) - h\|_{L^2} < \delta_1/2$. Set

$$\tau_0 = \inf\{\tau \mid \|g(t) - h\|_{C^{2,\alpha}} < \varepsilon, \forall t \in [t_{l_0}, \tau]\},$$

for $0 < \varepsilon < \min\{r_0, r_1\}$. It is clear from step 1 that $\tau_0 \geq t_{l_0} + t_0$. We *assert* that $\tau_0 = \infty$. Assume by contradiction that $\tau_0 < \infty$. For any $t \in [t_{l_0}, \tau_0]$, from the **claim** in step 2 we have

$$\begin{aligned}\|g(t) - h\|_{L^2} &\leq \|g(t_{l_0}) - h\|_{L^2} + \|g(t_{l_0}) - g(t)\|_{L^2} \\ &< \delta_1.\end{aligned}$$

This, together with step 1, implies that $[\tau_0, \tau_0 + t_0] \subset \{\tau \mid \|g(t) - h\|_{C^{2,\alpha}} < \varepsilon, \forall t \in [t_{l_0}, \tau]\}$. This is a contradiction.

The proof is complete ■

Now we note that Theorem 11.1 is already verified for the case $l = 0$, since Lemma 11.2 implies that the flow preserves the volume in this case. From the uniform global gradient bound in Proposition 11.2, u is uniformly bounded independent of t. Then by Propositions 11.2 and 11.7, $\|u\|_{C^2(M)}$ is bounded independent of t. So Theorem 11.1 for the case $l = 0$ follows from Proposition 11.4. To prove Theorem 11.1 for general case $l < k \leq n$, we only need to get C^0 estimates for u. To do that, we will make use of the result for case $l = 0$.

PROPOSITION 11.6. *Let (M, g_0) be a locally conformally flat manifold with $g_0 \in \Gamma_k^+$. We have*

(a). *When $k > n/2$, there is a constant $C_Q = C_Q(n, k) > 0$ depending only on n and k such that for any metric $g \in \mathcal{C}_k$.*

$$\int_M \sigma_k(g)vol(g) \leq C_Q vol(g)^{\frac{n-2k}{n}}.$$

(b). *When $k < n/2$, there is a constant $C_S = C_S(n) > 0$ such that for any metric $g \in \mathcal{C}_k$.*

$$\int_M \sigma_k(g)vol(g) \geq C_S vol(g)^{\frac{n-2k}{n}}.$$

(c). *If $k = n/2$ and g_0 is a metric of constant sectional curvature, then for any $g \in \mathcal{C}_k$*

$$\mathcal{E}_{n/2}(g) \geq \frac{1}{n} C_{MT}(\log vol(g) - \log vol(g_0)),$$

where $C_{MT} = \int_M \sigma_{n/2}(g_0) dg_0$.

Moreover, in cases (a) and (c) the equality holds if and only if g is a metric of constant sectional curvature.

Proof: When $k \geq n/2$, from [**70**] we know that (M, g_0) is conformally equivalent to a spherical space form. In this case, it was proved in [**126**] that any solution of (12.5) for $l = 0$ is of constant sectional curvature. By the results of Theorem 11.1 for the case $l = 0$ and [**20**] ($k = \frac{n}{2}$), for any $g \in \mathcal{C}_k$ there is a metric $g_e \in \mathcal{C}_k$ of constant sectional curvature with $vol(g) = vol(g_e)$ and

$$\tilde{\mathcal{F}}_k(g) \geq \tilde{\mathcal{F}}_k(g_e). \tag{11.23}$$

When $k > n/2$, (11.23) implies that

$$vol(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g) dg \leq vol(g_e)^{-\frac{n-2k}{n}} \int_M \sigma_k(g_e) dg_e,$$

and the equality holds if and only if (M, g) is a space form. It is clear that

$$vol(g_e)^{-\frac{n-2k}{n}} \int_M \sigma_k(g_e) dg_e$$

is a constant depending only on n, k. This proves (a).

(c) was already proved in [**20**]. For the completeness, we provide a proof here. When $k = n/2$, (11.23) implies that for any $g \in \mathcal{C}_{\frac{n}{2}}$ with $vol(g) = vol(g_0)$

$$\mathcal{E}_{n/2}(g) \leq 0.$$

For any $g \in \mathcal{C}_{\frac{n}{2}}$, choose a constant a such that $e^{-2a} g$ has volume $vol(g_0)$. It is easy to check that $a = \frac{1}{n}\{\log vol(g) - \log vol(g_0)\}$. By definition,

$$\mathcal{E}_{n/2}(e^{-2a} g) = \mathcal{E}_{n/2}(g) - a \int_M \sigma_{n/2}(g) dg.$$

Hence, we have

$$\begin{aligned} \mathcal{E}_{n/2}(g) &\geq a \int_M \sigma_{n/2}(g) dg \\ &= \frac{1}{n} \left(\int_M \sigma_{n/2}(g_0) dg_0 \right) \{\log vol(g) - \log vol(g_0)\}. \end{aligned}$$

This proves (c).

It remains to prove (b). For this case, we only need to prove that

$$\inf_{\mathcal{C}_k \cap \{vol(g)=1\}} \mathcal{F}_k(g) =: \beta_0 > 0.$$

Assume by contradiction that $\beta_0 = 0$. By Theorem 1 in [**72**], we can find a sequence of solutions $g_i = e^{-2u_i} g_0 \in \mathcal{C}_k$ of (3) with $vol(g_i) = 1$ and $\sigma_k(g_i) = \beta_i$ such that $\lim_{i\to\infty} \beta_i = 0$. $\sigma_k(g_l) = \beta_i$ means

$$\sigma_k(\nabla^2 u_i + du_i \otimes du_i - \frac{|\nabla u_i|^2}{2} g_0 + S_{g_0}) = \beta_i e^{-2ku_i}. \tag{11.24}$$

Consider the scaled metric $\tilde{g}_i = e^{-2\tilde{u}_i} g_0$ with $\tilde{u}_i = u_i - \frac{1}{2k} \log \beta_i$, which satisfies clearly that

$$\sigma_k(\nabla^2 \tilde{u}_i + d\tilde{u}_i \otimes d\tilde{u}_i - \frac{|\nabla \tilde{u}_i|^2}{2} g_0 + S_{g_0}) = e^{-2k\tilde{u}_i} \tag{11.25}$$

and

$$vol(\tilde{g}_i) = \beta_i{}^{\frac{n}{2k}} \to 0 \quad \text{as } i \to \infty.$$

By Corollary 1 in [**71**], we conclude that

$$\tilde{u}_i \to +\infty \text{ uniformly as } i \to \infty.$$

Hence $m_i := \inf_M \tilde{u}_i \to +\infty$ as $i \to \infty$. Now at the minimum point x_i of $\tilde{u}_i$, by (11.25),

$$\sigma_k(S_{g_0}) \le \sigma_k(\nabla^2 \tilde{u}_i + d\tilde{u}_i \otimes d\tilde{u}_i - \frac{|\nabla \tilde{u}_i|^2}{2} g_0 + S_{g_0}) = e^{-2km_i} \to 0.$$

This is a contradiction to the fact $g_0 \in \Gamma_k^+$. ■

Now we can prove the C^0 boundedness (and hence C^2 boundedness).

PROPOSITION 11.7. *Let $g = e^{-2u} g_0$ be a solution of flow (11.1) with $\sigma_k(g(t)) \in \Gamma_k^+$ on $M \times [0, T^*)$. Then there is a constant $c > 0$ depending only on v_0, g_0, k and n (independent of T^*) such that*

$$\|u(t)\|_{C^2} \le c, \quad \forall\, t \in [0, T^*). \tag{11.26}$$

Proof: We only need to show the boundedness of $|u|$. First we consider the case $l \ne n/2$. By Proposition 11.2 and the preservation of $\int \sigma_l(g) dg$, we have

$$\begin{aligned} c_l &= \int_M e^{(2l-n)u} \sigma_l(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0) dg_0 \\ &\le c_1 \int_M e^{(2l-n)u} dg_0. \end{aligned} \tag{11.27}$$

If $l < n/2$, then (11.27), together with (11.7), implies that $u < c$ for some constant $c > 0$. On the other hand, in this case Proposition 11.6 gives

$$vol(g) \le C(\int_M \sigma_l(g) dg)^{\frac{n}{n-2l}} = c_0 C,$$

which, together with (11.7) implies $u > c_1$, hence $|u| \le C$ in this case.

If $l > n/2$, (11.27) gives a lower bound of u. Suppose there is no upper bound, we have a sequence of u, with ∇u and $\nabla^2 u$ bounded, but $\sup u$ goes to infinity (so does $\inf u$). Set $v = u - \inf u$, so v is bounded and so is the C^2 norm of v. But, for $\tilde{g} = e^{-2v} g_0$, we get $\tilde{F}_l(\tilde{g})$ tends to 0. Take a subsequence, we get $\sigma_l(e^{-2v^*} g_0) = 0$ with v^* in $C^{1,1} \cap \overline{\Gamma}_k^+$. This is a contradiction to Lemma 11.1.

Then we consider the case $l = n/2$. In this case, $\mathcal{E}_{n/2}(g)$ is constant. First it is easy to check that $g_t = e^{-2tu} g_0 \in \Gamma^+_{\frac{n}{2}}$ when $0 \le t \le 1$ (using the fact $(1, \cdots, 1, -1) \in \overline{\Gamma}^+_{\frac{n}{2}}$ when n even). In particular, $\sigma_{\frac{n}{2}}(g_t) > 0$ for $t > 0$. From the expression of $\mathcal{E}_{n/2}(g)$,

$$-\sup(u) \int_M \sigma_{\frac{n}{2}}(g) dg \le \mathcal{E}_{n/2}(g) \le -\inf(u) \int_M \sigma_{\frac{n}{2}}(g) dg.$$

Since

$$\int_M \sigma_{\frac{n}{2}}(g) dg = \int_M \sigma_{\frac{n}{2}}(g_0) dg_0,$$

So we have

$$-\sup(u) \int_M \sigma_{\frac{n}{2}}(g_0) dg_0 \le \mathcal{E}_{n/2}(g) \le -\inf(u) \int_M \sigma_{\frac{n}{2}}(g_0) dg_0.$$

I.e., $\inf(u)$ is bounded from above and $\sup(u)$ is bounded from below. By (11.7) again, u is bounded from above and away from 0. Now we have proved boundedness of $|u|$ in all cases. Hence, we have obtained the C^2 bound for u (independent of T). ■

Theorem 11.1 now is proved for the general case.

Notes

The results in this chapter were proved in [**72, 73**]. The conformal flow we discussed here is a fully nonlinear version of the Yamabe flow treated by Ye [**132**]. When $l = 0$, Proposition 11.1 was proved by Viaclowsky [**126**], the argument there applies directly to the proof of Proposition XXXX. There is a general Liouville type theorem for conformally invariant equations in $\mathbb{R}^n$, proved by Li-Li in [**91, 92**] which implies Proposition 11.1 as a simple consequence.

The argument in the proof of global convergence follows from Simon [**117**]. Though Simon's argument originally designed for quasilinear flow, it was observed by Andrews [**8**]) that it can be adopted to deal with certain fully nonlinear flows evolving convex hypersurfaces in $\mathbb{R}^n$. Here, we adapted it to fully nonlinear conformal flow (11.1).

CHAPTER 12

Geometric inequalities

In this chapter, we are interested in certain global geometric quantities associated to the Schouten tensor and their relationship in conformal geometry. We recall some geometric functionals,

$$\mathcal{F}_k(g) = vol(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g)\, dg, \quad k = 0, 1, ..., n, \tag{12.1}$$

where dg is the volume form of g. When $k = 1$, $\sigma_k(g)$ is a constant multiple of the scalar curvature and $\mathcal{F}_1(g)$ is the Yamabe functional. If we pick a fixed background metric g_0, let $[g_0]$ be the conformal class of g_0. When (M, g_0) is a locally conformally flat manifold and $k \neq n/2$, the critical points of $\mathcal{F}_k$ in $[g_0]$ are the metrics g with

$$\sigma_k(g) = constant. \tag{12.2}$$

When $k = n/2$, $\mathcal{F}_{\frac{n}{2}}(g)$ is a constant in the conformal class. In this case, there is another functional defined by

$$\mathcal{E}_{n/2}(g) \quad = \quad -\int_0^1 \int_M \sigma_{n/2}(g_t) u dg_t dt, \tag{12.3}$$

where u is the conformal factor of $g = e^{-2u} g_0$ and $g_t = e^{-2tu} g_0$. Note that like $\mathcal{F}_k$, this functional is conformally invariant. Unlike $\mathcal{F}_k$, $\mathcal{E}_{n/2}$ depends on the choice of the background metric g_0. However, its derivative $\nabla \mathcal{E}_{n/2}$ *does not* depend on the choice of g_0. The critical points of $\mathcal{E}_{n/2}$ correspond to the metrics g satisfying (12.2) for $k = n/2$. Since any metric $g \in \Gamma^+_{n/2}$ is conformally equivalent to a metric of constant sectional curvature, in the rest of this paper, we will choose the latter metric as a background metric g_0 in (12.3) in this case.

The main objective of here is to establish a complete system of sharp inequalities for $\mathcal{F}_k$'s and $\mathcal{E}_{n/2}$ (if n is even) on locally conformally flat manifolds. The methods we use to establish such inequalities rely on the study of some fully nonlinear parabolic and elliptic equations associated to these geometric quantities. There are three types of inequalities depending on the range of k. More precisely, a *Sobolev type* inequality (12.4) is established for any $k < \frac{n}{2}$ and a *conformal quermassintegral type inequality* (12.7) for any $k \geq n/2$. And, for the exceptional case $k = n/2$, we establish a *Moser-Trudinger type inequality* (12.8) for $\mathcal{E}_{n/2}$.

Before giving precise results, let us first recall some notations and definitions. Let

$$\Gamma^+_k = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \,|\, \sigma_j(\Lambda) > 0, \forall\, j \leq k\}.$$

A metric g is said to be in Γ_k^+ if $\sigma_j(g)(x) > 0$ for $j \leq k$ and $x \in M$ (see [**70**]). For convenience, we set $\sigma_0(A) = 1$ and $\sigma_0(g) = 1$. We denote

$$\mathcal{C}_k = \{g \in [g_0] | g \in \Gamma_k^+\},$$

where $[g_0]$ is the conformal class of g_0.

We now state our main results.

THEOREM 12.1. *Suppose that (M, g_0) is a locally conformally flat manifold, $g_0 \in \Gamma_k^+$ and $g \in \mathcal{C}_k$. Let $0 \leq l < k \leq n$.*

(**A**). **Sobolev type inequality:** *If $0 \leq l < k < \frac{n}{2}$, then there is a positive constant $C_S = C_S([g_0], n, k, l)$ depending only on n, k, l and the conformal class $[g_0]$ such that*

$$(\mathcal{F}_k(g))^{\frac{1}{n-2k}} \geq C_S \left(\mathcal{F}_l(g)\right)^{\frac{1}{n-2l}}. \tag{12.4}$$

If we normalize $\int_M \sigma_l(g) dg = 1$, then the equality holds if and only if

$$\frac{\sigma_k(g)}{\sigma_l(g)} = C_S^{n-2k}. \tag{12.5}$$

There exists $g_E \in \mathcal{C}_k$ attaining the equality. Furthermore,

$$C_S \leq C_S(\mathcal{S}^n) = \binom{n}{k}^{\frac{1}{n-2k}} \binom{n}{l}^{\frac{-1}{n-2l}} \left(\frac{\omega_n^2}{2^n}\right)^{\frac{k-l}{(n-2k)(n-2l)}}, \tag{12.6}$$

where ω_n is the volume of the standard sphere $\mathcal{S}^n$.

(**B**). **Conformal quermassintegral type inequality:** *If $n/2 \leq k \leq n$, $1 \leq l < k$, then*

$$(\mathcal{F}_k(g))^{\frac{1}{k}} \leq \binom{n}{k}^{\frac{1}{k}} \binom{n}{l}^{-\frac{1}{l}} (\mathcal{F}_l(g))^{\frac{1}{l}}. \tag{12.7}$$

The equality in (12.7) holds if and only if (M, g) is a spherical space form.

(**C**). **Moser-Trudinger type inequality:** *If $k = n/2$, then*

$$(n - 2l)\mathcal{E}_{n/2}(g) \geq C_{MT} \left\{ \log \int_M \sigma_l(g) dg - \log \int_M \sigma_l(g_0) dg_0 \right\}, \tag{12.8}$$

where

$$C_{MT} = \int_M \sigma_{n/2}(g_0) dg_0 = \frac{\omega_n}{2^{\frac{n}{2}}} \binom{n}{\frac{n}{2}}.$$

The above inequality is also true for $l > k = n/2$, provided $g \in \mathcal{C}_l$. The equality holds if and only if (M, g) is a space form.

The geometric inequalities and the global estimates established in the chapter 10 will yield some consequences. In chapter 8, it was proved that positive Γ_k-curvature for some $k \geq n/2$ implies positive Ricci curvature. Hence, when the underlying manifold M is locally conformally flat, (M, g) is conformally equivalent to a spherical space form. Therefore, we restrict our attention to the case $k < n/2$.

Let Y_1 the Yamabe constant of $[g]$, i.e.,

$$Y_1([g]) = \inf_{g\in[g]} (vol(g))^{-\frac{n-2}{n}} \int_M \sigma_1(g) dvol(g).$$

Let

$$\mathcal{C}_j = \{g \in [g] \,|\, g \in \Gamma_j^+\}.$$

We define a new conformal invariant for $2 \le k \le n/2$ by

$$Y_k = \begin{cases} \inf\limits_{g\in\mathcal{C}_{k-1}} (vol(g))^{-\frac{n-2k}{n}} \displaystyle\int_M \sigma_k(g) dvol(g) & \text{if } \mathcal{C}_{k-1} \neq \emptyset, \\ -\infty, & \text{if } \mathcal{C}_{k-1} = \emptyset. \end{cases}$$

We note that if $k > n/2$ and $\mathcal{C}_{k-1} \neq \emptyset$, then (M, g) is conformally equivalent to a spherical space form (see [**70**]). Therefore, the only case $k \le n/2$ is of interest to us in this situation.

THEOREM 12.2. *Let (M, g_0) be a compact locally conformal flat n-dim manifold and $k \le n/2$. Assume that $Y_k([g_0]) > 0$, then there is a conformal metric $g \in \mathcal{C}_k$ such that*

$$\sigma_k(g) = 1.$$

If $Y_k([g_0]) = 0$, the either there is $g \in \mathcal{C}_k$ such that $\sigma_k(g) = 1$, or there is $g \in C^{1,1}$ in $\bar{\mathcal{C}}_k$ such that $\sigma_k(g) = 0$.

As an application of Theorem 12.2, we have the following

THEOREM 12.3. *Let (M, g) be an n-dimensional compact, oriented and connected locally conformally flat manifold and $n = 2m$. If g is a metric of positive Γ_{m-1}-curvature and*

$$\int_M \sigma_m(g) dvol(g) > 0, \tag{12.9}$$

then (M, g_0) is conformally equivalent to $\mathbb{S}^{2m}$.

The idea to prove Theorem 12.2 is to seek admissible solution of the following fully nonlinear equation:

$$\sigma_k(g) = \text{constant}, \tag{12.10}$$

for g in the conformal class. In [**29**], Chang-Gursky-Yang proved that if Y_1 and Y_2 (note that Y_1 positive implies $\mathcal{C}_1 \neq \emptyset$, and in the case $n = 4$, $\int_M \sigma_2(g) = Y_2$ for all g in the conformal class) are positive, then there equation (12.10) is solvable for $n = 4, k = 2$. This is an important result because the existence is obtained without the assumption on $\mathcal{C}_k \neq \emptyset$. Here we will deal with the case for higher dimension, but on the locally conformally flat manifolds. The key is to obtain some appropriate a priori estimates for (12.10) by making use of the positivity of Y_k.

Proof of Theorem 12.2. Let $g = e^{-2u} g_0 \in \Gamma_{k-1}^+$. We modify the homotopic approach in [**91**] to consider the following equation

$$f_t(u) = \sigma_k(tg^{-1}S_g + (1-t)\sigma_{k-1}^{1/(k-1)}(g^{-1}S_g)g) = 1. \tag{12.11}$$

Let

$$\Gamma_t = \{\Lambda \in \Gamma_{k-1}^+ \,|\quad t\Lambda + (1-t)\sigma_{k-1}^{1/(k-1)}(\Lambda)I \in \Gamma_k^+\}.$$

It is clear that $\Gamma_0 = \Gamma_{k-1}^+$ and $\Gamma_1 = \Gamma_k^+$. And for any $t \in [0,1]$, f_t satisfies the conditions Proposition 9.1 uniformly in t. From the proof of Theorem 9.2, we may take $g_0 \in \mathcal{C}_{k-1}$ with $\sigma_{k-1}(g_0) = 1$ and the degree of $\sigma_{k-1}(g) = 1$ is -1. From degree argument (e.g., see [**91**]), we only need to show a priori bound on solutions of equation (12.11) for all $0 \le t \le 1$.

For $g = e^{-2u} g_0$, and for any local orthonormal frame (with respect to g_0), we let S_{ij} be the Schouten tensor of g_0 and let $W_u = (u_{ij} + u_i u_j - \frac{|\nabla u|^2}{2}\delta_{ij} + S_{ij})$. Equation (12.11) then can be expressed as:

$$\sigma_k(tW_u + (1-t)\sigma_{k-1}^{1/(k-1)}(W_u)I) = e^{-2ku}. \tag{12.12}$$

By (9.95) in Proposition 9.1, there is C independent of t such that

$$\inf_M u \ge C, \quad \max_M |\nabla u| \le C, \quad \text{and} \quad \max_M |\nabla^2 u| \le C. \tag{12.13}$$

We now only need to obtain an upper bound of u. Set $\tilde{u} = u - \max_M u$. We have $W_{\tilde{u}} = W_u$. By (12.13), $\|\tilde{u}\|_{C^2(M)} \le \tilde{C}$ for some $\tilde{C}$ independent of t. $\tilde{u}$ satisfies equation

$$\sigma_k(tW_{\tilde{u}} + (1-t)\sigma_{k-1}^{1/(k-1)}(W_{\tilde{u}}I)) = e^{-2k\max_M u} e^{-2k\tilde{u}}. \tag{12.14}$$

Expand

$$\sigma_k(tW_{\tilde{u}} + (1-t)\sigma_{k-1}^{1/(k-1)}(W_{\tilde{u}})I) = \sum_{i=0}^{k} \binom{n-i}{n-k} t^i (1-t)^{k-i} \sigma_i(W_{\tilde{u}}) \sigma_{k-1}^{\frac{k-i}{k-1}}(W_{\tilde{u}}). \tag{12.15}$$

Since $W_{\tilde{u}} \in \Gamma_{k-1}$, we have

$$e^{-2k\max_M u} e^{-2k\tilde{u}} = \sigma_k(tW_{\tilde{u}} + (1-t)\sigma_{k-1}^{1/(k-1)}(W_{\tilde{u}})I) \ge t^k \sigma_k(W_{\tilde{u}}) + (1-t)^k \sigma_{k-1}^{\frac{k}{k-1}}(W_{\tilde{u}}). \tag{12.16}$$

Since $e^{\tilde{u}}$ is bound from below and above, integrating the above formula over M with respect to the metric $\tilde{g} = e^{-2\tilde{u}} g$, we get

$$e^{-2k\max_M u} \ge c_1(t^k Y_k + (1-t)^k ((Vol(g))^{2k-n-2} \int_M \sigma_{k-1}(g^{-1}S_g) dvol(g))^{\frac{k}{k-1}}) \tag{12.17}$$

for a constant independent of t, since $\inf_{g \in \mathcal{C}_l} (Vol(\tilde{g}))^{2l-n} \int_M \sigma_l(\tilde{g}^{-1} S_{\tilde{g}}) dvol(\tilde{g})$ is positive for $l \le n/2$ by Theorem 1 in [**73**]. This gives an upper bound of u. In turn $\|u\|_{C^2(M)}$ is bounded independent of t. By the Krylov-Evans theorem and standard elliptic theory, $\|u\|_{C^m(M)}$ is bounded for any m. The Theorem is proved for the case $Y_k > 0$.

If $Y_k = 0$, By (12.13), u is bounded from below, and the first and second derivatives of u are bounded independent of t. By (12.17), for any $t < 1$, u is bounded from above (depending on t). If $\sup u \to \infty$ for some sequence $t_j \to 1$, from (12.14) we obtain a $C^{1,1}$ solution $g \in \bar{\Gamma}_k^+$ with

$$\sigma_k(g) = 0.$$

If for some sequence $t_j \to 1$, $\sup u$ stay bounded, we obtain a solution $g \in \Gamma_k^+$ with

$$\sigma_k(g) = 1.$$

These two cases can not be happen at the same time by Lemma 2 in [**73**]. ■

We now prove the geometric inequalities. From the flow approach we developed in chapter 11, we have

PROPOSITION 12.1. *Let (M, g_0) be a compact, connect and oriented locally conformally flat manifold with $g_0 \in \Gamma_k^+$ and $0 \le l < k \le n$. There is a $g_E \in \mathcal{C}_k$ satisfying equation (12.5) such that*

$$\tilde{\mathcal{F}}_k(g_E) \le \tilde{\mathcal{F}}_k(g),$$

for any $g \in \mathcal{C}_k$ with $\tilde{\mathcal{F}}_l(g_E) = \tilde{\mathcal{F}}_l(g)$. Moreover, if (M, g_0) is conformally equivalent to a space form, then (M, g_E) is also a space form.

Proof: The case $l = 0$ has been treated in Proposition 11.6. We may assume $l \ge 1$ in the rest of proof. When (M, g_0) is conformally equivalent to a space form, Proposition 11.1 implies that any solutions of (12.5) are metrics of constant sectional curvature, and hence have the same $\tilde{\mathcal{F}}_k$ if they have been the same $\tilde{\mathcal{F}}_l$. Hence the Proposition follows from Theorem 11.1.

Now we remain to consider the case $k < n/2$ and (M, g_0) is not conformally equivalent to a space form. We will follow the same argument in the proof of Proposition 11.6. Here we need the local estimates in Theorem 8.1 for the quotient equation (12.5).

First we want to show

$$\inf_{g \in \mathcal{C}_k, \tilde{\mathcal{F}}_l(g)=1} \tilde{\mathcal{F}}_k(g) =: \beta_0 > 0. \tag{12.18}$$

Suppose $\beta_0 = 0$. By the result for flow (11.1), there is a sequence $g_i = e^{-2u_i} g_0 \in \mathcal{C}_k$ with $\tilde{\mathcal{F}}_l(g_i) = 1$ and

$$\frac{\sigma_k}{\sigma_l}(g_i) = \beta_i, \quad \lim_{i \to \infty} \beta_i = 0,$$

The scaled metric $\tilde{g}_i = e^{-2\tilde{u}_i} g_0$ with $\tilde{u}_i = u_i - \frac{1}{2(k-l)} \log \beta_i$ satisfies

$$\frac{\sigma_k}{\sigma_l}(\nabla^2 \tilde{u}_i + d\tilde{u}_i \otimes d\tilde{u}_i - \frac{|\nabla \tilde{u}_i|^2}{2} g_0 + S_{g_0}) = e^{-2(k-l)\tilde{u}_i}. \tag{12.19}$$

By Proposition 1,

$$C vol(\tilde{g}_i)^{\frac{n-2l}{n}} \le \tilde{\mathcal{F}}_l(\tilde{g}_i) = \beta_i^{\frac{n-2l}{2(k-l)}} \to 0 \quad \text{as } i \to \infty,$$

We want show that

$$m_i := \inf_M \tilde{u}_i \to +\infty \quad \text{as} \quad i \to \infty.$$

This can be done follows: suppose there exist a constant c_0 and a subsequence (which we will still denote as $\{m_i\}$) with $m_i \le c_0, \forall i$. At the minimum point x_i of $\tilde{u}_i$, we may pick a positive constant $b > 0$ depending only on n and c_0 such that for any geodesic ball (with respect to g_0) of radius $r < 1$,

$$vol(B_r) \ge b r^n, \tag{12.20}$$

and injectivity radius of g_0 is greater than $2be^{m_i}$. Now we take $r = be^{m_i}$, for $x \in B_r$, there is $x_i^* \in B_r$, such that

$$|e^{-n\tilde{u}_i(x)} - e^{-n\tilde{u}_i(x_i)}| \le |\nabla u(x_i^*)| r.$$

By gradient estimates in Theorem 8.1, since $e^{-m_i} > e^{-c_0}$, $\forall x \in B_r$,

$$|\nabla u(x)| \le \tilde{C} e^{-m_i},$$

where $\tilde{C}$ depending only on g_0 and c_0 (note that $f = 1$ in (12.19)). If $b > 0$ is chosen small enough (independent of i) so that $b < \frac{\tilde{C}e^{-nc_0}}{2}$, then for any $x \in B_r$, $e^{-n\tilde{u}_i(x)} \geq \frac{1}{2}e^{-nm_i}$. Together with (12.20),

$$0 \leftarrow vol(\tilde{g}_i) \geq \int_{B_r} e^{-n\tilde{u}_i} dg_0 \geq \frac{1}{2}e^{-nm_i} vol(B_r) \geq \frac{b}{2}e^{-nm_i} r^n = \frac{b^{n+1}}{2}.$$

This is a contradiction. Therefore, we must have $m_i \to +\infty$.

Now at the minimum point x_i of $\tilde{u}_i$, by equation (12.19),

$$\frac{\sigma_k}{\sigma_l}(S_{g_0}) \leq \frac{\sigma_k}{\sigma_l}(\nabla^2 \tilde{u}_i + d\tilde{u}_i \otimes d\tilde{u}_i - \frac{|\nabla \tilde{u}_i|^2}{2} g_0 + S_{g_0}) = e^{-2(k-l)m_i} \to 0.$$

This is a contradiction to the fact $g_0 \in \Gamma_k^+$.

Finally we prove the existence of an extremal metric in this case. From above argument, there is a minimization sequence $g_i \in \mathcal{C}_k$, with $\tilde{\mathcal{F}}_l(g) = 1$, and $\frac{\sigma_k(g_i)}{\sigma_l(g_i)} = \beta_i$, with β_i decreasing and bound below by a positive constant. As (M, g_0) is not conformally equivalent to $\mathcal{S}^n$ by assumption, it follows from Theorem 1.3 in [**64**] that the metrics converge (by taking a subsequence) to some g_E which attains the infimum C_S. ■

Proof of (**B**) *of Theorem 12.1.* The cases $l = n/2$ and $k = n/2$ were considered in [**72**] and [**70**]. Hence we assume that $k \neq n/2$ and $l \neq n/2$. Let us consider

$$\mathcal{F}_{k,l}(g) = \left(\int \sigma_l(g) dg\right)^{-\frac{n-2k}{n-2l}} \int_M \sigma_k(g) dg.$$

Since $\mathcal{F}_{k,l}$ is invariant under the transformation g to $e^{-2a}g$ for any constant a, Proposition 12.1 implies that for any $g \in \mathcal{C}_k$

$$\mathcal{F}_{k,l}(g) \leq \mathcal{F}_{k,l}(g_E) =: C(n,k,l).$$

It is clear that $C(n,k,l)$ depends only on n, k, l.

Hence, we may assume that Let $c_0 = \mathcal{F}_{k,l}(g_E)$. From Proposition 12.1, we have

$$\begin{aligned} \int_M \sigma_k(g) dg &\leq C(n,k,l) \left(\int_M \sigma_l(g) dg\right)^{\frac{n-2k}{n-2l}} \\ &= C(n,k,l) \left(\int_M \sigma_l(g) dg\right)^{\gamma k} \left(\int_M \sigma_l(g) dg\right)^{\frac{k}{l}}, \end{aligned} \tag{12.21}$$

where $\gamma = \frac{n-2k}{k(n-2l)} - \frac{1}{l}$. It is clear that $\gamma > 0$ when $l > n/2$ and $\gamma < 0$ when $l < n/2$.

We first consider the case $l > n/2$. In this case, by Proposition 11.6 we have

$$\int_M \sigma_l(g) dg \leq c_1 vol(g)^{\frac{n-2l}{n}},$$

where $c_1 = \mathcal{F}_l(g_e)$. It follows that

$$\left(\int_M \sigma_l(g) dg\right)^{\gamma} \leq c_0^{\gamma} vol(g)^{\frac{l-k}{kl}}. \tag{12.22}$$

Hence

$$\begin{aligned}(\mathcal{F}_k(g))^{1/k} &= \left(vol(g)^{-\frac{n-2k}{n}}\int_M \sigma_k(g)dg\right)^{\frac{1}{k}}\\ &\leq c_0^{\frac{1}{k}}\left(vol(g)^{-\frac{n-2l}{n}}\int_M \sigma_l(g)dg\right)^{\frac{1}{l}}\\ &= c_0^{\frac{1}{k}}(\mathcal{F}_l(g))^{1/l}.\end{aligned}$$

The equality holds if and only if g is a metric of constant sectional curvature.

Consider the case $l < n/2$. In this case, by Proposition 11.6 again we have

$$\int_M \sigma_l(g)dg \geq c_1 vol(g)^{\frac{n-2l}{n}},$$

where $c_1 = \mathcal{F}_l(g_e)$. Since $\gamma < 0$, we have (12.22). The same argument given in the previous case gives the same conclusion.

Finally, since $k \geq n/2$, (M, g_0) is conformally equivalent to a space form ([**70**]). The existence of the extremal metric which attains the equality case follows the uniqueness result in Proposition 11.1. And the constant $C(n,k,l)$ is easy to calculate. ■

Proof of (**A**) *of Theorem 12.1.* Inequality (12.4) follows from (12.18) in the proof of Proposition 12.1. The existence of the extremal metric has also proved there. The inequality $C_S \leq C_S(\mathcal{S}^n)$ will be established later (Theorem 12.4). The constant $C_S(\mathcal{S}^n)$ can be computed easily. ■

Proof of (**C**) *of Theorem 12.1.* Let us first consider the case $l < n/2$. Let $g \in \mathcal{C}_{n/2}$. Choose a such that $\int_M \sigma_l(e^{-2a}g)dvol(e^{-2a}g) = \int_M \sigma_l(g_0)dg_0$. It is easy to see that

$$a = \frac{1}{n-2l}\{\log\int_M \sigma_l(g)dg - \log\int_M \sigma_l(g_0)dg_0\}.$$

By Proposition 12.1, we have

$$\begin{aligned}\mathcal{E}_{n/2}(g) &= \mathcal{E}_{n/2}(e^{-2a}g) + a\int_M \sigma_{n/2}(g)dg\\ &\geq a\int_M \sigma_{n/2}(g_0)dg_0\\ &= \frac{1}{n-2}\int_M \sigma_{n/2}(g_0)dg_0\left\{\log\int_M \sigma_l(g)dg - \log\int_M \sigma_l(g_0)dg_0\right\}.\end{aligned}$$

This proves the Theorem for the case $l < n/2$.

Now we consider the case $l > n/2$. 12.1. For any $g \in \mathcal{C}_l$ we choose

$$a = (\int_M \sigma_{n/2}(g)dg)^{-1}\mathcal{E}_{n/2}(g)$$

such that $\mathcal{E}_{n/2}(e^{-2a}g) = \mathcal{E}_{n/2}(g_0)$. Recall that $\tilde{\mathcal{F}}_{n/2} = \mathcal{E}_{n/2}$. By Proposition 12.1 again, we have

$$\begin{aligned}
\tilde{\mathcal{F}}_l(g) &= \frac{1}{n-2l}\int_M \sigma_l(g)dg \\
&= \frac{1}{n-2l}e^{-(2l-n)a}\int_M \sigma_l(e^{-2a}g)dvol(e^{-2a}g) \\
&\geq \frac{1}{n-2l}e^{-(2l-n)a}\int_M \sigma_l(g_0)dg_0 \\
&= \frac{1}{n-2l}\exp\left\{(n-2l)\left(\int_M \sigma_{n/2}(g)dg\right)^{-1}\mathcal{E}_{n/2}(g)\right\}\int_M \sigma_l(g_0)dg_0.
\end{aligned}$$

Since (M, g_0) is conformally equivalent to a space form in this case, the existence of the extremal metric can be proved along the same line as in part (**B**) of the Theorem. Note that since n is even, (M, g_0) is the standard sphere. The computation of C_{MT} is straightforward. ■

We now address the question of the best constant in part (**A**) of Theorem 12.1. As in the Yamabe problem (i.e., $k = 1$ and $l = 0$), for $0 \leq l < k < n/2$ we define

$$Y_{k,l}(M, [g_0]) = \inf_{g\in\mathcal{C}_k}(\mathcal{F}_l(g))^{-\frac{n-2k}{n-2l}}\mathcal{F}_k(g) = \inf_{g\in\mathcal{C}_k}(\int_M \sigma_l(g)dg)^{-\frac{n-2k}{n-2l}}\int_M \sigma_k(g)dg.$$

It is clear that $Y_{k,l}(M, [g_0]) = C_s^{n-2k}$. In this section we prove

THEOREM 12.4. *For any compact, oriented locally conformally flat manifold (M, g_0), we have*

$$Y_{k,l}(M, [g_0]) \leq Y_{k,l}(\mathcal{S}^n, g_{\mathcal{S}^n}), \tag{12.23}$$

where $g_{\mathcal{S}^n}$ is the standard metric of the unit sphere.

When $k = 1$ and $l = 0$, this was proven by Aubin (e.g., see [**11**]) for general compact manifolds. To prove Theorem 12.4 we need to construct a sequence of "blow-up" functions which belong to $\mathcal{C}_k$. This is a delicate part of the problem.

We need two Lemmas.

LEMMA 12.1. *Let D be the unit disk in $\mathbb{R}^n$ and ds^2 the standard Euclidean metric. Let $g_0 = e^{-2u_0}ds^2$ be a metric on D of positive Γ_k-curvature with $k < n/2$. Then there is a conformal metric $g = e^{-2u}ds^2$ on $D\backslash\{0\}$ of positive Γ_k-curvature with the following properties:*

1). *$\sigma_k(g) > 0$ in $D\backslash\{0\}$.*
2). *$u(x) = u_0(x)$ for $r = |x| \in (r_0, 1]$.*
3). *$u(x) = a + \log r$ for $r = |x| \in (0, r_3)$ and some constant a.*

for some constants r_0 and r_3 with $0 < r_3 < r_0 < 1$.

Proof: Let v be a function on D and $\tilde{g} = e^{-2v} g_0$. By the transformation formula of the Schouten tensor, we have

$$\begin{aligned} S(\tilde{g})_{ij} &= \nabla^2_{ij}(v+u_0) + \nabla_i(v+u_0)\nabla_j(v+u_0) - \frac{1}{2}|\nabla(v+u_0)|^2\delta_{ij} \\ &= \nabla^2_{ij} v + \nabla_i v \nabla_j v + \nabla_i v \nabla_j u_0 + \nabla_j v \nabla_i u_0 \\ &\quad + (\frac{1}{2}|\nabla v|^2 + \nabla v \nabla u_0)\delta_{ij} + S_{g_0} \end{aligned} \tag{12.24}$$

Here ∇ and ∇^2 are the first and the second derivatives with respect to the standard metric ds^2. Let $r = |x|$. We want to find a function $v = v(r)$ with $\tilde{g} \in \Gamma_k^+$ and

$$v' = \frac{\alpha(r)}{r},$$

where $\alpha = 1$ near 0 and $\alpha = 0$ near 1. From (12.24) we have

$$S(\tilde{g})_{ij} = \frac{2\alpha - \alpha^2}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2} + S(g_0)_{ij} + O(|\nabla u_0|)\frac{\alpha}{r}, \tag{12.25}$$

where $O(|\nabla u_0|)$ is a term bounded by a constant C_1 depending only on $\max|\nabla u_0|$. Let $A(r)$ be an $n \times n$ matrix with entry $a_{ij} = S(\tilde{g})_{ij} - S(g_0)_{ij}$. Hence

$$\sigma_k(\tilde{g}) = e^{-2k(v+u_0)}\sigma_k\left(A + S(g_0)\right).$$

To our aim, we need to find α such that $A + S(g_0) \in \Gamma_k^+$. Let $\varepsilon \in (0, 1/2)$ and $r_0 = \min\{\frac{1}{2}, C_1\varepsilon\}$. We will choose α such that

$$\alpha(r) \in [0,1] \text{ and } \alpha(r) = 0, \text{ for } r \in [r_0, 1]. \tag{12.26}$$

Since $\sigma_k(\tilde{g}) = e^{2k(v+u_0)}\sigma_k(A(r) + S(g_0))$, we want to find α such that $\sigma_k(A(r) + S(g_0)) > 0$. It is clear to see that for $r \in [0, r_0]$

$$A(r) \geq \left(\frac{2\alpha - \alpha^2 - \varepsilon\alpha}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2}\right),$$

as a matrix. This implies that

$$\sigma_k(A(r)) \geq \frac{(n-1)!}{k!(n-k)!}\left(\frac{2\alpha - \alpha^2 - \varepsilon\alpha}{2r^2}\right)^k \left(n - 2k + 2\frac{r\alpha' - \varepsilon\alpha}{2\alpha - \alpha^2 - \varepsilon\alpha}\right). \tag{12.27}$$

One can easily check that for any small $\delta > 0$,

$$\alpha(r) = \frac{2(1-\varepsilon)\delta}{\delta + r^{\frac{1-\varepsilon}{2}}} \tag{12.28}$$

is a solution of

$$(2-\varepsilon)\alpha - \alpha^2 = -4(r\alpha' - \varepsilon\alpha).$$

Now we can finish our construction of α. Since $S(g_0) \in \Gamma_k^+$, by the openness of Γ_k^+ we can choose $r_1 \in (0, r_0)$ and an non-increasing function $\alpha : [r_1, r_0] \subset [0, 1)$ such that $\sigma_k(\tilde{g}) > 0$ and $\alpha(r_1) > 0$. Now we choose a suitable $\delta > 0$ and α in the form (12.28).Then find $r_2 \in (0, r_1)$ with $\alpha(r_2) = 1$. It is clear that $\sigma_k(A(r)) > 0$ on $[r_2, r_1]$. Define $\alpha(r) = 1$ on $[0, r_2]$. We may smooth α such that the new resulted conformal metric g satisfying all conditions in Lemma 12.1. ■

REMARK 12.1. *From Lemma 12.1, one can prove that the connected sum of two locally conformally flat manifolds* (M_1, g_1) *and* (M_2, g_2) *with* $g_1, g_2 \in \Gamma_k$ $(k < n/2)$ *admits a locally conformally flat structure with a metric in* Γ_k^+. *This is also true for general manifolds, which will appear in a forthcoming paper.*

LEMMA 12.2. *For any small constants* $\delta > 0$ *and* $\varepsilon > 0$, *there exists a function* $u : \mathbb{R}^n \backslash \{0\} \to 0$ *satisfying:*

1. *The metric* $g = e^{-2u} dx^2$ *has positive* Γ_k*-curvature.*
2. $u = \log(1 + |x|^2) + b_0$ *for* $|x| \geq \delta$, *i.e.,* $(\{x \in \mathbb{R}^n \,|\, |x| \geq \delta\}, g)$ *is a part of a sphere.*
3. $u = \log |x|$ *for* $|x| \leq \delta_1$, *i.e.,* $(\{x \in \mathbb{R}^n \,|\, 0 < |x| \leq \delta_1\}, g)$ *is a cylinder.*
4. $vol(B_\delta \backslash B_{\delta_1}, g) \leq C\delta^{-\frac{2n}{1-\varepsilon_0}}$.
5. $\int_{B_\delta \backslash B_{\delta_1}} \sigma_k(g) dvol(g) \leq C\delta^{-\frac{2(n-2k)}{1-\varepsilon_0}}$, *for any* $k < n/2$,

where C *is a constant independent of* δ, $\delta_1 = \delta^{\frac{3-\varepsilon_0}{1-\varepsilon_0}}$ *and* $b_0 \sim \frac{3-\varepsilon_0}{1-\varepsilon_0} \log \delta$.

Proof: Let $\delta \in (0, 1)$ be any small constant. For any small constant $\varepsilon_0 > 0$, we define u by

$$u(r) = \begin{cases} \log(1 + r^2) + b_0, & r \geq \delta \\ -\dfrac{2}{1-\varepsilon_0} \log \dfrac{1 + \delta^{3-\varepsilon_0} r^{-(1-\varepsilon_0)}}{2} + \dfrac{3-\varepsilon_0}{1-\varepsilon_0} \log \delta & r \in (\delta_1, \delta) \\ \log r, & r \leq \delta_1, \end{cases},$$

where $\delta_1 = \delta^{\frac{3-\varepsilon_0}{1-\varepsilon_0}}$ and

$$b_0 = -\log(1 + \delta^2) - \frac{2}{1-\varepsilon_0} \log \frac{1 + \delta^2}{2} + \frac{3-\varepsilon_0}{1-\varepsilon_0} \log \delta.$$

As in the proof of Lemma 5, we write $u'(r) = \frac{\alpha(r)}{r}$. It is easy to see that $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\alpha(r) = \begin{cases} \dfrac{2r^2}{1 + r^2}, & r \geq \delta, \\ \dfrac{2\delta^{3-\varepsilon_0}}{\delta^{3-\varepsilon_0} + r^{1-\varepsilon_0}}, & r \in (\delta_1, \delta), \\ 1, & r \leq \delta_1. \end{cases}$$

One can check all conditions in the Lemma, except the smoothness of u, which is $C^{1,1}$. We first check (1). By a direct computation, see for example (13), we have

$$\sigma_k(e^{-2u}|dx|^2) = e^{2ku(r)} \frac{(n-1)!}{k!(n-k)!} \left(\frac{2\alpha - \alpha^2}{2r^2} \right)^k \left(n - 2k + 2\frac{r\alpha'}{2\alpha - \alpha^2} \right).$$

In the interval (δ_1, δ), $\alpha \in (0, 2)$ satisfies

$$\frac{2r\alpha'}{2\alpha - \alpha^2} = -(1 - \varepsilon_0).$$

Since $k < n/2$, we have $\sigma_k(e^{-2u}|dx|^2) > 0$. One can also directly to check (4) and (5). Here we only check (5). A direct computation gives

$$\begin{aligned}\int_{B_\delta \setminus B_{\delta_1}} \sigma_k(g) dvol(g) &\leq c\int_{\delta_1}^{\delta} e^{-(n-2k)u(r)} r^{-2k} r^{n-1} dr \\ &\leq c\delta^{-(n-2k)\frac{3-\varepsilon_0}{1-\varepsilon_0}} \int_{\delta_1}^{\delta} r^{n-2k-1} dr \\ &\leq c\delta^{-\frac{2(n-2k)}{1-\varepsilon_0}}.\end{aligned}$$

From our construction, we only have $u \in C^{1,1}$. But, for $\delta > 0$ fixed, we can smooth α so that $u \in C^\infty$ satisfies all conditions (1)-(5). ■

Proof of Theorem 12.4. Let $p \in M$ and U a neighborhood of p such that (U, g) is conformally flat, namely $(U, g) = (D, e^{-2u_0}|dx|^2)$. Applying Lemma 12.1, we obtain a conformal metric u satisfying conditions 1)-3) in Lemma 12.1 with constants r_0, r_3 and a. By adding a constant we may assume $a = 0$. Now applying Lemma 12.2 for any small constant $\delta > 0$ we have a conformal metric $g_\delta = e^{-2u_\delta}|dx|^2$ on $\mathbb{R}^n \setminus \{0\}$. Consider the rescaled function

$$\tilde{u}_\delta = u_\delta(\frac{\delta_1}{r_3}x) - \log\frac{\delta_1}{r_3}.$$

Now u and $\tilde{u}_\delta$ are the same in $\{0 < |x| < r_3\}$. Consider the following conformal transformation

$$f(x) = \frac{r_3^2}{2}\frac{x}{|x|^2},$$

which maps $\{r_3/2 \leq |x| \leq r_3\}$ into itself and maps one of boundary components to another with opposite orientations. Now we define a new function on M by

$$w_\delta(x) = \begin{cases} 0, & |x| \geq r_0, \\ u - u_0, & r_3/2 \leq |x| \leq r_0, \\ \tilde{u}_\delta(f(x)) + 2\log|x| - \log\frac{r_3^2}{2} - u_0, & |x| \leq r_3/2. \end{cases}$$

Since u and $\tilde{u}_\delta$ are the same in $\{0 < |x| < r_3\}$, it clear that $w_\delta(x)$ is smooth on M. Consider the conformal metric $g_\delta = e^{-2w_\delta}g$ and compute, using Lammas 12.1 and 12.2

$$\begin{aligned}\int_M \sigma_k(g_\delta) dvol(g_\delta) &= \int_{\{|x|\leq r_3/2\}} \sigma_k(g_\delta) dg_\delta + O(1) \\ &= e^{-(n-2k)b_0} \int_{\mathbb{R}^n\setminus\{|x|\leq\delta\}} \sigma_k(g_{\mathcal{S}^n}) dvol(g_{\mathcal{S}^n}) + O(1)\delta^{-\frac{2(n-2k)}{1-\varepsilon_0}} \\ &= \delta^{-\frac{3-\varepsilon_0}{1-\varepsilon_0}(n-2k)} \int_{\mathbb{R}^n\setminus\{|x|\leq\delta\}} \sigma_k(g_{\mathcal{S}^n}) dvol(g_{\mathcal{S}^n}) + o(\delta^{-\frac{3-\varepsilon_0}{1-\varepsilon_0}(n-2k)})\end{aligned}$$

and

$$\int_M \sigma_l(g_\delta) dvol(g_\delta) = \delta^{-\frac{3-\varepsilon_0}{1-\varepsilon_0}(n-2l)} vol(\mathbb{R}^n\setminus\{|x| \leq \delta\}, g_{\mathcal{S}^n}) + o(\delta^{-\frac{3-\varepsilon_0}{1-\varepsilon_0}(n-2l)}),$$

where $g_{\mathcal{S}^n} = \frac{1}{(1+|x|^2)^2}|dx|^2$ is the standard metric of the sphere and $O(1)$ is a term bounded by a constant independent of δ. Now it is readily to see

$$Y_{k,l}(M) \leq \lim_{\delta \to 0} Y_{k,l}(g_\delta) \to Y_{k,l}(\mathcal{S}^n),$$

as $\delta \to 0$. ■

Notes

The main results in this chapter appeared in [**73**], as an application of conformal curvature flow studied in [**72, 73**].

When (M, g_0) is a locally conformally flat manifold and $k \neq n/2$, it was proved in [**126**] that the critical points of $\mathcal{F}_k$ in $[g_0]$ are the metrics g satisfying (12.2). When $k = n/2$, $\mathcal{F}_{\frac{n}{2}}(g)$ is a constant in the conformal class [**126**]. In this case, the functional (12.3 was found in [**20**], see also [**31**].

When $l = 0$ and $k = 1$, inequality (12.4) is the standard Sobolev inequality (e.g., see [**11**]). Inequality (12.7) is of reminiscent in form to the classical quermassintegral inequality (e.g., see [**72**] for the discussion), which is one of the motivations of this paper. In the case $n = 4, k = 2$ and $l = 1$, inequality (12.7) was proved earlier by Gursky in [**76**] for general 4-dimensional manifolds. Some cases of the inequality were also verified in [**72**] and [**70**] for locally conformally flat manifolds. (12.8) is similar to the Moser-Trudinger inequality on compact Riemannian surfaces (see [**101**] and [**80**]). When $l = 0$, (12.8) was proven by Brendle-Viaclovsky and Chang-Yang in [**20**] and [**31**] using a result in [**72**] on a fully nonlinear conformal flow. We also refer to [**14**] for a different form of Moser-Trudinger inequality in higher dimensions. We suspect (12.6) should be true on general compact manifolds.

Note that $\int_M \sigma_m(g) dvol(g)$ is a conformal invariant for $m = n/2$. When $n = 4$, Theorem 12.3 was proved in [**76**]. A similar result was obtained for $n = 6$ in [**76**] under a weaker condition.

The connected sums technique for locally conformally flat manifolds was devised in [**114**] in the case of positive scalar curvatures.

CHAPTER 13

Appendix: Basic facts about concave symmetric functions

We first start with elementary symmetric functions and Garding's theory of hyperbolic polynomials. We recall the definition of k-symmetric functions: For $1 \leq k \leq n$, and $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$,

$$\sigma_k(\lambda) = \sum \lambda_{i_1} ... \lambda_{i_k}, \tag{13.1}$$

where the sum is taken over all strictly increasing sequences $i_1, ..., i_k$ of the indices from the set $\{1, ..., n\}$. The definition can be extended to symmetric matrices by letting $\sigma_k(W) = \sigma_k(\lambda(W))$, where $\lambda(W) = (\lambda_1(W), ..., \lambda_n(W))$ are the eigenvalues of the symmetric matrix W. We also set $\sigma_0 = 1$ and $\sigma_k = 0$ for $k > n$. The following proposition gives explicit algebraic formulas for $\sigma_k(W)$.

Proposition 13.1. *If $W = (W_{ij})$ is an $n \times n$ symmetric matrix, let $F(W) = \sigma_k(W)$ for $1 \leq k \leq n$. Then the following relations hold.*

$$\begin{aligned}
\sigma_k(W) &= \frac{1}{k!} \sum_{\substack{i_1,...,i_k=1 \\ j_1,...,j_k=1}}^{n} \delta(i_1, ..., i_k; j_1, ..., j_k) W_{i_1 j_1} \cdots W_{i_k j_k}, \\
F^{\alpha\beta} &:= \frac{\partial F}{\partial W_{\alpha\beta}}(W) \\
&= \frac{1}{(k-1)!} \sum_{\substack{i_1,...,i_{k-1}=1 \\ j_1,...,j_{k-1}=1}}^{n} \delta(\alpha, i_1, ..., i_{k-1}; \beta, j_1, ..., j_{k-1}) W_{i_1 j_1} \cdots W_{i_{k-1} j_{k-1}} \\
F^{ij,rs} &:= \frac{\partial^2 F}{\partial W_{ij} \partial W_{rs}}(W) \\
&= \frac{1}{(k-2)!} \sum_{\substack{i_1,...,i_{k-2}=1 \\ j_1,...,j_{k-2}=1}}^{n} \delta(i, r, i_1, ..., i_{k-2}; j, s, j_1, ..., j_{k-2}) W_{i_1 j_1} \cdots W_{i_{k-2} j_{k-2}},
\end{aligned}$$

where the Kronecker symbol $\delta(I; J)$ for indices $I = (i_1, ..., i_m)$ and $J = (j_1, ..., j_m)$ is defined as

$$\delta(I; J) = \begin{cases} 1, & \text{if } I \text{ is an even permutation of } J; \\ -1, & \text{if } I \text{ is an odd permutation of } J; \\ 0, & \text{otherwise.} \end{cases}$$

The Newton-MacLaurin inequality for the elementary symmetric functions is fundamental:

$$(n-q+1)(q+1)\sigma_{q-1}(\Lambda)\sigma_{q+1}(\Lambda) \leq q(n-q)\sigma_q^2(\Lambda). \quad \text{(Newton-MacLaurin inequality)} \tag{13.2}$$

We now introduce Garding's theory of hyperbolic polynomials [**51**] and treat the elementary symmetric functions in that category. We will follow the arguments in [**51**] and [**81**] closely.

Definition 13.1. *Let P be a homogeneous polynomial of degree m in a finite vector space V. For $\theta \in V$ we say P is hyperbolic at θ if $P(\theta) \neq 0$ and the equation $P(x+t\theta) = 0$ (as a polynomial of $t \in \mathbb{C}$) has only real roots for every $x \in V$. We say P is complete if $P(x+ty) = P(x)$ for all x, t implies $y = 0$.*

Proposition 13.2. *Suppose P is hyperbolic at θ, then the component Γ of θ in $\{x \in V; P(x) \neq 0\}$ is a convex cone, the zeros of $P(x+ty)$ (as a polynomial in t) are real if $x \in V$ and $y \in \Gamma$. The polynomial $\frac{P(x)}{P(\theta)}$ is real, and it is positive when $x \in \Gamma$. Furthermore, $(\frac{P(x)}{P(\theta)})^{\frac{1}{m}}$ is concave and homogeneous of degree 1 in Γ, equal to 0 on the boundary of Γ.*

Proof. We may assume $P(\theta) = 1$. Then

$$P(x+t\theta) = (t-t_1) \times ... \times (t-t_m),$$

with real t_j. So $P(x) = (-t_1) \times ... \times (-t_m)$ is real. Set

$$\Gamma_\theta = \{x \in V; P(x+t\theta) \neq 0, t \geq 0\}.$$

Then Γ_θ is open and $\theta \in \Gamma_\theta$ since $P(\theta + t\theta) = (1+t)^m P(\theta)$ only has the zero $t = -1$. Since Γ_θ is open and closed in $\{x \in V; P(x) \neq 0\}$. If $x \in \bar{\Gamma}_\theta$, then $P(x+t\theta) \neq 0$, when $t > 0$. So

$$\Gamma_\theta = \{x \in \bar{\Gamma}_\theta, P(x) \neq 0\}.$$

Also, Γ_θ is connected, for if $x \in \Gamma_\theta$, then $x + t\theta \in \Gamma_\theta$ when $t > 0$. Hence $\lambda x + \mu\theta \in \Gamma_\theta$ for all $\lambda > 0, \mu > 0$. This proves that Γ_θ is starshaped with respect to θ and $\Gamma_\theta = \Gamma$.

If $y \in \Gamma$ and $\delta > 0$ is fixed, then

$$E_{y,\delta} = \{x \in V; P(x + i\delta\theta + isy) \neq 0, Re(s) \geq 0\}$$

is open, and $0 \in E_{y,\delta}$ since for $s \neq 0$, $P(i\delta + isy) = (is)^m P(\frac{\delta\theta}{s} + y) = 0$ implies $s < 0$. If $x \in \bar{E}_{y,\delta}$, then $P(x + i\delta\theta + isy) \neq 0$ by Hurwitz' theorem if$Res > 0$, and this is still true when $Re(s) = 0$ since $x + isy$ is real then. Therefore, $E_{y,\delta}$ is both open and closed, so $E_{y,\delta} = V$ Thus,

$$P(x + i(\delta\theta + y)) \neq 0, \forall x \in \mathbb{R}^n, y \in \Gamma, \delta > 0.$$

Since Γ is open, this remains true for $\delta = 0$. So the equation $P(x+ty) = 0$ has only real roots, for if $t = t_1 + it_2$ is a root with $t_2 \neq 0$ we would get $P(\frac{x+t_1 y}{t_2} + iy) = 0$. This means that y can play the role of θ, so Γ is starshaped with respect to every point in Γ. Γ is convex. We also have $P(y) > 0$ for all $y \in \Gamma$.

We now prove the concavity statement in the proposition. As $P(x+ty)$ has only real roots for $y \in \Gamma$, there are $t_j \in \mathbb{R}$, $j = 1, ..., m$,

$$P(x+ty) = P(y)(t-t_1) \times ... \times (t-t_m).$$

In turn,

$$P(sx+y) = P(y)(1-st_1) \times ...(1-st_m).$$

If $sx + y \in \Gamma$, we must have $1 - st_j > 0$ for every j. If $f(s) = logP(sx + y)$, then

$$f^{'}(s) = -\sum \frac{t_j}{1 - st_j}, \quad f^{"}(s) = -\sum \frac{t_j^2}{(1 - st_j)^2}.$$

Therefore, by Cauchy-Schwarz inequality,

$$m^2 e^{-\frac{f(s)}{m}} \frac{d^2(e^{\frac{f(s)}{m}})}{ds^2} = f^{'}(s)^2 + mf^{"}(s)$$
$$= (\sum \frac{t_j}{1 - st_j})^2 - m \sum \frac{t_j^2}{(1 - st_j)^2} \leq 0.$$

■

We wish to construct some examples of hyperbolic polynomials. If P is a homogeneous polynomial of degree m. For $x^l = (x_1^l, ..., x_n^l) \in V$, $l = 1, ..., m$, we denote $< x^l, \frac{\partial}{\partial x} >= \sum_1^n x_j^l \frac{\partial}{\partial x_j}$ as a vector field. We define the complete polarization of P as

$$\tilde{P}(x^1, ..., x^m) = \frac{1}{m!} < x^1, \frac{\partial}{\partial x} > ... < x^m, \frac{\partial}{\partial x} > P(x).$$

It is a multilinear and symmetric in $x^1, ..., x^m \in V$, independent of x, and that

$$\tilde{P}(x, ..., x) = \frac{1}{m!} \frac{d^m}{dt^m} P(tx) = P(x), \forall x \in V.$$

And

$$P(t_1 x^1 + ... + t_m x^m) = m! t_1 ... t_m \tilde{P}(x^1, ..., x^m) + ...$$

where the dots denote terms not containing all the factors t_j.

Lemma 13.1. *If P is hyperbolic at θ and $m > 1$, then for any $y = (y_1, ..., y_n) \in \Gamma$,*

$$Q(x) = \sum_1^n y_j \frac{\partial}{\partial x_j} P(x)$$

is also hyperbolic at θ. In general, if $x^1, ..., x^l \in \Gamma$ for some $l < m$, then

$$\tilde{Q}_l(x) = \tilde{P}(x^1, ..., x^l, x, ..., x)$$

is hyperbolic at θ.

The proof is immediate. It follows Rolle's theorem. If we repeat the argument, the polynomials $\{P_l\}_1^m$ defined by $P(x + s\theta)$ is hyperbolic at θ if P is.

Corollary 13.1.

1. *The polynomial $P = (x_1)^2 - (x_2)^2 - ... - (x_n)^2$ is hyperbolic at $(1, 0, ..., 0)$.*
2. *The polynomial $P = x_1 ... x_n$ is complete hyperbolic at any θ with $P(\theta) \neq 0$. The positive cone Γ of P at $(1, ..., 1)$ is*
$$\Gamma = \{x = (x_1, ..., x_n); x_j > 0, \quad \forall j\}.$$
3. *In general the elementary symmetric function $\sigma_k(x)$ is complete hyperbolic at $(1, ..., 1)$, the corresponding positive cone Γ_k is*
$$\Gamma_k = \{\sigma_l(x) > 0, \forall l \leq k\}.$$

4. *Let $\mathcal{S}$ denote set of all real $n \times n$ symmetric matrices. Then $\sigma_k(W), W \in \mathcal{S}$ is complete hyperbolic at the identity matrix, the corresponding positive cone is*
$$\Gamma_k = \{\sigma_l(W) > 0, \forall l \leq k\}.$$
5. *, For $W^1, ..., W^l \in \Gamma_k$, $l < k$, then $Q_l(W) = \tilde{P}(W^1, ..., W^l, W, ..., W)$ is complete hyperbolic in Γ_k.*

PROPOSITION 13.3. *Suppose P a homogenous polynomial of degree m, suppose it is hyperbolic at θ and $P(\theta) > 0$, then*
$$P(x^1, ..., x^m) \geq P(x^1)^{\frac{1}{m}} ... P(x^m)^{\frac{1}{m}}, \forall x^1, ..., x^m \in \Gamma. \tag{13.3}$$
If P is complete, the equality holds if and only if all x^j are pairwise proportional. This is also equivalent that for $x, y \in \Gamma$ not proportional, the function $h(t) = P(x + ty)^{\frac{1}{m}}$ is strictly concave in $t > 0$. If P is complete, then $\tilde{Q}_l(X) = \tilde{P}(x^1, ..., x^l, x, ..., x)$ is complete if $m - l \geq 2$ and $x^1, ..., x^l \in \Gamma$. In particular, $\tilde{P}(x^1, ..., x^m) > 0$ if $x^1 \in \bar{\Gamma}$ and $x^j \in \Gamma$ when $m \geq 2$.

Proof. Since $P^{\frac{1}{m}}(X)$ is concave in Γ, it follows that for any $x, y \in \Gamma$, $h(t) = P(x + ty)^{\frac{1}{m}}$ is concave in $t > 0$. So, $h^{"}(t) \leq 0$. A direct computation yields
$$h^{"}(0) = (m-1)(\tilde{P}(y, y, x, ..., x)P(X) - \tilde{P}(y, x, ..., x)^2)P(x)^{\frac{1}{m}-2}.$$
We get the inequality
$$\tilde{P}(y, y, x, ..., x)P(X) \leq \tilde{P}(y, x, ..., x)^2.$$
In turn, it implies
$$\tilde{P}(y, x, ..., x)^m \geq P(y)P(x)^{m-1}.$$
We now apply induction argument. Take $y = x^1$ and assuming that (13.3) is already proved for hyperbolic polynomials of degree $m - 1$. Let $Q(x) = \tilde{P}(y, x, ..., x)$, we get
$$\begin{aligned} \tilde{P}(x^1, ..., x^m) &\geq (Q(x^2)...Q(x^m))^{\frac{1}{(m-1)}} \\ &\geq (P(x^1)P(x^2)^{m-1}...P(x^1)P(x^m)^{m-1})^{\frac{1}{m(m-1)}}, \end{aligned}$$
which proves (13.3).

To prove the last statement in the proposition, it suffices to show that if $m \geq 3$, Q (defined above) is complete. suppose $Q(x) = Q(x + tz)$ for all x, t. In particular, $Q(y + tz) = Q(y)$. That means that $Q(ty + z) = Q(ty)$, so $P(ty + z) - P(ty) = a$ is independent of t. Since the zeros of $P(ty) + a = t^m P(y) + a$ must all be real, it follows that $a = 0$. This $P(y + sz) = P(y) \neq 0$ for all s, so it follows that $y + sz \in \Gamma$. Hence,
$$\frac{(sx + y + sz)}{(s + 1)} \in \Gamma, \forall x \in \Gamma, s > 0.$$
Letting $s \to \infty$, we conclude that $x + z \in \bar{\Gamma}$ for all $x \in \Gamma$. This implies $x + z \in \Gamma$. We can replace z by tz for any t, so $x + tz \in \Gamma$ for all t and $x \in \Gamma$. Thus $P(z + sx)$ can not have any zeros $\neq 0$, so $P(z + sx) = s^m P(x)$. That is $P(x + tz) = P(x)$ for all t and all $x \in \Gamma$. Since P is analytic, that means $P(x + tz) = P(x)$ for all t and all $x \in V$. By the completeness assumption on P, $z = 0$.

Finally, we discuss the equality case in (13.3). By the above, we may assume $m = 2$. If the equality holds, we have $P(y)P(x) = \tilde{P}(y,x)^2$. This implies the roots of the second order polynomial $p(t) = P(x+ty)$ are equal, i.e., $t_1 = t_2 = -\lambda \neq 0$. In turn, for all t,

$$P(y + (t+\lambda)^{-1}(x - \lambda y)) = (t+\lambda)^{-2}P(ty + x) = P(y).$$

That is both roots of the polynomial $f(s) = P(sy + (x - \lambda y))$ are vanishing.

Lemma 13.2. *Suppose P is a second order complete hyperbolic polynomial. Suppose both roots of $f(s) = P(sy + w)$ vanishing for some $y \in \Gamma$ and $w \in V$. Then, all the roots of $g(s) = P(sz + w)$ are vanishing for any $z \in \Gamma$.*

Proof of the lemma. Since $P(y + tw) = P(y) \neq 0$ for all t, we must have $y + tw \in \Gamma$. By the convexity of Γ, we have $z + tw \in \Gamma$ for all t. So, $P(z + tw) \neq 0$. For any $z \in \Gamma$ and all t,

$$P(z)(1 + t\lambda_1)(1 + t\lambda_2) = P(z + tw) \neq 0,$$

λ_1, λ_2 are the roots of $P(sz + w)$. Since t is arbitrary, this gives $\lambda_1 = \lambda_2 = 0$. ■

From the lemma, we have $P(z+t(x-\lambda y)) = P(z)$ for all $z \in \Gamma$ and all t. Since Γ is open and P is analytic, $P(z+t(x-\lambda y)) = P(z)$ for all z and all t. By the completeness of P, $x - \lambda y = 0$. That is, x and y are proportional. ■

Corollary 13.2. *Let $F = \sigma_k^{1/k}$, then the matrix $\frac{\partial F}{\partial W_{ij}}$ is positive definite for $W \in \Gamma_k^+$. where W_{ij} are the entries of W. If $W \in \Gamma_q^+$, then $(W|i) \in \Gamma_{q-1}, \forall q = 0, 1, \cdots, n$, $i = 1, 2, \cdots, n$, where $(W|i)$ is the matrix with i-th column and i-th row deleted.*

The above follows from the strictly concavity of F. The following facts regarding the quotients of elementary symmetric functions will be used in later chapters.

Proposition 13.4.

$$\frac{n!k}{(k-1)!(n-k+1)!(n-k+1)}\sigma_k^{k-1}(\Lambda) \leq \sigma_{k-1}^k(\Lambda), \quad \forall \Lambda \in \Gamma_k^+. \tag{13.4}$$

Γ_q^+ is convex and if $W \in \Gamma_q^+$, then $(W|i) \in \Gamma_{q-1}, \forall q = 0, 1, \cdots, n$, $i = 1, 2, \cdots, n$, where $(W|i)$ is the matrix with i-th column and i-th row deleted. Let $F = (\frac{\sigma_k}{\sigma_l})^{\frac{1}{k-l}}$, then $\frac{\partial F}{\partial w_{ij}}$ is positive definite for $W = (w_{ij}) \in \Gamma_k^+$ and it is semi-positive definite for $W = (w_{ij}) \in \Gamma_{k-1}^+$, and $\sum_j F^{jj} \geq 1$. The function F is concave in Γ_{k-1}^+. If $W = (w_{ij})$ is diagonal with $W = \Lambda$. Then, $\forall i$ fixed,

$$F^{ii} = F^* \sigma_{l-1}(\Lambda_i)\{\sigma_l(\Lambda)\frac{\sigma_{k-1}(\Lambda_i)}{\sigma_{l-1}(\Lambda_i)} - \sigma_k(\Lambda)\}, \tag{13.5}$$

where $F^* = \frac{1}{k-l}\left(\frac{\sigma_k(\Lambda)}{\sigma_l(\Lambda)}\right)^{\frac{1}{k-l}-1}\frac{1}{\sigma_l^2(\Lambda)}$.

Lemma 13.3. *For $F(W) = \frac{\sigma_k(W)}{\sigma_l(W)}$ defined on symmetric matrices with $w \in \Gamma_k^+$, let $F^{ij} = \frac{\partial F(W)}{\partial w_{ij}}$. Suppose W is diagonal, and $w_{ii} = \lambda_i, \forall i = 1, \cdots, n$. Then*

$$F^{ii} \leq F^{jj}, \quad \text{if} \quad \lambda_i \geq \lambda_j.$$

If in addition, $\Lambda_{ij} \in \Gamma_{k-1}^+$, then $F^{ii}\lambda_i^2 \geq F^{jj}\lambda_j^2$ for $\lambda_i \geq \lambda_j$.

Proof: The first statement follows from (13.5) and the monotonicity of σ_{l-1} and $\frac{\sigma_{k-1}}{\sigma_{l-1}}$. We now check $F^{ii}\lambda_i^2 \geq F^{jj}\lambda_j^2$, under the condition that $\Lambda_{ij} \in \Gamma_{k-1}^+$. It is easy to check that for any $m = 1, \cdots, n$,

$$\begin{aligned} \sigma_m(\Lambda_i) &= \sigma_m(\Lambda_{ij}) + \lambda_j \sigma_{m-1}(\Lambda_{ij}), \\ \sigma_m(\Lambda) &= \sigma_m(\Lambda_{ij}) + (\lambda_i + \lambda_j)\sigma_{m-1}(\Lambda_{ij}) + \lambda_i\lambda_j\sigma_{m-2}(\Lambda_{ij}). \end{aligned} \tag{13.6}$$

By (13.6), we compute

$$\begin{aligned} F^{ii}\lambda_i^2 - F^{jj}\lambda_j^2 &= (\lambda_i^2 - \lambda_j^2)[\sigma_l(\Lambda_{ij})\sigma_{k-1}(\Lambda_{ij}) - \sigma_k(\Lambda_{ij})\sigma_{l-1}(\Lambda_{ij})] \\ &\quad + (\lambda_i - \lambda_j)\lambda_i\lambda_j[\sigma_l(\Lambda_{ij})\sigma_{k-2}(\Lambda_{ij}) - \sigma_k(\Lambda_{ij})\sigma_{l-2}(\Lambda_{ij})]. \end{aligned}$$

As $\Lambda_{ij} \in \Gamma_{k-1}^+$, both terms in $[\cdots]$ are positive by the Newton-MacLaurin inequality. ■

The following Garding's inequality is also valid for the quotient of hessians.

LEMMA 13.4. *Let* $\Lambda = (\lambda_1, \cdots, \lambda_n), \Lambda_0 = (\mu_1, \cdots, \mu_n) \in \Gamma_k^+$,

$$F(\Lambda) = \left(\frac{\sigma_k(\Lambda)}{\sigma_l(\Lambda)}\right)^{\frac{1}{k-l}}.$$

Then,

$$\sum_i \{\frac{\sigma_{k-1}(\Lambda_i)}{\sigma_k(\Lambda)} - \frac{\sigma_{l-1}(\Lambda_i)}{\sigma_l(\Lambda)}\}\mu_i \geq (k-l)\frac{F(\Lambda_0)}{F(\Lambda)}.$$

Proof. The main argument of the proof follows from [**26**]. For $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k^+$, set

$$F(\Lambda) = \left(\frac{\sigma_k(\Lambda)}{\sigma_l(\Lambda)}\right)^{\frac{1}{k-l}}.$$

From the concavity of F in Γ_k^+, for $\Lambda, \Lambda_0 = (\mu_1, \cdots, \mu_n) \in \Gamma_k^+$ we have

$$\begin{aligned} F(\Lambda_0) &\leq F(\Lambda) + \sum_l (\mu_i - \lambda_i)\frac{\partial F(\Lambda)}{\partial \lambda_i} \\ &= F(\Lambda) + \frac{1}{k-l}F(\Lambda)\sum_i \{\frac{\sigma_{k-1}(\Lambda_i)}{\sigma_k(\Lambda)} - \frac{\sigma_{l-1}(\Lambda_i)}{\sigma_l(\Lambda)}\}(\mu_i - \lambda_i) \\ &= \frac{1}{k-l}F(\Lambda)\sum_i \{\frac{\sigma_{k-1}(\Lambda_i)}{\sigma_k(\Lambda)} - \frac{\sigma_{l-1}(\Lambda_i)}{\sigma_l(\Lambda)}\}\mu_i. \end{aligned}$$

In the last equality, we have used the fact that F is homogeneous of degree one. Hence, we have

$$\sum_i \{\frac{\sigma_{k-1}(\Lambda_i)}{\sigma_k(\Lambda)} - \frac{\sigma_{l-1}(\Lambda_i)}{\sigma_l(\Lambda)}\}\mu_i \geq (k-l)\frac{F(\Lambda_0)}{F(\Lambda)}.$$

■

We now treat general concave symmetric functions. Let $\Psi \subset \mathbb{R}^n$ be an open symmetric domain and f is a C^2 symmetric function defined in Ψ, denote

$$Sym(n) = \{n \times n \quad \text{real symmetric matrices}\},$$

set

$$\tilde{\Psi} = \{A \in Sym(n) : \lambda(A) \in \Psi\}. \tag{13.7}$$

We extend f to $F : \tilde{\Psi} \to R$ by $F(A) = f(\lambda(A))$. We define $\tilde{F}(A) = -F(A^{-1})$ whenever $A^{-1} \in \tilde{\Psi}$. We define $\dot{f}^k = \frac{\partial f}{\partial \lambda_k}$, $\ddot{f}^{kl} = \frac{\partial^2 f}{\partial \lambda_k \partial \lambda_l}$, $F^{\alpha\beta} = \frac{\partial F}{\partial A_{\alpha\beta}}$ and $F^{\alpha\beta,rs} = \frac{\partial^2 F}{\partial A_{\alpha\beta} \partial A_{rs}}$. The following lemma is well known (e.g., see [**13, 9, 53, 125**]).

Lemma 13.5. (a). *The at any diagonal $A \in \tilde{\Psi}$ with distinct eigenvalues, let $\ddot{F}(B, B)$ be the second derivative of F in direction $B \in Sym(n)$, then*

$$\ddot{F}(B, B) = \sum_{j,k=1}^{n} \ddot{f}^{jk} B_{jj} B_{kk} + 2 \sum_{j<k} \frac{\dot{f}^j - \dot{f}^k}{\lambda_j - \lambda_k} B_{jk}^2. \tag{13.8}$$

(b). *If $\tilde{F}(A) = -F(A^{-1})$ is concave near a positive definite matrix A, then*

$$\sum_{j,k,p,q=1}^{n} (F^{kl,pq}(A) + 2F^{jp}(A) A^{kq}) X_{jk} X_{pq} \geq 0 \tag{13.9}$$

for every symmetric matrix X.

We deduce the following form of Lemma 13.5.

Corollary 13.3. *Assume F satisfies condition in Lemma 13.5(b). Suppose $A \in \tilde{\Psi}$, A is semipositive definite and diagonal. Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ and $\lambda_i > 0, \forall i \geq n - l + 1$. Then*

$$\sum_{j,k=n-l+1}^{n} \ddot{f}^{jk}(A) X_{jj} X_{kk} + 2 \sum_{n-l+1 \leq j<k} \frac{\dot{f}^j - \dot{f}^k}{\lambda_j - \lambda_k} X_{jk}^2 + 2 \sum_{i,k=n-l+1}^{n} \frac{\dot{f}^i(A)}{\lambda_k} X_{ik}^2 \geq 0 \tag{13.10}$$

for every symmetric matrix $X = (X_{jk})$ with $X_{jk} = 0$ if $j \leq n - l$.

Proof. (13.10) follows directly from (13.8) and (13.9) if A is positive definite. For semi-definite A, it follows by approximating. ■

We now further assume that $\Psi = \Gamma$ is a convex cone such that

$$\Gamma \subseteq \Gamma_1. \tag{13.11}$$

and the symmetric function f satisfies the following conditions in Γ:

$$\frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \quad \text{for} \quad i = 1, 2, \ldots, n \quad \text{and} \quad \lambda \in \Gamma, \tag{13.12}$$

$$f \quad \text{is concave in } \Gamma, \tag{13.13}$$

and for $M > 0$, there is $\delta_M > 0$ such that for $\lambda \in \Gamma$ with $F(\lambda) \leq M$,

$$\sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i}(\lambda) \geq \delta_M. \tag{13.14}$$

Set

$$\tilde{\Gamma} = \{W| \quad W \text{ is a symmetric matrix whose eigenvalues } \lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma\}.$$

Define $F(W) = f(\lambda(W))$ for $W \in \tilde{\Gamma}$. We note that since $\Gamma \subset \Gamma_1$, for $W \in \tilde{\Gamma}$, the eigenvalues λ_i of W satisfies $|\lambda_i| \leq (n-1)\lambda_{max}$, where λ_{max} is the largest eigenvalue of W. From a result in section 3 in [**26**], F is concave in Γ implies F is concave in $\tilde{\Gamma}$ and condition (13.12) implies $(\frac{\partial F}{\partial W_{ij}})$ is positive definite for all $W = (W_{ij}) \in \tilde{\Gamma}$. If there is no confusion, we will also simply write Γ for $\tilde{\Gamma}$.

Remark 13.1. *We note that $\sigma_k^{\frac{1}{k}}$ and general quotient operator $(\frac{\sigma_k}{\sigma_l})^{\frac{1}{k-l}}$ $(0 \leq l < k \leq n)$ satisfy the* structure conditions *(13.11)-(13.14) with $\Gamma = \Gamma_k$ and one may take $\delta_M = 1$ for all $M > 0$.*

The condition (13.12) is a monotonicity condition which is natural for the ellipticity of equation (3.9) we will treat in later chapters, as we will see that the concavity condition (13.13) is also crucial for C^2 and $C^{2,\alpha}$ estimates. The condition (13.14) appears artificial, but it follows from some natural conditions on F. For example, in order that equation (3.9) has an admissible solution for some $\tilde{\varphi}$ with $\sup \tilde{\varphi} = M$, there must exist $W \in \Gamma$ such that $F(W) = M$. We may assume $M = 1$. By conditions (13.11)-(13.13), we have

$$F(t_0 I) \geq 1, \quad \text{for some} \quad t_0 > 0, \tag{13.15}$$

where I is the identity matrix.

Lemma 13.6. *Suppose that f satisfies (13.11), (13.13) and (13.15). Set $F^{ij}(W) = \frac{\partial F(W)}{\partial W_{ij}}$ for $W = (W_{ij}) \in \Gamma$.*

(**a**). *Let t_0 be the number in (13.15), then for all $W \in \Gamma$ with $f(W) \leq 1$,*

$$\sum_{i,j} F^{ij}(W) W_{ij} \leq t_0 \sum_i F^{ii}(W). \tag{13.16}$$

(**b**). *Suppose further that f satisfies (13.12), then there is $C > 0$ such that $\forall W \in \Gamma$ with $f(W) \geq 1$, the following is true:*

$$\sigma_1(W) \geq C f(W). \tag{13.17}$$

(**c**). *If in addition, f satisfies*

$$\forall \gamma \in \Gamma, \quad \overline{\lim}_{t \to +\infty} f(t\gamma) > 1; \quad \textit{and for all } p \in \partial\Gamma \quad \overline{\lim}_{\lambda \to p} f(\lambda) < 1, \tag{13.18}$$

then $\sum_{i,j} F^{ij}(W) W_{ij} \geq 0, \forall W \in \Gamma$. And for any compact set K in Γ, there is a $t_K > 0$, such that

$$f(t\gamma) > 1, \quad \textit{for all} \quad \gamma \in K, t \geq t_K. \tag{13.19}$$

Moreover there is $\delta > 0$ such that for all $W \in \mathcal{G}$ with $f(W) \leq 1$, the following is true

$$\delta \leq \delta + \sum_{i,j} F^{ij}(W)W_{ij} \leq 2t_0 \sum_i F^{ii}(W). \tag{13.20}$$

(**d**). *If F satisfies (13.15) and*

$$\overline{\lim}_{t\to+\infty} F(tW) > -\infty, \text{ for all } W \in \Gamma, \tag{13.21}$$

then there is $\delta_M > 0$ depending on F and t_0 in (13.15) such that (13.14) is true.

(**e**). *If F satisfies*

$$\overline{\lim}_{t\to+\infty} F(tW_1 + W_2) > -\infty, \text{ for all } W_1, W_2 \in \Gamma, \tag{13.22}$$

then $\sum_{i,j} F^{ij}(W)W_{ij} > 0$ for all $W \in \Gamma$.

Proof. Let I be the identity matrix. By the concavity of f,

$$f(tI) \leq f(W) + \sum_{i,j} F^{ij}(W)(t\delta_{ij} - W_{ij}). \tag{13.23}$$

By (13.15), $f(t_0 I) \geq 1$. Since $f(W) \leq 1$, (13.16) follows from (13.23).

To prove (13.17), we note $\sigma_1(W)$ is invariant under symmetrization (i.e., symmetrization of eigenvalues of W), while $f(W)$ is non-decreasing under symmetrization by the concavity of f. So we only need to check that if $f(t, \cdots, t) \geq 1$, then $\sigma_1(t, \cdots, t) \geq Cf(t, \cdots, t)$. By (13.12), $f(t, \cdots, t) \geq 1$ implies $t \geq t_0$. From the concavity of f,

$$f(t, \cdots, t) \leq f(t_0 I) + (t - t_0) \sum_i f_{\lambda_i}(t_0, \cdots, t_0) \leq A\sigma_1(t, \cdots, t),$$

if we pick $A \geq \frac{f(t_0 I)}{\sigma_1(t_0, \cdots, t_0)} + \sum_i f_{\lambda_i}(t_0, \cdots, t_0)$.

We note that by concavity assumption on f and the first condition in (13.18), for any $\gamma \in \Gamma$, $f(t\gamma)$ is an increasing function for $t > 0$. This implies

$$\sum_{i,j} F^{ij}(W)W_{ij} \geq 0.$$

By the monotonicity of $f(t\gamma)$ and the first condition in (13.18), for any $\gamma \in \Gamma$, there is $t_\gamma < \infty$ such that $f(t\gamma) > 1$ for all $t \geq t_\gamma$. Then (13.19) follows from the continuity of f and compactness of K in Γ.

By the first condition in (13.18) again, there exists $\delta > 0$ such that $f(2t_0 I) \geq 1 + \delta$ (this also follows from the monotonicity condition (13.12)). Since $f(W) \leq 1$, (13.20) follows from (13.23).

The concavity condition (13.13) and (13.21) implies that $\frac{d}{dt}F(tW) \geq 0$ for all $W \in \Gamma$. That is $\sum_{i,j} F^{ij}(W)W_{ij} \geq 0$ for all $W \in \Gamma$. By the monotonicity condition (13.12), there exists $\epsilon > 0$ such that $F(2t_0 I) \geq M + \epsilon$. Since $F(W) \leq M$, (13.14) follows from (13.23) by letting $t = 2t_0$.

We now prove the last statement in the lemma. Since Γ is open, for each $W \in \Gamma$, there is $\delta > 0$ such that $\tilde{W} = W - \delta I \in \Gamma$. In turn, $t\tilde{W} + \delta I \in \Gamma$ for all $t > 0$. Set $g(t) = F(t\tilde{W} + \delta I)$. By concavity of F and condition (13.22), we have $g'(1) \geq 0$, that is, $\sum_{i,j} F^{ij}(W)\tilde{W}_{ij} \geq 0$. In turn, by condition (13.12) we get $\sum_{i,j} F^{ij}(W)W_{ij} \geq \delta \sum_i F^{ii}(W) > 0$. ∎

Notes

The theory of hyperbolic polynomial was developed by Garding [**51**], our presentation here follows mainly from Garding's original treatment, see also [**81**]. Some important properties of concave symmetric functions were discussed in [**26**].

Bibliography

[1] A. Aeppli, *On the uniqueness of compact solutions for certain elliptic differential equations*, Proc. Amer. Math. Soc. **11**, (1960), 832-836.

[2] A. D. Alexandrov, *Zur theorie der gemishchten volumina von konvexen korpern, II*, Mat. Sbornik, **2**, (1937), 1205-1238.

[3] A.D. Alexandrov, *Zur Theorie der gemischten Volumina von konvexen korpern, III. Die Erweiterung zweeier Lehrsatze Minkowskis uber die konvexen polyeder auf beliebige konvexe Flachen (in Russian)*, Mat. Sbornik N.S. **3**, (1938), 27-46.

[4] A.D. Alexandrov, *Uniqueness theorems for surfaces in the large. I* (Russian), Vestnik Leningrad. Univ. **11** (1956), 5–17. English translation: AMS Translations, series 2, **21**, (1962), 341-354.

[5] A.D. Alexandrov, *Existence and uniqueness of a convex surface with a given integral curvature*, Doklady Acad. Nauk Kasah SSSR **36** (1942), 131-134.

[6] A.D. Alexandrov, *Uniqueness theorems for surfaces in the large I*, Vestnik Leningrad Univ. **11** (1956), 5-17 = Amer. Soc. Trans. Ser.2 **21** (1962), 341-354.

[7] O. Alvarez, J.M. Lasry and P.-L. Lions, *Convexity viscosity solutions and state constraints*, J. Math. Pures Appl. **76**, (1997), 265-288.

[8] B. Andrews, *Monotone quantities and unique limits for evolving convex hypersurfaces,* Internat. Math. Res. Notices **1997** (1997), 1001–1031.

[9] B. Andrews, *Pinching estimates and motion of hypersurfaces by curvature functions*, Preprint, 2004.

[10] B. Andrews and X.N. Ma, *A curvature flow approach to the Christoffel-Minkowski problem*, Preprint, 2002.

[11] T. Aubin, *Équations différentilles non linéaires et probléme de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. **55** (1976), 269-296.

[12] I. Bakelman and B. Kantor, *Existence of spherically homeomorphic hypersurfaces in Euclidean space with prescribed mean curvature*, Geometry and Topology, Leningrad, **1**, (1974), 3-10.

[13] J.M. Ball, *Differentiability properties of symmetric and isotropic functions*, Duke Math. J. **51**, (1984), 699-728.

[14] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math., **138** (1993), 213–242.

[15] A. Besse, *Einstein manifolds*, Springer-Verlag, Berlin, 1987.

[16] G. Besson, G. Courtois and S. Gallot, *Volume et entropie minimale des espaces localement symétriques,* Invent. Math., **103** (1991), 417–445. MR 92d:58027

[17] G. Besson, G. Courtois and S. Gallot, *Entropies et rigidités des espaces localement symétriques de courbure strictement négative,* Geom. Funct. Anal., **5** (1995), 731–799. MR 96i:58136

[18] J. Bourgain and V. Milman, *New volume ratio properties for convex symmetric bodies in* R^n, Invent. Math. **88** (1987), 319–340.

[19] J.-P. Bourguignon, *Les variétés de dimension* 4 *à signature non nulle dont la courbure est harmonique sont d'Einstein,* Invent. Math. **63** (1981), 263–286.

[20] S. Brendle and J. Viaclovsky, *A variational characterization for* $\sigma_{n/2}$, preprint, 2002.

[21] Y.D. Burago and V.A. Zalgaller, *Geometric Inequalities*, Springer, Berlin, 1988.

[22] L.A. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations.* Ann. of Math., (2) **130** (1989), no. 1, 189–213. MR90i:35046

[23] L.A. Caffarelli and Xavier Cabré, *Fully nonlinear elliptic equations*, Amer. Math. Soc. Colloq. Publ., **43**, Amer. Math. Soc., Providence, RI, 1995. MR 96h:35046

[24] L. Caffarelli and A. Friedman, *Convexity of solutions of some semilinear elliptic equations*, Duke Math. J. **52**, (1985), 431-455.

[25] L. Caffarelli, P. Guan and X. Ma, *A convexity principle for solutions of fully nonlinear elliptic equations*, preprint, 2004.

[26] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian*, Acta Math. **155**, (1985),261 - 301.

[27] L.A. Caffarelli, L. Nirenberg and J. Spruck, *Nonlinear second order elliptic equations IV: Starshaped compact Weingarten hypersurfaces*, Current topics in partial differential equations, Y.Ohya, K.Kasahara and N.Shimakura (eds), Kinokunize, Tokyo, (1985), 1-26.

[28] A. Chang, F. Hang and P. Yang, *On a class of locally conformally flat manifolds.* Int. Math. Res. Not. **2004**, no. 4, (2004) 185–209.

[29] A. Chang, M. Gursky and P. Yang, *An equation of Monge-Ampere type in conformal geometry, and four manifolds of positive Ricci curvature*, Ann. of Math. **155** (2002), 709–787.

[30] A. Chang, M. Gursky and P. Yang, *An apriori estiamtes for a fully nonlinear equation on four manifolds*, Dedicated to the memory of Thomas H. Wolff. J. Anal. Math. **87** (2002), 151–186.

[31] A. Chang and P. Yang, *The inequality of Moser and Trudinger and applications to conformal geometry,* special issue in memory of J. Moser, Comm. Pure Appl. Math. **56** (2003), 1135–1150.

[32] C.C. Chen and C.S. Lin, *Local Behavior of Singular Positive Solutions of Semilinear Elliptic Equations With Sobolev Exponent,* Duke Math. J. **78** (1995), 315-334.

[33] C.C. Chen and C.S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes*, Comm. Pure Appl. Math. **50** (1997), 971-1017.

[34] Q.-M., Cheng, Compact locally conformally flat Riemannian manifolds. Bull. London Math. Soc. 33 (2001), no. 4, 459–465.

[35] S.Y. Cheng and S.T. Yau, *On the regularity of the solution of the n-dimensional Minkowski problem* Comm. Pure Appl. Math. **29**, (1976), 495-516.

[36] S.S. Chern, *Integral formulas for hypersurfaces in euclidean space and their applications to uniqueness theorems*, J. Math. Mech. **8-r**, (1959), 947-955.

[37] K.S. Chou(Tso), *On the existence of convex hypersurfaces with prescribed mean curvature*, Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4) **16**, (1989), 225-243.

[38] B. Chow, *The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature,* Comm. Pure Appl. Math. **45** (1992), no. 8, 1003–1014.

[39] T. Colding, *Shape of manifolds with positive Ricci curvature,* Invent. Math., **124** (1996), 175–191.

[40] MR 96k:53067 T. Colding, *Large manifolds with positive Ricci curvature,* Invent. Math., **124** (1996), 193–214. MR 96k:53068

[41] P. Delanoë, *Plongements radiaux* $\mathbb{S}^n \to \mathbb{R}^{n+1}$ *a courbure de Gauss positive prescrite*, Ann. Sci.Ecole Norm. Sup (4) **18**, (1985), 635-649.

[42] L.C. Evans, *Classical solutions of fully nonlinear, convex, second order elliptic equations*, Comm. Pure Appl. Math. **35** (1982), 333-363.

[43] C. Fefferman and C. R. Graham, *Conformal invariants*, Astrisque (1985), Numero Hors Serie, 95–116.

[44] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc., **93** (1959), 418-491.

[45] W. Fenchel and B. Jessen, *Mengenfunktionen und konvexe korper*, Det. Kgl. Danske Videnskab. Selskab, Math.-fys. Medd. **16**(3), (1938), 1-31.

[46] W.J. Firey, *The determination of convex bodies from their mean radius of curvature functions*, Mathematik, **14-r**, (1967), 1-14.

[47] W. J. Firey, *Intermediate Christoffel-Minkowski problems for figures of revolution*, Israel J. Math., **8** (1970), 384-390.

[48] P. Gajer, *Concordances of metrics of positive scalar curvature,* Pacific J. Math. **157** (1993), 257–268.

[49] L. Gao and S.T. Yau, *The existence of negatively Ricci curved metrics on three-manifold.* Invent. Math. **85** (1986), 637–652. MR 87j:53061

[50] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243.

[51] L. Garding, *An inequality for hyperbolic polynomials*, J. Math. Mech, **8**, (1959), 957-965.

[52] C. Gerhardt, *Closed Weingarten hypersurfaces in space forms*, Geometric Analysis and the Calculus of Variation, Fort, F., (eds), International Press, Boston, (1996), 71-98.

[53] C. Gerhardt, *Closed Weingarten hypersurfaces in Riemannian manifolds*, J. Diff. Geometry, **43**, (1996), 612-641.

[54] D. Gilgarg and N. Trudinger *Elliptic Partial Differential Equations of Second Orders*, Springer, (2001).

[55] M. Gromov, *Volume and bounded cohomology,* Publ. Math. I.H.E.S., **56** (1981) 213–307. MR 84h:53053

[56] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces,* Progress in Math., **152** Birkhäuser Boston, Inc., Boston, MA, 1999. MR 2000d:53065

[57] B. Guan and P. Guan, *Hypersurfaces with prescribed curvatures*, Annals of Mathematics, **156** (2002), 655-674.

[58] P. Guan, C^2 *a priori estimates for degenerate Monge-Ampere equations*, Duke Math. J., **86**, (1997), 323-346.

[59] P. Guan and Y.Y. Li, $C^{1,1}$ *estimates for solutions of a problem of Alexandrov*, Comm. Pure and Appl. Math., **50**, (1997), 789-811.

[60] P. Guan and Y.Y. Li, *unpublished notes*, 1995.

[61] P. Guan and C.S. Lin, *On the equation* $det(u_{ij} + \delta_{ij}u) = u^p f(x)$ *on* $\mathbb{S}^n$, preprint, NCTS in Tsing-Hua University, 2000.

[62] P. Guan, C.S. Lin and X.N. Ma, *The Christoffel-Minkowski problem II: Weingarten curvature equations*, preprint, 2002.

[63] P. Guan, C.S. Lin and X.N. Ma, *The Christoffel-Minkowski problem: prescribing curvature measure equation*, preprint, 2002.

[64] P. Guan, C.S. Lin and G. Wang, *Applications of method of moving planes to conformally invariant equations,*, Mathematische Zeitschrift, **247** (2004), pp. 1-19.

[65] P. Guan, C.S. Lin and G. Wang, *Schouten tensor and some topological properties*, to appear in Comm. Geometric Analysis.

[66] P. Guan, C.S. Lin and G. Wang, *Local gradient estimates for conformal quotient equations*, preprint, 2002.

[67] P. Guan and X. Ma, *Cheristoffel-Minkowski problem I: convexity of solutions of a Hessian Equation*, Inventiones Mathematicae, **151** (2003), 553-577.

[68] P. Guan, X.N. Ma and F. Zhou, *The Christoffel-Minkowski problem III: existence and convexity of admissible solutions*, to appear in comm. Pure & Applied Math.

[69] P. Guan, X. Ma, N. Trudinger and X. Zhu, *A form of Alexandrov-Fenchel inequality*, Preprint, Max-Planck Institute for Mathematics in Sciences, Leipzig, (2001).

[70] P. Guan, J. Viaclovsky and G. Wang, *Some properties of Schouten tensor and applications to conformal geometry*, Transactions of American Math. Society, **355** (2003), 925-933.

[71] P. Guan and G. Wang, *Local estimates for a class of fully nonlinear equations arising from conformal geometry*, International Mathematics Research Notices, **2003**, Issue 26(2003), 1413-1432 .

[72] P. Guan and G. Wang, *A fully nonlinear conformal flow on locally conformally flat manifolds*, Journal fur die reine und angewandte Mathematik, **557** (2003), 219-238.

[73] P, Guan and G,. Wang, *Geometric inequalities on locally conformally flat manifolds*, Duke Math. Journal, **124**, (2004), 177-212.

[74] P. Guan and G. Wang, *Conformal deformations of the smallest eigenvalue of the Ricci tensor*, preprint, http://xxx.lanl.gov/archive/math: math.DG/0505083.

[75] P. Guan and X.J. Wang, *On a Monge-Ampère equation arising in geometric optics*, J. Differential Geom. **48** (1998), 205–223.

[76] M. Gursky, *The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE*, Comm. Math. Phys. **207** (1999), 131–143.

[77] M. Gursky and J. Viaclovsky, *A conformal invariant related to some fully nonlinear equations*, preprint, 2002.

[78] M. J. Gursky and J. Viaclovsky, *Fully nonlinear equations on Riemannian manifolds with negative curvature*, Indiana Univ. Math. J. **52** (2003), 399–419.

[79] M. J. Gursky and J. Viaclovsky, *A fully nonlinear equation on four-manifolds with positive scalar curvature.* J. Differential Geom. **63** (2003), 131–154.

[80] C. Hong, *A best constant and the Gaussian curvature,* Proc. Amer. Math. Soc. 97 (1986), 737–747.

[81] L. Hörmander, *Notions of Convexity*, Birkhauser, Boston, (1994).

[82] G. Huisken and C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces.* Acta Math. **183**, (1999), 45–70.

[83] S. Ilias, *Constantes explicites pour les inégalitès de Sobolev sur les variétés riemanniennes compactes.* Ann. Inst. Fourier (Grenoble), **33** (1983), no. 2, 151–165. MR 85b:58127

[84] N. Ivochkina, *Solution of the curvature problem for equation of curvature of order m*, Soviet Math. Dokl., **37**, (1988), 322-325.

[85] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Volume **II**, Interscience Publishers, John Wiley & Sons, New York, 1969.

[86] N.J. Korevaar and J. Lewis, *Convex solutions of certain elliptic equations have constant rank hessians*, Arch. Rational Mech. Anal. **97**, (1987), 19-32.

[87] N.V. Krylov, *Boundedly inhomogeneous elliptic and parabolic equations in a domain*, Izv. Akad. Nauk SSSR **47** (1983), 95-108.

[88] N.V. Krylov, *On the general notion of fully nonlinear second-order elliptic equations*, Trans. Amer. Math. Soc., **347** (1995), no. 3, 857–895.

[89] N.V. Krylov and M.V. Safonov, *An estimate of the probability that a diffusion process hits a set of positive measure*, Soviet Math. Dokl., **20**,(1979), 253-257. MR 80b:60101

[90] J. Lafontaine, *Conformal geometry from the Riemannian viewpoint,* Conformal geometry (Bonn, 1985/1986), 65–92, Aspects Math., E12, Vieweg, Braunschweig, 1988.

[91] A. Li and Y.Y. Li, *On some conformally invariant fully nonlinear equations*, Comm. Pure Appl. Math. 56 (2003), 1416-1464.

[92] A. Li and Y.Y. Li, *On some conformally invariant fully nonlinear equations, Part II: Liouville, Harnack and Yamabe*, preprint, 2004.

[93] Y.Y. Li, *Degree theory for second order nonlinear elliptic operators and its applications*, Comm. PDE, **14** (1989), no. 11, 1541-1578.

[94] J. Lohkamp, *Metrics of negative Ricci curvature,* Ann. of Math., (2) **140** (1994), 655–683. 1998. MR 95i:53042

[95] H. Minkowski, *Allgemeine Lehrsätze über die konvexen Polyeder*, Nachr.Ges. Wiss. Gottingen, (1897), 198-219.

[96] S. Nayatani, *Patterson-Sullivan measure and conformally flat metrics* Math. Z. **225** (1997), 115–131.

[97] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure Applied Math., **6** (1953), 337-394.

[98] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Lecture Notes 1973/74, Courant Institute.

[99] V.I. Oliker, *Hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed Gaussian curvature and related equations of Monge-Ampère type*, Comm. Partial Diff. Equations, **9**, (1984), 807-838.

[100] V.I. Oliker, *Existence and uniqueness of convex hypersurfaces with prescribed Gaussian curvature in spaces of constant curvature*, Sem. Inst. Mate. Appl. "Giovanni Sansone", Univ. Studi Firenze, 1983.

[101] E., Onofri, *On the positivity of the effective action in a theory of random surfaces,* Comm. Math. Phys. **86** (1982) 321–326.

[102] K. Grove and P. Petersen (eds), *Comparison geometry,* Math. Sci. Res. Inst. Publ., **30**, Cambridge Univ. Press, Cambridge, 1997. MR 97m:53001

[103] P. Petersen, *On eigenvalue pinching in positive Ricci curvature,* Invent. Math., **138** (1999), 1–21. MR 01e:53032

[104] M.H. Protter and H.F. Weinberger, *Maximum principles in differential equations,* Prentice-Hall, Inc., Englewood Cliffs, N.J. 1967. MR 36 #2935

[105] A. V. Pogorelov, *Regularity of a convex surface with given Gaussian curvature*, Mat. Sb., **31** (1952), 88-103.

[106] A.V. Pogorelov, *Extrinsic geometry of convex surfaces*, Transl. Math. Monographs, **35** AMS, Providence, R.I., 1973.

[107] A.V. Pogorelov, *The Minkowski multidimensional problem*, Wiley, New York, (1978).

[108] M.V. Safonov, *On the classical solution of Bellman's elliptic equation*, Soviet Math. Dokl. **30**, (1984), 482-485. MR 86f:35081

[109] M.V. Safonov, *Classical solution of second-order nonlinear elliptic equations*, Math. USSR-Izv., **33** (1989), no. 3, 597-612. MR 90d:35104

[110] R. Schneider, *Convex bodies: The Brunn-Minkowski theory*, Cambridge University, (1993).

[111] R. Schoen, *Conformal deformation of a Riemannian metric to constant curvature,* J. Diff. Geome., **20** (1984), 479-495.

[112] R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics,* Topics in calculus of variations, 120–154, Lecture Notes in Math., 1365, Springer, Berlin, 1989.

[113] R. Schoen, *On the number of constant scalar curvature metrics in a conformal class,* in: Differential geometry, 311–320, Pitman Monogr. Surveys Pure Appl. Math., 52, Longman Sci. Tech., Harlow, 1991.

[114] R. Schoen and S. T. Yau, *On the structure of manifolds with positive scalar curvature.* Manuscripta Math. **28** (1979), 159–183.

[115] R. Schoen and S. T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature,* Invent. Math. **92** (1988), no. 1, 47–71.

[116] W. Sheng, N. Trudinger and X.J. Wang, *Convex hypersurfaces of prescribed Weingarten curvature*, Comm. in Analysis and Geometry, **12** (2004), 213-232.

[117] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems,* Ann. of Math. **118** (1983), 525–571.

[118] I. Singer,B. Wong, S.T. Yau, and Stephen S.-T. Yau, *An estimate of the gap of the first two eigenvalues in the Schrdinger operator.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **12** (1985), 319–333.

[119] M. Tani, *On a conformally flat Riemannian space with positive Ricci curvature.* Tohoku Math. J. (2) **19** 1967 227–231.

[120] A. Treibergs, *Existence and convexity of hypersurfaces of prescribed mean curvature*, Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4) **12**, (1985), 225-241.

[121] A. Treibergs and S.W. Wei, *Embedded hypersurfaces with prescribed mean curvature*, J. Diff. Geometry, **18**, (1983), 513-521.

[122] N.S. Trudinger, *On the Dirichlet problem for Hessian equations*, Acta Math., **175** (1995), 151–164.

[123] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa **22**, (1968), 265-274.

[124] J. Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Zeit. **205**, 1990, 355-372.

[125] J. Urbas, *An expansion of convex hypersurfaces.* J. Diff. Geom. **30**, (1991), 91 - 125.

[126] J. Viaclovsky, *Conformal geometry, contact geometry and the calculus of variations*, Duke Math. J. **101** (2000), no. 2, 283–316.

[127] J. Viaclovsky, *Conformal invariant Monge-Ampère equations: global solutions*, Trans. AMS, **352** (2000), 4371-4379.

[128] J. Viaclovsky, *Estimates and some existence results for some fully nonlinear elliptic equations on Riemannian manifolds,* Comm. Anal. Geom. **10** (2002), 815–846.

[129] L. Wang, *On the regularity of fully nonlinear parabolic equations I & II*, Comm. Pure Appl. Math., **45** (1992), 27-76 & 141-178. MR 90g: 35028 & MR 92m:35126

[130] X. J. Wang, *On the design of a reflector antenna*, Inverse Problems **12** (1996), 351–375.

[131] S. T. Yau, *On the Ricci curvature of a complex Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure and Appl. Math., **31** (1978), 339-411.

[132] R. Ye, *Global existence and convergence of Yamabe flow,* J. Differential Geom. **39** (1994), no. 1, 35–50

[133] X.P. Zhu, *Multiple convex hypersurfaces with prescribed mean curvature.* Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4) **21**, (1994), 175-191.