A MEAN CURVATURE TYPE FLOW IN SPACE FORMS

PENGFEI GUAN AND JUNFANG LI

ABSTRACT. In this article, we introduce a new type of mean curvature flow (1.3) for bounded star-shaped domains in space forms and prove its longtime existence, exponential convergence without any curvature assumption. Along this flow, the enclosed volume is a constant and the surface area evolves monotonically. Moreover, for a bounded convex domain in $\mathbb{R}^{n+1}$, the quermassintegrals evolve monotonically along the flow which allows us to prove a class of Alexandrov-Fenchel inequalities of quermassintegrals.

1. INTRODUCTION

There have been extensive interests on flows of hypersurfaces evolving by functions of their mean curvature in the past 30 years. Brakke [4] used geometric measure theory to study surfaces driven by their mean curvature. The more differential geometric approach was given by Huisken [13] who studied the mean curvature flow,

$$(1.1) \qquad \frac{\partial X}{\partial t} = -H\nu,$$

where X, ν and H are the position function, the outward unit normal vector and the mean curvature of the hypersurface respectively.

Huisken in [13] proved that flow (1.1) is a contracting flow which contracts convex hypersurfaces into a round point. In contrast, another model type of flow, the inverse mean curvature flow, is an expanding flow introduced by Gerhardt [7], Urbas [25], which expands star-shaped mean convex hypersurfaces out to a round sphere. Note that, in general, flow (1.1) would develop singularities when the initial data is a star-shaped mean convex hypersurface.

Hypersurface flows can be used to prove geometric inequalities, old and new. For example, Gage and Hamilton [6] used curve shortening flow to prove isoperimetric inequality for convex planar domains. Andrews [1] used affine mean curvature flow to prove affine isoperimetric inequality. Topping [24] studied generalized isoperimetric inequalities for convex domains on Riemann surfaces using Grayson's curve shortening flow. Various geometric inequalities were proved in [11, 12] using fully nonlinear conformal

flows. Schulze [23] used a non-linear mean curvature type flow to prove the isoperimetric inequality in $\mathbb{R}^{n+1}$ with $n \leq 7$. The authors used the inverse mean curvature type of flows to prove Alexandrov-Fenchel quermassintegral inequalities for starshaped k-convex domains in [9].

In this paper, we introduce a new type of mean curvature flow in space forms. For example, in Euclidean space $\mathbb{R}^{n+1}$ it reads as

$$(1.2) \qquad \frac{\partial X}{\partial t} = (n - Hu)\nu.$$

where X is the position vector, H is the mean curvature, and $u = <X, \nu>$ the support function of the hypersurface. In general space forms, the flow equation is given by

$$(1.3) \qquad \partial_t X = (n\phi'(\rho) - Hu)\nu,$$

while more details of the definition of this flow in general space forms can be found in section 3, c.f. (3.2). We will show this flow evolves star-shaped domains (*without any curvature assumption*) in space forms into spheres. The key of discovering this flow is the Minkowski identity which also plays crucial role in the proof. We note that the assumption of starshapedness (i.e., $u > 0$) is necessary for flow (1.3) to be parabolic.

One of the main theorems of this article is the following.

Theorem 1.1. *Let M_0 be a smooth compact, star-shaped hypersurface with respect to the origin in the space forms $N^{n+1}(K)$ with $n \geq 2$. Then the evolution equation (1.3) has a smooth solution for $t \in [0, \infty)$. Moreover, the surfaces converges exponentially to a sphere as $t \to \infty$ in the C^∞ topoloty.*

Another nice feature of flow (1.3) is that along the flow, volume of the enclosed domain is constant and the surface area is monotonically decreasing. This property yields that the flow of the hypersurface M_t converges to a solution of the isoperimetric problem in the space forms.

If the initial hypersurface encloses a (weakly) convex bounded domain in Euclidean space $\mathbb{R}^{n+1}$, then by arguments in Bian-Guan [3], the convexity is preserved along the flow (1.2) and the domain becomes strictly convex instantly. Moreover, we will show all the quermassintegrals

$$\int_M \sigma_k(\kappa) d\mu_g$$

are non-increasing along the flow, where the quermassintegral is surface area, total mean curvature, total scalar curvature, etc. repectively for different $k = 0, 1, 2, \cdots, n - 1$. This yields a proof to the following well-known Alexandrov-Fenchel inequalities of quermassintegrals from convex geometry.

Corollary 1.2. *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a bounded convex domain with smooth boundary $\partial\Omega$. Then the following inequality holds for any $0 \leq k < n - 1$,*

$$(1.4) \qquad V(\Omega)^{\frac{1}{n+1}} \leq c_{n,k} \Big(\int_{\partial\Omega} \sigma_k(\kappa) d\mu \Big)^{\frac{1}{n-k}},$$

where $c_{n,k}$ is a constant depending only on n and k, and $\sigma_k(\kappa)$ is the kth elementary symmetric function of the principal curvatures. The "=" is attained if and only if Ω is a ball.

When $k = 0$, we have used the convention $\sigma_0 \equiv 1$. Note that in this case, the inequality is reduced to the classical isoperimetric inequality. In fact, in this case the convexity condition can be replaced by star-shapedness.

Remark 1.3. *In our second paper [10], we will present a fully-nonlinear generalization of the mean curvature type of flow which can be used to prove other types of quermassintegral inequalities in Euclidean space.*

In appearance, flow (1.2) resembles to the normalized mean curvature flow of Huisken [13],

$$(1.5) \qquad \frac{\partial X}{\partial t} = (c(t)u - H)\nu,$$

where $c(t) = \frac{1}{n}\frac{\int_M H^2 d\mu}{\int_M d\mu}$ for mean curvature H and also the volume preserving mean curvature flow of Huisken [15],

$$(1.6) \qquad \frac{\partial X}{\partial t} = (c(t) - H)\nu,$$

where $c(t) = \frac{\int_M H d\mu}{\int_M d\mu}$. However, there are several differences among these equations. The longtime existence of solutions of (1.5) or (1.6) is not clear if the initial surface is not convex, while (1.2) requires no curvature assumption except starshapedness. Note that normalized mean curvautre flow (1.5) preserves the surface area while the volume preserving mean curvature flow (1.6) and our new flow (1.2) preserves the volume and evolve the surface area monotonically. Moreover, (1.2) evovles all the quermassintegrals monotonically.

On the other hand, the presence of normalization factor $c(t)$ makes (1.5) or (1.6) an integral-differential equation and a priori estimates are not easy to obtain. Thus, for the quasilinear equation (1.5) or (1.6), one needs to do the more involved curvature estimates rather than just gradient estimate. In contrast, for flow (1.2), the normalization function is a fixed constant n thanks to the Minkowski identity, and the principal part of the resulting parabolic equation is of divergent form, which makes the a priori estimates easier by the classical theory of parabolic equations in divergent form (e.g., [18]).

We note there is also a volume preserving mean curvature flow defined in hyperbolic space, see [5].

When $n = 1$, Theorem 1.1 holds for curves as well. We will leave the details for the readers. The rest of this paper is organized as follows. In Section 2, we give the preliminaries for hypersurface theory in space forms and prove the important Minkowski type identities. In Section 3, we introduce the flow equation in space forms, derive the evolution equations for various geometric quantities, and prove the monotonicity formulas. In Section 4, we convert the problem to an initial value problem on $\mathbb{S}^n$ and prove the C^0 estimate. In Section 5, we establish the gradient estimate and prove the exponential convergence and the main theorems. In the last section, we show the convexity is preserved along the flow in Euclidean space and prove Corollary 1.2.

2. Hypersurfaces in space forms

Let $N^{n+1}(K)$ be a space form of sectional curvature $K = -1, 0$, or $+1$ and $n \geq 2$. Let $g^N := ds^2$ denote the Riemannian metric of $N^{n+1}(K)$. Then it is known that the space forms can be viewed as Euclidean space $\mathbb{R}^{n+1}$ equipped with a metric tensor, i.e., $N^{n+1}(K) = (\mathbb{R}^{n+1}, ds^2)$ with proper choice ds^2. More specifically, let $\mathbb{S}^n$ be the unit sphere in Euclidean space $\mathbb{R}^{n+1}$ with standard induced metric dz^2, then

$$(2.1) \qquad g^N := ds^2 = d\rho^2 + \phi^2(\rho)dz^2,$$

where N^{n+1} is the Euclidean space $\mathbb{R}^{n+1}$ if $\phi(\rho) = \rho$, $\rho \in [0, \infty)$; N^{n+1} is the elliptic space $\mathbb{S}^{n+1}$ with constant sectional curvature $+1$ if $\phi(\rho) = \sin(\rho)$, $\rho \in [0, \pi)$; and N^{n+1} is the hyperbolic space $\mathbb{H}^{n+1}$ with constant sectional curvature -1 if $\phi(\rho) = \sinh(\rho)$, $\rho \in [0, \infty)$.

We define

$$(2.2) \qquad \Phi(\rho) = \int_0^\rho \phi(r)dr.$$

Then $\Phi(\rho)$ is $\frac{\rho^2}{2}$, $\cosh \rho$, $-\cos \rho$ respectively. Consider the vector field $V = \phi(\rho)\frac{\partial}{\partial \rho}$ on $N^{n+1}(K)$. We first show that V is a conformal killing field. The following lemma holds for general warped product manifolds, see for example, [20]. For convenience, we include a proof here.

Lemma 2.1. *The vector field V satisfies $D_i V_j = \phi'(\rho)g_{ij}^N$, where D is the covariant derivative with respect to the metric g^N.*

Proof. We first note that the Lie derivative of $d\rho$ is

$$\mathcal{L}_V d\rho = \mathcal{L}_{\phi(\rho)\frac{\partial}{\partial \rho}} d\rho = \phi'(\rho)d\rho.$$

Thus

$$\mathcal{L}_V d\rho^2 = 2\phi'(\rho)d\rho^2.$$

Note

$$\mathcal{L}_V \phi^2(\rho)dz^2 = V(\phi^2)dz^2 = 2\phi^2(\rho)\phi'(\rho)dz^2.$$

We have
$$\mathcal{L}_V ds^2 = 2\phi'(\rho)ds^2.$$
The proof is now complete since $\frac{1}{2}\mathcal{L}_V g^N = DV$.

$\square$

Let $M^n \subset N^{n+1}$ be a closed hypersurface with induced metric g.

Lemma 2.2. *Let $M^n \subset N^{n+1}$ be a closed hypersurface with induced metric g. Let Φ and V be defined as in (2.2). Then $\Phi|_M$ satisfies,*

$$(2.3) \qquad \nabla_i \nabla_j \Phi = \phi'(\rho)g_{ij} - h_{ij}\langle V, \nu \rangle,$$

where ∇ is the covariant derivative with respect to g, ν is the outward unit normal and h_{ij} is the second fundamental form of the hypersurface. Moreover, the Laplacian of Φ satisfies,

$$(2.4) \qquad \Delta \Phi = n\phi'(\rho) - H\langle V, \nu \rangle,$$

where H is the mean curvature.

Proof. Let $e_1, \cdots, e_n$ be a basis of the tangent space of M.

$$(2.5) \qquad \begin{aligned} \nabla_i \nabla_j \Phi &= D^2\Phi(e_i, e_j) - h_{ij}\langle V, \nu \rangle \\ &= \langle D_{e_i}V, e_j \rangle - h_{ij}\langle V, \nu \rangle \\ &= \phi'g_{ij} - h_{ij}\langle V, \nu \rangle, \end{aligned}$$

where the last step we have used Lemma 2.1.

$\square$

Note that in Euclidean space the vector field $V = \rho \frac{\partial}{\partial \rho}$ is equivalent to the position vector field X. Thus $\langle V, \nu \rangle = \langle X, \nu \rangle$ is just the Euclidean support function of the hypersurface M. In this sense, let us still call the inner product
$$u := \langle V, \nu \rangle$$
to be the support function of a hypersurface in N^{n+1}. An immediate corollary of Lemma 2.2 is the following Minkowski identity in N^{n+1}.

Proposition 2.3. *Let M be a closed hypersurface in N. Then*

$$(2.6) \qquad \int_M Hu = n \int_M \phi'(\rho).$$

Proof. Applying divergence theorem to (2.4), we obtain

$$0 = \int_M \Delta\Phi = n \int_M \phi'(\rho) - \int_M Hu.$$

This finishes the proof.

$\square$

The eigenvalues of the Weingarten tensor $h^i_j := g^{ik}h_{kj}$ are the principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_n)$. If we denote the kth elementary symmetric functions of an n-vector by σ_k, then $\sigma_1(\kappa)$ is the mean curvature (unnormalized), $\sigma_2(\kappa)$ is the scalar curvature, etc. For convenience, we also use $\sigma_k(A_{ij})$ to denote the k-th elementary function of the eigenvalues of a symmetric matrix A_{ij}. Recall in [21, 22], Reilly proved a formula for the

r-th Newton operator $T_r(h)$ of the second fundamental form in general Riemannanian manifolds, see Proposition 1 in [22]. In space form, since second fundamental form is Codazzi, Reilly's formula is equivalent to say the Newton operator T_r is divergent free. More specifically, in local coordinates, the Newton operator T_r of h_j^i is a symmetric matrix $(T_r)_{ij} = \frac{\partial \sigma_{r+1}}{\partial h_j^i} =: \sigma_{r+1}^{ij}$, and

$$\nabla_i (T_r)_{ij} = 0.$$

Using this property, we can generalize (2.4) to the kth Weingarten curvature $\sigma_k(\kappa)$.

Lemma 2.4. *Let $M^n \subset N^{n+1}$ be a closed hypersurface with induced metric g. Let Φ and V be defined as in (2.2). Then $\Phi|_M$ satisfies,*

$$(2.7) \qquad \nabla_i((T_k)_{ij}\nabla_j\Phi) = (n-k)\phi'\sigma_k(\kappa) - (k+1)\sigma_{k+1}(\kappa)\langle V, \nu\rangle.$$

Proof. We multiply (2.3) with $(T_k)_{ij}$, then sum over i, j. By divergent free property of T_k, we have the left hand side is $\nabla_i((T_k)_{ij}\nabla_j\Phi)$. The following formulas are well-known, see e.g. [14],

$$\begin{aligned} (T_k)_{ij}g_{ij} &= (n-k)\sigma_k \\ (T_k)_{ij}h_{ij} &= (k+1)\sigma_{k+1}. \end{aligned}$$

Put these identities together, we complete the proof. $\qquad\square$

Prove as in Proposition 2.3, we have the following generalized Minkowski identities.

Proposition 2.5. *Let M be a closed hypersurface in N. Then, for $k = 0, 1, \cdots, n-1$,*

$$(2.8) \qquad (k+1)\int_M \sigma_{k+1}(\kappa)u = (n-k)\int_M \phi'(\rho)\sigma_k(\kappa),$$

where we use the convention that $\sigma_0 \equiv 1$.

Next, we derive the gradient and hessian of the support function u under the induced metric g on M.

Lemma 2.6. *The support function u satisfies*

$$(2.9) \qquad \begin{aligned} \nabla_i u &= g^{kl}h_{ik}\nabla_l\Phi, \\ \nabla_i\nabla_j u &= g^{kl}\nabla_k h_{ij}\nabla_l\Phi + \phi'h_{ij} - (h^2)_{ij}u, \end{aligned}$$

where $(h^2)_{ij} := g^{kl}h_{ik}h_{jl}$.

Proof. Using another model of the space forms, these formulas were first proved in [2]. The same formulas in hyperbolic space were also proved by a direct computation in [17] using the expression of support function u for a radial graph in hyperbolic space, see the expression of u for a radial graph in (4.1).

By the tensorial properties of both sides, we only need to prove them at any point under orthonormal coordinates. Thus, we have $g_{ij} = \delta_{ij}$ and

$\nabla_i u = D_i \langle V, \nu \rangle = \langle D\Phi, D_i \nu \rangle = h_{ik} D_k \Phi$. By the tensorial property of both sides, we proved the first identity.

Differentiate ∇u one more time and apply (2.3), we obtain

$$
\begin{aligned}
\nabla_i \nabla_j u &= \nabla_i h_{jk} \nabla_k \Phi + h_{jk} \nabla_i \nabla_k \Phi \\
&= \nabla_i h_{jk} \nabla_k \Phi + h_{jk}(\phi' g_{ik} - h_{ik} u) \\
&= \nabla_k h_{ij} \nabla_k \Phi + \phi' h_{ij} - (h^2)_{ij} u,
\end{aligned}
$$

where in the last step we have used the fact that second fundamental form of hypersurfaces in space forms is Codazzi. $\qquad \square$

3. A FLOW OF MEAN CURVATURE TYPE

Let $M(t)$ be a smooth family of closed hypersurfaces in a space form N. Let $X(\cdot, t)$ denote a point on $M(t)$. In general, we have the following evolution property.

Lemma 3.1. *Let $M(t)$ be a smooth family of closed hypersurfaces in $N^{n+1}(K)$ evolving along the flow*

$$\partial_t X = f(X(\cdot, t))\nu,$$

where ν is the unit outward normal vector field and f is a function defined on $M(t)$. Then, we have the following evolution equations.

$$
\text{(3.1)} \qquad
\begin{aligned}
\partial_t g_{ij} &= 2f h_{ij} \\
\partial_t d\mu_g &= f d\mu_g \\
\partial_t h_{ij} &= -\nabla_i \nabla_j f + f(h^2)_{ij} - K f g_{ij} \\
\partial_t H &= -\Delta f - f|h|^2 - nKf,
\end{aligned}
$$

where $|h|^2$ is the norm square of second fundamental form and $d\mu_g$ is the volume element of the metric $g(t)$.

Proof. Proof is standard, see for example, [13]. $\qquad \square$

Now, we consider the following mean curvature type of flow in (N, ds^2) where $ds^2 = d\rho^2 + \phi^2(\rho)dz^2$,

$$\text{(3.2)} \qquad \partial_t X = (n\phi'(\rho) - uH)\nu.$$

Note that $\phi'(\rho)$ is 1, $\cos(\rho)$, and $\cosh(\rho)$ in $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$, and $\mathbb{H}^{n+1}$ respectively.

Proposition 3.2. *Let $M(t)$ be a smooth family of closed hypersurfaces in the space form $N^{n+1}(K)$ evolve along the flow (3.2). Then the mean curvature evolves as the following*

$$\text{(3.3)} \qquad \partial_t H = u\Delta H + H\nabla H \nabla \Phi(\rho) + 2\nabla H \nabla u + (H^2 - n|h|^2)\phi'.$$

Proof. From Lemma 3.1, we have

(3.4)
$$
\begin{aligned}
\partial_t H &= u\Delta H + 2\nabla u \nabla H + H\Delta u - n\Delta\phi' - (n\phi' - Hu)(|h|^2 + nK) \\
&= u\Delta H + 2\nabla u \nabla H + H(\nabla H\nabla\Phi + H\phi' - |h|^2 u) - n\Delta\phi' \\
&\quad -(n\phi' - Hu)(|h|^2 + nK) \\
&= u\Delta H + 2\nabla u \nabla H + H\nabla H\nabla\Phi + (H^2 - n|h|^2)\phi' \\
&\quad -n(\Delta\phi' + K\Delta\Phi),
\end{aligned}
$$

where we have used (2.9). It's easy to see that $\Delta\phi' = -K\Delta\Phi$ holds for all the three different K. This proves the assertion. □

The following estimate will be useful in later sections.

Corollary 3.3. *Let $N^{n+1}(K)$ be hyperbolic space. Then along flow (3.2), the mean curvature is uniformly bounded from above, i.e.,*

(3.5)
$$
H(\cdot, t) \leq \max_{x \in M(0)} H(x, 0).
$$

Proof. The proof is an immediate corollary of Proposition 3.2 by applying maximum principle to H. Notice that $\phi' = \cosh\rho > 0$ in this case and $(H^2 - n|h|^2) \leq 0$ by Newton-McLaurin inequality. □

Remark 3.4. *We need the uniform upper bound of mean curvature H just in the hyperbolic case.*

Applying the Minkowski identities, we have the following monotonicity formulas,

Proposition 3.5. *Along the flow (3.2), the volume $V(t)$ of the enclosed domain by the closed hypersurface is a constant and surface area evolves as follows,*

(3.6)
$$
\frac{d}{dt}A(t) = -\frac{1}{n-1}\int_{M(t)}\sum_{i<j}(\kappa_i(x) - \kappa_j(x))^2 u\, d\mu_g,
$$

where $\kappa(x)$ are the principal curvatures of the hypersurface at point $x \in M(t)$.

Proof. In general, along flow equation $\partial_t X = f\nu$, the volume evolution is $V'(t) = \int_{M(t)} f\, d\mu_g$ and surface area satisfies $A'(t) = \int_{M(t)} f\, d\mu_g$. Thus, by Minkowski identity (2.6),

$$
V'(t) = \int_M (n\phi' - Hu)d\mu = 0,
$$

By the general Minkowski identity (2.8), we have

$$
\begin{aligned}
A'(t) &= \int_M (n\phi' - Hu)H d\mu_g \\
&= \int_M (n\phi'H - \frac{2n}{n-1}\sigma_2(\kappa)u)d\mu + \int_M (\frac{2n}{n-1}\sigma_2 - H^2)u d\mu \\
&= -\int_M (H^2 - \frac{2n}{n-1}\sigma_2)u d\mu \\
&= -\int_M (\sum_i \kappa_i^2 - \frac{2}{n-1}\sigma_2)u d\mu \\
&= -\frac{1}{n-1}\int_M \sum_{i<j}(\kappa_i - \kappa_j)^2 u d\mu.
\end{aligned}
$$

$\square$

Next we consider the special case of a bounded convex domain in $\mathbb{R}^{n+1}$. We first show that the weak convexity of the initial hypersurface is preserved and improved along the flow (1.2).

Lemma 3.6. *Let M_0 be the smooth boundary of a bounded convex domain in $\mathbb{R}^{n+1}$ which may have flat side. If M_t is a solution of flow (1.2) with initial hypersurface to be M_0, then M_t is strictly convex for any $t > 0$ whenever the flow exists.*

Proof. The proof follows immediately from Theorem 1.4 of Bian-Guan [3].
$\square$

Proposition 3.7. *Let M_0 be the smooth boundary of a bounded convex domain in $\mathbb{R}^{n+1}$. If M_t is a solution of flow (1.2) with initial hypersurface to be M_0, then for any $k = 0, 1, \cdots, n-2$,*

$$
(3.7) \qquad \frac{d}{dt}\int_M \sigma_k(\kappa)d\mu \leq 0,
$$

where $\kappa(x)$ are the principal curvatures of the hypersurface at point $x \in M(t)$.

Proof. Recall for a general evolution equation,

$$
\partial_t X = F\nu,
$$

with speed function $F(X, t)$, we have the evolution equation of the qumermassintegrals, c.f. Proposition 4 of [9],

$$
(3.8) \qquad \frac{d}{dt}\int_M \sigma_k(\kappa)d\mu = (k+1)\int_M F\sigma_{k+1}(\kappa)d\mu.
$$

For any $t > 0$, $\sigma_l > 0$ for all $l = 0, 1, \cdots, n$ by Lemma 3.6. By Newton-McLaurin inequality, we have the following,

$$
(3.9) \qquad \sigma_1\sigma_{k+1} \geq c_1\sigma_{k+2}.
$$

where $c_1 = \frac{n(k+2)}{n-k-1}$. Let $F = n - Hu$ in (3.8) and apply inequality (3.9), we have

$$
\begin{aligned}
\frac{d}{dt} \int_M \sigma_k(\kappa) d\mu &= (k+1) \int_M (n\sigma_{k+1} - u\sigma_1\sigma_{k+1}) d\mu \\
&\leq (k+1) \int_M (n\sigma_{k+1} - c_1 u \sigma_{k+2}) d\mu \\
&= 0,
\end{aligned}
$$

where in the last step we used Minkowski identity (2.8). This completes the assertion. $\qquad\square$

4. RADIAL GRAPH AND C^0 ESTIMATE

Let (M, g) be a hypersurface in $N^{n+1}(K)$ with induced metric g. We now give the local expressions of the induced metric, second fundamental form, Weingarten curvatures etc when M is a graph of a smooth and positive function $\rho(z)$ on $\mathbb{S}^n$. Let $\partial_1, \cdots, \partial_n$ be a local frame along M and ∂_ρ be the vector field along radial direction. Then the support function, induced metric, inverse metric matrix, second fundamental form can be expressed as follows ([19]). For simplicity, all the covariant derivatives with respect to the standard spherical metric e_{ij} will also be denoted as ∇ when there is no confusion in the context.

(4.1)
$$
\begin{aligned}
u &= \frac{\phi^2}{\sqrt{\phi^2 + |\nabla\rho|^2}} \\
g_{ij} &= \phi^2 \delta_{ij} + \rho_i\rho_j, \quad g^{ij} = \frac{1}{\phi^2}(e^{ij} - \frac{\rho_i\rho_j}{\phi^2 + |\nabla\rho|^2}) \\
h_{ij} &= (\sqrt{\phi^2 + |\nabla\rho|^2})^{-1}(-\phi\nabla_i\nabla_j\rho + 2\phi'\rho_i\rho_j + \phi^2\phi' e_{ij}) \\
h^i_j &= \frac{1}{\phi^2\sqrt{\rho^2 + |\nabla\rho|^2}}(e^{ik} - \frac{\rho_i\rho_k}{\phi^2 + |\nabla\rho|^2})(-\phi\nabla_k\nabla_j\rho + 2\phi'\rho_k\rho_j + \phi^2\phi' e_{kj})
\end{aligned}
$$

where all the covariant derivatives ∇ and ρ_i are w.r.t. the spherical metric e_{ij}.

It will be convenient if we introduce a new variable γ satisfying

$$
\frac{d\gamma}{d\rho} = \frac{1}{\phi(\rho)}.
$$

Let $\omega := \sqrt{1 + |\nabla\gamma|^2}$, one can compute the unit outward normal $\nu = \frac{1}{\omega}(1, -\frac{\rho_1}{\phi^2}, \cdots, -\frac{\rho_n}{\phi^2})$ and the general support function $u = \langle D\phi, \nu \rangle = \frac{\phi}{\omega}$. Moreover,

(4.2)
$$
\begin{aligned}
g_{ij} &= \phi^2 \delta_{ij} + \rho_i\rho_j, \quad g^{ij} = \frac{1}{\phi^2}(e^{ij} - \frac{\gamma_i\gamma_j}{\omega^2}) \\
h_{ij} &= \frac{\phi}{\omega}(-\gamma_{ij} + \phi'\gamma_i\gamma_j + \phi' e_{ij}) \\
h^i_j &= \frac{1}{\phi\omega}(e^{ik} - \frac{\gamma_i\gamma_k}{\omega^2})(-\gamma_{kj} + \phi'\gamma_k\gamma_j + \phi' e_{kj})
\end{aligned}
$$

where all the covariant derivatives are w.r.t. the spherical metric e_{ij}.

We also have for the mean curvature,

$$(4.3) \qquad H = \frac{1}{\phi\omega}(e^{ij} - \frac{\gamma_i\gamma_j}{\omega^2})(-\gamma_{ij} + \phi'\gamma_i\gamma_j + \phi'e_{ij}).$$

We now consider the flow equation (3.2) of radial graphs over $\mathbb{S}^n$ in $N^{n+1}(K)$. It is known ([8]) if a closed hypersurface which is a radial graph and satisfies

$$\partial_t X = f\nu,$$

then the evolution of the scalar function $\rho = \rho(X(z,t),t)$ satisfies

$$\partial_t \rho = f\omega.$$

Thus we only need to consider the following parabolic initial value problem on $\mathbb{S}^n$,

$$(4.4) \qquad \begin{cases} \partial_t\rho = (n\phi' - Hu)\omega, \text{for} \quad (z,t) \in \mathbb{S}^n \times [0,\infty) \\ \rho(\cdot,0) = \rho_0, \end{cases}$$

where ρ_0 is the radial function of the initial star-shaped hypersurface.

Equivalently, the equation for γ satisfies

$$(4.5) \qquad \partial_t\gamma = (n\phi' - Hu)\frac{\omega}{\phi}$$

We next show that the radial function ρ is uniformly bounded from above and below.

Proposition 4.1. *Let M_0 be a radial graph of function ρ_0 embedded in space form $N^{n+1}(K)$. If $\rho(z,t)$ solves the initial value problem (4.4), then for any $(z,t) \in \mathbb{S}^n \times [0,T]$,*

$$\min_{x\in M}\rho(x,0) \le \rho(x,t) \le \max_{x\in M}\rho(x,0).$$

Proof. Note $\nabla\gamma = \frac{1}{\phi}\nabla\rho$, thus γ attains maximum or minimum at the same point as ρ. At critical points of γ and ρ, we have the following critical point conditions,

$$\nabla\gamma = 0, \nabla\rho = 0, \omega = 1.$$

Then it follows from (4.2) that, at critical points of γ and ρ, $H = \frac{1}{\phi}(-\Delta\gamma + n\phi')$. Together with (4.5), at critical points,

$$\gamma_t = \frac{1}{\phi}\Delta\gamma, \quad \text{and} \quad \partial_t\rho = \frac{1}{\phi}\Delta\rho.$$

By standard maximum principle, this proves the uniform upper and lower bounds for γ and ρ. $\qquad\square$

5. Gradient estimate and convergence

We need the following technical lemma.

Lemma 5.1. *Let γ be a solution of (4.5) and assume $|\nabla\gamma|^2$ attains maximum value at point P, then at P, the following holds,*
(5.1)
$$\mathcal{L}\big(\frac{|\nabla\gamma|^2}{2}\big) = n\frac{\phi''\phi - (\phi')^2}{\phi\omega}|\nabla\gamma|^4 - \frac{n-1}{\phi\omega}|\nabla\gamma|^2 - \frac{1}{\phi\omega}|\nabla\nabla\gamma|^2 - \frac{\phi'}{\phi\omega}\Delta\gamma|\nabla\gamma|^2,$$

where $\mathcal{L}$ is a strictly parabolic operator defined as follows: for any function $f(z,t)$,
$$\mathcal{L}(f) := f_t - \frac{1}{\phi\omega}\big(e^{ij} - \frac{\gamma_i\gamma_j}{\omega^2}\big)f_{ij}$$

Proof. By critical point condition, at P we have
$$\nabla|\nabla\gamma|^2 = 0, \nabla\omega = 0.$$

From (4.2) and (4.5), using critical point condition, we have
$$\partial_t\gamma = n\omega\frac{\phi'}{\phi} - \frac{1}{\phi\omega}\big(-\Delta\gamma + \frac{\gamma_i\omega_i}{\omega} + n\phi'\big).$$

Thus, at P
(5.2)
$$\begin{aligned}
\partial_t\frac{|\nabla\gamma|^2}{2} = {} & \gamma_k\nabla_k\gamma_t \\
= {} & \gamma_k\nabla_k\big(n\omega\frac{\phi'}{\phi} - \frac{1}{\phi\omega}(-\Delta\gamma + \frac{\gamma_i\omega_i}{\omega} + n\phi')\big) \\
= {} & \gamma_k\nabla_k\big(\frac{n\phi'}{\phi\omega}|\nabla\gamma|^2 + \frac{\Delta\gamma}{\phi\omega} - \frac{\gamma_i\omega_i}{\omega^2}\big) \\
= {} & n\frac{\phi''\phi - (\phi')^2}{\phi\omega}|\nabla\gamma|^4 - \frac{\phi'}{\phi\omega}\Delta\gamma|\nabla\gamma|^2 + \frac{1}{\phi\omega}\nabla_k\Delta\gamma\nabla_k\gamma - \frac{\omega_{ki}\gamma_i\gamma_k}{\phi\omega^2}
\end{aligned}$$

Recall on sphere, tensor $A_{ij} := f_{ij} + fe_{ij}$ is Codazzi for any function $f(z)$, where e_{ij} is the standard spherical metric and the covariant derivatives are w.r.t. e_{ij}. Thus
$$\nabla_i(\gamma_{jk} + \gamma e_{jk}) = \nabla_k(\gamma_{ij} + \gamma e_{ij}).$$

Applying this property, we have
(5.3)
$$\begin{aligned}
\nabla_k\Delta\gamma\nabla_k\gamma = {} & \nabla_i\gamma_{ki}\nabla_k\gamma - (n-1)|\nabla\gamma|^2 \\
= {} & \Delta\frac{|\nabla\gamma|^2}{2} - |\nabla\nabla\gamma|^2 - (n-1)|\nabla\gamma|^2.
\end{aligned}$$

Plug (5.3) into (5.2), we have
(5.4)
$$\begin{aligned}
\partial_t\frac{|\nabla\gamma|^2}{2} = {} & \frac{1}{\phi\omega}\Delta\frac{|\nabla\gamma|^2}{2} - \frac{\omega_{ik}\gamma_i\gamma_k}{\phi\omega^2} \\
& + n\frac{\phi''\phi - (\phi')^2}{\phi\omega}|\nabla\gamma|^4 - \frac{n-1}{\phi\omega}|\nabla\gamma|^2 - \frac{1}{\phi\omega}|\nabla\nabla\gamma|^2 - \frac{\phi'}{\phi\omega}\Delta\gamma|\nabla\gamma|^2
\end{aligned}$$

Note that
$$\frac{1}{\phi\omega}\big(e^{ij} - \frac{\gamma_i\gamma_j}{\omega^2}\big)\big(\frac{|\nabla\gamma|^2}{2}\big)_{ij} = \frac{1}{\phi\omega}\Delta\frac{|\nabla\gamma|^2}{2} - \frac{\omega_{ik}\gamma_i\gamma_k}{\phi\omega^2}.$$

This completes the proof. $\qquad\square$

5.1. $\mathbb{R}^{n+1}$ and $\mathbb{S}^{n+1}$ cases. In this subsection, we give a priori estimates for $\mathbb{R}^{n+1}$ and $\mathbb{S}^{n+1}$ cases which are relatively easier.

Proposition 5.2. *Let M_0 be a radial graph of function ρ_0 over $\mathbb{S}^n$ in space form $N^{n+1}(K)$ for $K = 0$ or $+1$. If $\rho(z,t)$ solves the initial value problem (4.4) on interval $[0,T]$, then for any $(z,t) \in \mathbb{S}^n \times [0,T]$,*

$$|\nabla \rho|(z,t) \leq C$$

where C is a uniform constant only depends on the initial values and the covariant derivatives are w.r.t. the spherical metric on $\mathbb{S}^n$. Moreover, the following exponential convergence holds, for any $(z,t) \in \mathbb{S}^n \times [0,T]$,

$$(5.5) \qquad e^{\alpha t}|\nabla \gamma|^2(z,t) \leq \max_{z \in \mathbb{S}^n} |\nabla \gamma|^2(z,0),$$

where $\alpha > 0$ is a uniform constant depending only on the initial graph.

Proof. By Lemma 5.1, we have at the maximal point P,
$$(5.6)$$
$$\begin{aligned}
\mathcal{L}(\tfrac{|\nabla \gamma|^2}{2}) =\ & n\tfrac{\phi''\phi - (\phi')^2}{\phi\omega}|\nabla \gamma|^4 - \tfrac{n-1}{\phi\omega}|\nabla \gamma|^2 - \tfrac{1}{\phi\omega}|\nabla\nabla\gamma|^2 - \tfrac{\phi'}{\phi\omega}\Delta\gamma|\nabla\gamma|^2 \\
\leq\ & n\tfrac{\phi''\phi - (\phi')^2}{\phi\omega}|\nabla \gamma|^4 - \tfrac{n-1}{\phi\omega}|\nabla \gamma|^2 - \tfrac{1}{n\phi\omega}(\Delta\gamma)^2 - \tfrac{\phi'}{\phi\omega}\Delta\gamma|\nabla\gamma|^2,
\end{aligned}$$

where we have used Cauchy-Schwartz inequality in the last step. By completing the square, we obtain
$$(5.7)$$
$$\begin{aligned}
\mathcal{L}(\tfrac{|\nabla \gamma|^2}{2}) \leq\ & n\tfrac{\phi''\phi - (\phi')^2}{\phi\omega}|\nabla \gamma|^4 - \tfrac{n-1}{\phi\omega}|\nabla \gamma|^2 \\
& -\tfrac{1}{n\phi\omega}\big(\Delta\gamma + \tfrac{n}{2}\phi'|\nabla\gamma|^2\big)^2 + \tfrac{n}{4\phi\omega}(\phi')^2|\nabla\gamma|^4 \\
=\ & -\tfrac{n}{\phi\omega}(\tfrac{3}{4}(\phi')^2 - \phi''\phi)|\nabla\gamma|^4 - \tfrac{n-1}{\phi\omega}|\nabla\gamma|^2 - \tfrac{1}{n\phi\omega}\big(\Delta\gamma + \tfrac{n}{2}\phi'|\nabla\gamma|^2\big)^2
\end{aligned}$$

In $\mathbb{R}^{n+1}$, $\tfrac{3}{4}(\phi')^2 - \phi''\phi = \tfrac{3}{4}$. In $\mathbb{S}^{n+1}$, $\tfrac{3}{4}(\phi')^2 - \phi''\phi = \tfrac{3}{4} + \tfrac{1}{4}\sin^2(\rho)$. Thus in both these two cases, we have

$$(5.8) \qquad \mathcal{L}(\tfrac{|\nabla\gamma|^2}{2}) \leq -\tfrac{n-1}{\phi\omega}|\nabla\gamma|^2 \leq 0.$$

Standard parabolic maximal principle yields that $|\nabla\gamma|^2(z,t) \leq C = \max |\nabla\gamma|^2(z,0)$. Since $|\nabla\gamma| = \tfrac{|\nabla\rho|}{\phi}$, and ϕ is uniformly bounded from above and below by positive constants, we have the uniform bound for $|\nabla\rho|$ as well.

Next, we prove the exponential estimate. Since we have shown that $|\nabla\gamma|^2$ is uniformly bounded from above, so is ω. Thus there exists a uniform constant $\alpha \geq \tfrac{2(n-1)}{\phi\omega} > 0$ which depends only on the upper bound of ω and ϕ. By (5.8),

$$\mathcal{L}(\tfrac{|\nabla\gamma|^2}{2}) \leq -\alpha\tfrac{|\nabla\gamma|^2}{2}.$$

Standard maximal principle then proves (5.5).

$\square$

5.2. **a Priori estimates : $\mathbb{H}^{n+1}$ case.** In this section, we derive gradient estimate and exponential convergence of the parabolic initial value problem (4.4) in $\mathbb{H}^{n+1}$. The difficulty in $\mathbb{H}^{n+1}$ is because in (5.7), $\frac{3}{4}(\phi')^2 - \phi''\phi = \frac{3}{4} - \frac{1}{4}\sinh^2(\rho)$ and a priorily, it may not be positive for all initial graph. Since this term is the leading term, we have to find other means to get around.

To obtain a uniform gradient bound, we have to use the fact that the mean curvature is uniformly bounded from above.

Proposition 5.3. *Let M_0 be a radial graph of function ρ_0 over $\mathbb{S}^n$ in space form $\mathbb{H}^{n+1}$. If $\rho(z,t)$ solves the initial value problem (4.4) on interval $[0,T]$, then for any $(z,t) \in \mathbb{S}^n \times [0,T]$,*

$$|\nabla \rho|(z,t) \leq C$$

where C is a uniform constant only depends on the initial values and the covariant derivatives are w.r.t. the spherical metric on $\mathbb{S}^n$.

Proof. As in the proof of Proposition 5.2, we carry out computations at the maximum point P of the test function $|\nabla\gamma|^2$. Recall in (5.6), we have obtained
(5.9)
$$\mathcal{L}(\frac{|\nabla\gamma|^2}{2}) = n\frac{\phi''\phi - (\phi')^2}{\phi\omega}|\nabla\gamma|^4 - \frac{n-1}{\phi\omega}|\nabla\gamma|^2 - \frac{1}{\phi\omega}|\nabla\nabla\gamma|^2 - \frac{\phi'}{\phi\omega}\Delta\gamma|\nabla\gamma|^2$$

On the other hand, from (4.3), at the critical point P, we have

$$H = \frac{1}{\phi\omega}(-\Delta\gamma + n\phi').$$

Or equivalently,

(5.10)
$$-\frac{1}{\phi\omega}\Delta\gamma = H - \frac{n\phi'}{\phi\omega}.$$

Plug (5.10) into (5.9), after simplifications, we have

$$\begin{aligned}\mathcal{L}(\frac{|\nabla\gamma|^2}{2}) =& \ n\frac{\phi''\phi-(\phi')^2}{\phi\omega}|\nabla\gamma|^4 - \frac{1}{\phi\omega}\big(n-1+n(\phi')^2 - \phi\phi'\omega H\big)|\nabla\gamma|^2 - \frac{1}{\phi\omega}|\nabla\nabla\gamma|^2 \\ =& \ -\frac{1}{\phi\omega}\big[\omega(n\omega - H\cosh\rho\sinh\rho) + n\cosh^2\rho - 1\big]|\nabla\gamma|^2 - \frac{1}{\phi\omega}|\nabla\nabla\gamma|^2,\end{aligned}$$

where we have used $\omega^2 = 1 + |\nabla\gamma|^2$, $\phi = \sinh\rho$, and $\phi''\phi - (\phi')^2 = -1$. At P, if $n\omega \leq H\cosh\rho\sinh\rho$, by the uniform upper bound of H, (c.f. Corollary 3.5), and the uniform upper bound of ρ, we have $\omega \leq C$ and the proof is done. Now we can assume that $n\omega - H\cosh\rho\sinh\rho > 0$. Note $\cosh\rho \geq 1$, thus

(5.11)
$$\begin{aligned}\mathcal{L}(\frac{|\nabla\gamma|^2}{2}) =& \ -\frac{1}{\phi\omega}\big[\omega(n\omega - H\cosh\rho\sinh\rho) + n\cosh^2\rho - 1\big]|\nabla\gamma|^2 - \frac{1}{\phi\omega}|\nabla\nabla\gamma|^2 \\ \leq& \ 0.\end{aligned}$$

By the standard maximal principle, we prove the assertion. $\qquad\square$

5.3. **Long-time existence and exponential convergence.** By direct simplification, we obtain the following lemma by (4.2), (4.3), and (4.5).

Lemma 5.4. *Recall $\rho(z)$, $z \in \mathbb{S}^n$, is a positive function on unit sphere and $\phi(\rho)$ is $\sin(\rho)$, ρ, or $\sinh(\rho)$ when $K = +1, 0, -1$ respectively. Equation (4.5) is then a parabolic equation of divergent form, i.e.,*

$$\gamma_t = \mathrm{div}\left(\frac{1}{\phi\omega}\nabla\gamma\right) + \frac{2\phi'|\nabla\gamma|^2}{\phi\omega},$$

where $\omega = \sqrt{1 + |\nabla\gamma|^2}$, and $\gamma(z) := \gamma(\rho(z))$ satisfies $\frac{\partial\gamma}{\partial\rho} = \frac{1}{\phi(\rho)}$.

Equivalently, $\rho(z,t)$ solves the following divergent parabolic equation,

$$\rho_t = \mathrm{div}\left(\frac{1}{\phi\tilde{\omega}}\nabla\rho\right) + \frac{2\phi'|\nabla\rho|^2}{\phi^2\tilde{\omega}},$$

where $\tilde{\omega} = \sqrt{\phi^2 + |\nabla\rho|^2}$ and all the covariant derivatives are w.r.t. the standard metric on unit sphere.

Proof. (**Proof of Theorem 1.1 for cases : $\mathbb{R}^{n+1}$ and $\mathbb{S}^{n+1}$.**) Since (4.4) or (4.5) is a divergent quasi-linear parabolic equation, by the classical theory of parabolic equation in divergent form (e.g., [18]), the higher regularity a priori estimates of the solution follow from the uniform gradient estimates in Proposition 5.2 and Proposition 5.3. Moreover, the solution of $\rho(\cdot, t)$ exists for all $t \in [0, \infty)$. Now we are ready to prove our main theorem for the Euclidean and elliptic cases, that is the exponential convergence. This follows from the exponential estimate (5.5) in Proposition 5.2. That is, $e^{\beta t}|\nabla\rho(t)| \to 0$ as $t \to \infty$ for each $\beta < \alpha$, and the graph converges to a sphere exponentially fast. This proves our main theorem for the Euclidean and elliptic spaces. $\qquad\square$

What's left is the exponential convergence in hyperbolic space $\mathbb{H}^{n+1}$. Since the gradient estimate implies uniform bounds for all the a priori estimates because the equation is quasilinear divergent parabolic, we now know that the principal curvatures, surface area element, and other geometric quantities are all uniformly bounded from above.

Next, we will show that when t is large enough, different principal curvatures of the radial graph at the same point are comparable uniformly for arbitrary small ϵ.

Proposition 5.5. *Let $\rho(z,t)$ solves the initial value problem (4.4). Then for any $\epsilon > 0$, there exists a $T_0 > 0$, such that for any $t > T_0$,*

(5.12) $$\max_{i<j,z\in\mathbb{S}^n} |\kappa_i(z,t) - \kappa_j(z,t)| < \epsilon,$$

where $\kappa(z,t)$ are the principal curvatures of the radial graph $\rho(z,t)$ at $(z,t) \in \mathbb{S}^n \times [0, \infty)$.

Proof. Let $A(t)$ be the surface area of the radial graph at time $t > 0$, then

$$(5.13) \qquad\qquad A(t) - A(0) = \int_0^t A'(s)ds$$

Since $A'(s) \leq 0$ from Proposition 3.5 and also $A(t)$ is uniformly bounded from above and below, by our regularity estimates, we have $\int_0^\infty A'(t)dt < \infty$. By Proposition 3.5,

$$|A'(t)| = -\frac{1}{n-1}\int_{M(t)}\sum_{i<j}(\kappa_i(x) - \kappa_j(x))^2 u d\mu_g.$$

Since we have established the uniform a priori estimates for all the derivatives of ρ both with respect to z and t for any order and also $0 < C_1 \leq u \leq C_2$ uniformly, $A'(t)$ is uniformly continuous with respect to t and $d\mu_g$ is also uniformly bounded from both up and below. This proves the assertion when T is big enough. $\qquad\square$

Applying Proposition 5.5, we will prove the following exponential convergence theorem which proves that the solution of the initial value problem (4.4) for the hyperbolic case also converges to a standard sphere exponentially.

Proposition 5.6. *Let M_0 be a radial graph of function ρ_0 over $\mathbb{S}^n$ in space form $\mathbb{H}^{n+1}$. If $\gamma(z,t)$ solves (4.5) on interval $[0,T]$, then there exist a uniform constant $\alpha > 0$ which depends only on the initial graph, such that for any $(z,t) \in \mathbb{S}^n \times [0,T]$,*

$$e^{\alpha t}|\nabla\gamma|^2(z,t) \leq \max_{z\in\mathbb{S}^n}|\nabla\gamma|^2(z,0)$$

where the covariant derivatives are w.r.t. the spherical metric on $\mathbb{S}^n$.

Proof. Let $\epsilon > 0$ be a small enough constant to be determined and $T_0 > 0$ be defined as in Proposition 5.5. For any $T > T_0$, assume $\frac{|\nabla\gamma|^2}{2}$ achieves maximum on the interval $[T_0, T]$ at point P. Recall in the proof of Proposition 5.3, we have proved (5.11) at P,
(5.14)
$$\mathcal{L}(\frac{|\nabla\gamma|^2}{2}) \quad \leq \quad -\frac{1}{\omega\sinh\rho}(n\omega^2 - \omega\cosh\rho\sinh\rho H + n\cosh^2\rho - 1)|\nabla\gamma|^2.$$

At the critical point P, we can simplify the Weingarten tensor $h_j^i = \frac{1}{\omega\sinh\rho}(-\gamma_{ij} + \cosh\rho\delta_{ij})$, where δ_{ij} is the standard spherical metric if we choose orthonormal coordinates at this point. By rotating the coordinates, we can assume $\gamma_1 = |\nabla\gamma|$ and $\gamma_j = 0$ for $2 \leq j \leq n$. Then by critical point condition, $\gamma_{1j} = 0$ for all $j = 1, \cdots, n$. Now fixing the $i = 1$ direction, we can diagonalize the rest $(n-1) \times (n-1)$ matrix. Finally, γ_{ij} becomes a diagonal matrix with $\gamma_{11} = 0$. Since the Weingarten tensor $h_j^i = \frac{1}{\omega\sinh\rho}(-\gamma_{ij} +$

$\cosh\rho\delta_{ij})$ at P, it is also diagonalized so the eigenvalues are h_i^i which are also the principal curvatures. Thus at P, the principal curvatures

(5.15) $$\kappa_i = h_i^i = \frac{1}{\omega\sinh\rho}(-\gamma_{ii} + \cosh\rho).$$

By Proposition 5.5 and (5.15), we have

(5.16) $$|\gamma_{ii} - \gamma_{jj}| \leq \epsilon C_1$$

where C_1 is a uniform constant and we have used the uniform boundedness of ρ and ω. Since $\gamma_{11} = 0$, using (5.16), we have $|\gamma_{jj}| = |\gamma_{11} - \gamma_{jj}| \leq \epsilon C_1$ for $2 \leq j \leq n$. Recall at P,

$$-\omega\sinh\rho H = \sum \gamma_{ii} - n\cosh\rho \geq -(n-1)\epsilon C_1 - n\cosh\rho.$$

For small enough ϵ which depends on the uniform bounds of ρ, we have

(5.17) $$\begin{aligned} & n\omega^2 - \omega\cosh\rho\sinh\rho H + n\cosh^2\rho - 1 \\ \geq\ & (n-1) + (n - (n-1)\epsilon C_1\cosh\rho)|\nabla\gamma|^2 \\ \geq\ & n-1. \end{aligned}$$

Now if we plug (5.17) into (5.14), we have

(5.18) $$\mathcal{L}(\frac{|\nabla\gamma|^2}{2}) \leq -\frac{n-1}{\omega\sinh\rho}|\nabla\gamma|^2 \leq -\alpha\frac{|\nabla\gamma|^2}{2},$$

where $\alpha > 0$ is a constant independent of time t. Standard maximal principle proves that

$$\max_{(z,t)\in\mathbb{S}^n\times[T_0,T]} e^{\alpha t}|\nabla\gamma|^2 \leq C_1.$$

Since $\max_{(z,t)\in\mathbb{S}^n\times[0,T_0]} e^{\alpha t}|\nabla\gamma|^2 \leq e^{\alpha T_0}C_2$, where C_2 is the uniform upper bound of $|\nabla\gamma|^2$, we have

$$\max_{(z,t)\in\mathbb{S}^n\times[0,T]} e^{\alpha t}|\nabla\gamma|^2 \leq C,$$

where C is a uniform constant depending on C_1, C_2, and T_0. Since all the constants including T_0 are uniform, we finish the proof.

$\square$

Proof. (**Proof of Theorem 1.1 :** $\mathbb{H}^{n+1}$**.**) With all the a priori estimates and exponential convergence established, the proof is now standard. $\square$

6. Preserving convexity

In this section, we prove flow (1.2) preserves convexity in $\mathbb{R}^{n+1}$. The proof follows arguments in the proof of Theorem 1.4 in Bian-Guan [3]. Since the statement in Theorem 1.4 in [3] does not imply directly the convexity of flow (1.2) (even though its proof does), we state the following general result. We denote $\mathcal{S}^+_n$ to be the set of all positive definite $n \times n$ matrices, $u = X \cdot \nu$, g and h the first and the second fundamental forms of surface M respectively.

The proof in [3] can be adapted by considering u as an independent variable and making use of Lemma 2.9 in the case of $N = \mathbb{R}^{n+1}$.

Theorem 6.1. *Suppose $F(A, X, \nu, u, t)$ is monotone (i.e., $\frac{\partial F}{\partial A_{ij}} > 0$) in A and $F(A^{-1}, X, \nu, u, t)$ is locally convex in $(A, X) \in \mathcal{S}^+{}_n \times \mathbb{R}^{n+1}$ for each fixed $\nu \in \mathbb{S}^n$, $u \in \mathbb{R}^+$, $t \in [0, T]$ for some $T > 0$. Let $M(t)$ be an oriented immersed connected hypersurface in $\mathbb{R}^{n+1}$ with a nonnegative definite second fundamental form $h(t)$ satisfying equation*

$$(6.1) \qquad\qquad X_t = -F(g^{-1}h, X, \nu, u, t)\nu.$$

Then $h(t)$ is of constant rank $l(t)$ for each $t \in (0, T]$ and $l(s) \leq l(t)$ for all $0 < s \leq t \leq T$. Moreover the null space of h is parallel for each t. In particular, if M_0 is compact and convex, then $M(t)$ is strictly convex for all $t \in (0, T)$.

Proof. The proof follows the similar lines of arguments in the proof of Proposition 5.1 in [3], here we will just indicate some modifications needed when u is considered as an independent variable. We use the same notations as in [3]. For $\epsilon > 0$, let $W = (g^{im}h_{mj} + \epsilon\delta_{ij})$, let $l(t)$ be the minimal rank of $h(t)$. For a fixed $t_0 \in (0, T)$, let $x_0 \in M$ such that $h(t_0)$ attains minimal rank at x_0. Set $\phi(x, t) = \sigma_{l+1}(W(x, t)) + \frac{\sigma_{l+2}}{\sigma_{l+1}}(W(x, t))$. It is proved in section 2 in [3] that ϕ is in $C^{1,1}$. The first part of Theorem will follow if we can establish that there are constants C_1, C_2 independent of ϵ such that

$$(6.2) \qquad F^{ij}\phi_{ij} - \phi_t \leq C_1\phi + C_2|\nabla\phi|, \quad \text{near } (x_0, t_0).$$

Denote $h^2 = (h_l^i h_j^l)$. Note that under (6.1), the Weingarten form $h_j^i = g^{im}h_{mj}$ satisfies the equation

$$(6.3) \qquad\qquad \partial_t h_j^i = \nabla^i \nabla_j F + F(h^2)_j^i.$$

As in [3], near (t_0, x_0), the index set $\{1, \cdots, n\}$ can be divided in to two subset B, G, where for $i \in B$, the eigenvalues of (W_{ij}), λ_i is small and for $j \in G$, λ_j is strictly positive away from 0. As in [3], we may assume at each point of computation, $(W_{ij}$ is diagonal. Notice that $W_{ii} \leq \phi$ for all $i \in B$, so we have the following inequality as corresponding to inequality (5.3) in

[3]:

$$\sum F^{\alpha\beta}\phi_{\alpha\beta} - \phi_t = O(\phi + \sum_{i,j\in B}|\nabla W_{ij}|) - \frac{1}{\sigma_1(B)}\sum_{\alpha,\beta}\sum_{i,j\in B, i\neq j}F^{\alpha\beta}W_{ij\alpha}W_{ij\beta}$$

$$-\frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta}\sum_{i\in B}F^{\alpha\beta}(W_{ii\alpha}\sigma_1(B) - W_{ii}\sum_{j\in B}v_{jj\alpha})(W_{ii\beta}\sigma_1(B) - W_{ii}\sum_{j\in B}v_{jj\beta})$$

$$-\sum_{i\in B}[\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}][\sum_{\alpha,\beta,\gamma,\eta\in G}F^{\alpha\beta,\gamma\eta}(\Lambda)W_{i\alpha\beta}W_{i\gamma\eta} + \sum_{\alpha}F^{X^\alpha}X_{ii}^\alpha$$

$$+2\sum_{\alpha\beta\in G}F^{\alpha\beta}\sum_{j\in G}\frac{1}{\lambda_j}W_{ij\alpha}W_{ij\beta} + 2\sum_{\alpha,\beta\in G}\sum_{\gamma=1}^{n+1}F^{\alpha\beta,X^\gamma}W_{i\alpha\beta}X_i^\gamma + \sum_{\gamma,\eta=1}^{n+1}F^{X^\gamma,X^\eta}X_i^\gamma X_i^\eta$$

$$+2\sum_{\alpha,\beta\in G}\sum_{\gamma=1}^{n+1}F^{\alpha\beta,u}W_{i\alpha\beta}u_i + 2\sum_{\gamma=1}^{n+1}F^{u,X^\gamma}u_iX_i^\gamma + F^u u_{ii} + F^{u,u}u_i^2].$$

The term involving X_{ii} and the terms involving u_i, u_{ii} ($i \in B$) can be controlled by W_{ii} and ∇W_{ii} using the Weingarten formula and Lemma 2.9. We obtain

$$\sum F^{\alpha\beta}\phi_{\alpha\beta} - \phi_t = O(\phi + \sum_{i,j\in B}|\nabla W_{ij}|) - \frac{1}{\sigma_1(B)}\sum_{\alpha,\beta}\sum_{i,j\in B, i\neq j}F^{\alpha\beta}W_{ij\alpha}W_{ij\beta}$$

$$-\frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta}\sum_{i\in B}F^{\alpha\beta}(W_{ii\alpha}\sigma_1(B) - W_{ii}\sum_{j\in B}v_{jj\alpha})(W_{ii\beta}\sigma_1(B) - W_{ii}\sum_{j\in B}v_{jj\beta})$$

$$-\sum_{i\in B}[\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}][\sum_{\alpha,\beta,\gamma,\eta\in G}F^{\alpha\beta,\gamma\eta}(\Lambda)W_{i\alpha\beta}W_{i\gamma\eta}$$

$$+2\sum_{\alpha\beta\in G}F^{\alpha\beta}\sum_{j\in G}\frac{1}{\lambda_j}W_{ij\alpha}W_{ij\beta} + 2\sum_{\alpha,\beta\in G}\sum_{\gamma=1}^{n+1}F^{\alpha\beta,X^\gamma}W_{i\alpha\beta}X_i^\gamma + \sum_{\gamma,\eta=1}^{n+1}F^{X^\gamma,X^\eta}X_i^\gamma X_i^\eta].$$

The analysis in the proof of Theorem 3.2 in [3] can be used to show the right hand side of above inequality can be controlled by $\phi + |\nabla\phi| - C\sum_{i,j\in B}|\nabla W_{ij}|$. Inequality (6.2) is verified. The first part of theorem follows from the standard strong maximum principle for parabolic equations.

For the second part of theorem, as in [3], we may approximate M_0 by a strictly convex M_0^ϵ. By continuity, there is $\delta > 0$ (independent of ϵ), such that there is a solution $M^\epsilon(t)$ to (6.1) with $M^\epsilon(0) = M_0^\epsilon$ for $t \in [0,\delta]$. We argue that $M^\epsilon(t)$ is strictly convex for $t \in [0,\delta]$. If not, there is $t_0 > 0$ so that $M^\epsilon(t)$ is strictly convex for $0 \le t < t_0$, but there is one point x_0 such that $(h_{ij}(x_0,t_0))$ is not of full rank. This contradicts the first part of theorem. Taking $\epsilon \to 0$, we conclude that $M(t)$ is convex for all $t \in [0,\delta]$. This implies that the set t where $M(t)$ is convex is open. It is obviously closed. Therefore, $M(t)$ is convex for all $t \in [0,T]$. Again, by first part of theorem, $M(t)$ is strictly convex for all $t \in (0,T)$. $\square$

In the case of flow (1.2), $F(g^{-1}h, X, \nu, u, t) = u\sigma_1(g^{-1}h) - n$, it is clear F satisfies conditions in Theorem 6.1.

Proof. (**Proof of Corollary 1.2.**) The proof follows from our main theorem, the monotonicity of the quermassintegrals (Proposition 3.7) and Theorem 6.1. □

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DEPARTMENT OF MATHEMATICS, MCGILL UNIVERSITY, MONTREAL, QUEBEC. H3A 2K6, CANADA.

E-mail address: guan@math.mcgill.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, AL 35294

E-mail address: jfli@uab.edu