A GEODESIC EQUATION IN THE SPACE OF SASAKIAN METRICS

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Dedicated to Professor S.T. Yau on the occasion of his 60th birthday

This paper is to draw attention to a geodesic equation on space of Sasakian metrics. The equation is connected to some interesting geometric properties of Sasakian manifolds. It can be viewed as a parallel case of a well-known geodesic equation for the space of Kähler metrics introduced in [11, 15, 5].

Let us first recall the definition of the space of Kähler metrics. For a given compact Kähler manifold (M, g) with Kähler metric g. Let

$$\mathcal{H}^{K} = \{ \phi \in C^{2}(M) | (g_{i\bar{j}} + \phi_{i\bar{j}}) > 0 \}$$

be the space of all Kähler metrics in cohomology class of g. Mabuchi [11] introduced a natural Riemannian structure on \mathcal{H}^{K} . A formal calculation in [11] yields that \mathcal{H}^{K} is a non-positive curved symmetric space of non-compact type. Semmes [15] studied the geodesic equation on the Kähler space \mathcal{H}^{K} relating it to a Dirichlet problem of the homogenous complex Monge-Ampere equation

(1)
$$\det(g_{i\bar{j}} + \phi_{i\bar{j}}) = 0$$

This is a degenerate equation. The corresponding elliptic equation on compact Kähler manifold is the famous complex Monge-Ampère equation related to the Calabi conjecture. The solution was established by Yau in this celebrated work [16]. In [5], Donaldson conjectured that the space \mathcal{H}^K of Kähler metrics is geodesically convex by smooth geodesics and that it is a metric space. Donaldson also outlined a strategy to relate the geometry of this infinite dimensional space to the existence problems in Kähler geometry. In [3], Chen obtained important $C^{1,1}$ regularity of the associated Dirichlet problem for the homogeneous complex Monge-Ampère equation. He proved the Kähler metric space is a metric space and partially validated the geodesic conjecture of Donaldson. We also refer [2, 13] for further works.

Back to Sasakian geometry. Let (M, g) be a connected oriented (2n + 1)dimensional smooth Riemannian manifold. (M, g) is said to be a Sasakian manifold if the cone manifold $(C(M), \tilde{g}) = (M \times R^+, r^2g + dr^2)$ is Kähler. Sasakian geometry was introduced by Sasaki [14] and is often described as an odd dimensional counterparts of Kähler geometry. There is an extensive literature on of Sasakian manifolds, in particular the work by Boyer, Galicki and their collaborators. We refer [1] for an up to date account. Recently, some interesting connections of Sasakian geometry with superstring theory in mathematical physics have been found (e.g., see [12] and references therein).

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We list some basic geometric properties associated to Sasakian manifold, they can be found in book [1]. The following equivalent conditions provide three alternative characterizations of the Sasakian property.

(1) There exists a Killing vector field ξ of unit length on M so that the (1,1) type tensor field Φ , defined by $\Phi(X) = \nabla_X \xi$ satisfies the condition

(2)
$$(\nabla_X \Phi)(Y) = \langle \xi, Y \rangle_g X - \langle X, Y \rangle_g \xi$$

for any pair of vector fields X and Y on M.

(2) There exists a Killing vector field ξ of unit length on M so that the Riemann curvature satisfies the condition

(3)
$$R(X,\xi)Y = <\xi, Y >_g X - < X, Y >_g \xi$$

for any pair of vector fields X and Y on M.

- (3) There exists a Killing vector field ξ of unit length on M so that the sectional curvature of every section containing ξ equal one.
- (4) The metric cone $(M \times R^+, r^2g + dr^2)$ is Kähler.

Set $\eta(X) = \langle X, \xi \rangle_g$ for any vector field X on M. Let Φ be (1,1) tensor field which defines a complex structure on the contact sub-bundle $D = ker\{\eta\}$ which annihilates ξ . In view of the above equivalent conditions, (ξ, η, Φ) is called a contact structure on (M, g). The Killing vector field ξ is called the characteristic or Reeb vector field, η is called the contact 1-form. Sasakian manifolds can be studied from many view points as they have many structures. They have a one dimensional foliation, called the Reeb foliation, which has a transverse Kähler structure; they also have a contact structure. Sasakian geometry is a special kind of contact metric geometry such that the structure transverse to the Reeb vector field ξ is Kähler and invariant under the flow of ξ .

Let (ξ, η, Φ, g) be a Sasakian structure on manifold M. Let \mathcal{F}_{ξ} is the characteristic foliation generated by ξ . In order to consider the deformations of Sasakian structures, we consider the quotient bundle of the foliation \mathcal{F}_{ξ} , $\nu(\mathcal{F}_{\xi}) = TM/L\xi$. The metric g gives a bundle isomorphism between \mathcal{F}_{ξ} and the contact sub-bundle D. By this isomorphism, $\Phi|_D$ induces a complex structure \overline{J} on $\nu(\mathcal{F}_{\xi})$. $(D, \Phi|_D, d\eta)$ give M a transverse Kähler structure with Kähler form $\frac{1}{2}d\eta$ and metric g^T defined by $g^T(\cdot, \cdot) = \frac{1}{2}d\eta(\cdot, \Phi \cdot)$. A p-form θ on Sasakian manifold (M, g) is called **basic** if

(4)
$$i_{\mathcal{E}}\theta = 0, \quad L_{\mathcal{E}}\theta = 0,$$

where i_{ξ} is the contraction with the Killing vector field ξ , L_{ξ} is the Lie derivative with respect to ξ . It is easy to see that the exterior differential preserves **basic** forms. Namely, if θ is a **basic** form, so is $d\theta$. Let $\wedge_B^p(M)$ be the sheaf of germs of **basic** p-forms and $\Omega_B^p(M) = \Gamma(M, \wedge_B^p(M))$ the set of all section of $\Lambda_B^p(M)$. The **basic** cohomology can be defined in a usual way.

Let D^C be the complexification of the sub-bundle D, and decompose it into its eigenspaces with respect to $\Phi|_D$, that is

(5)
$$D^C = D^{1,0} \oplus D^{0,1}$$

Similarly, we have a splitting of the complexification of the bundle $\wedge^1_B(M)$ of basic one forms on M,

(6)
$$\wedge^1_B(M) \otimes C = \wedge^{1,0}_B(M) \oplus \wedge^{0,1}_B(M).$$

Let $\wedge_B^{i,j}(M)$ denote the bundle of basic forms of type (i, j). Accordingly, the following decomposition is in place,

(7)
$$\wedge^p_B(M) \otimes C = \bigoplus_{i+j=p} \wedge^{i,j}_B(M).$$

Define ∂_B and $\overline{\partial}_B$ by

(8)
$$\frac{\partial_B}{\partial_B} : \wedge_B^{i,j}(M) \to \wedge_B^{i+1,j}(M); \\ \frac{\partial_B}{\partial_B} : \wedge_B^{i,j}(M) \to \wedge_B^{i,j+1}(M);$$

which is the decomposition of d. Let $d_B^c = \frac{1}{2}\sqrt{-1}(\overline{\partial}_B - \partial_B)$ and $d_B = d|_{\wedge_B^p}$. We have $d_B = \overline{\partial}_B + \partial_B$, $d_B d_B^c = \sqrt{-1}\partial_B\overline{\partial}_B$, $d_B^c = (d_B^c)^2 = 0$.

Fix a Sasakian structure (ξ, η, Φ, g) on compact manifold M. A function φ on M is called **basic** if $\xi \varphi \equiv 0$. We denote the space of all smooth **basic** real function on M by $C_B^{\infty}(M)$. Define

(9)
$$\mathcal{H} = \{ \varphi \in C^{\infty}_B(M) : \eta_{\varphi} \wedge (d\eta_{\varphi})^n \neq 0 \},$$

where

(10)
$$\eta_{\varphi} = \eta + d_B^c \varphi, \quad d\eta_{\varphi} = d\eta + \sqrt{-1} \partial_B \overline{\partial}_B \varphi.$$

A basic (1,1) form θ is called positive at point p if the matrix $(\theta_{i\overline{j}})$ is positive, where $\theta_{i\overline{j}} = -\sqrt{-1}\theta(X_i, \overline{X}_j)$ and $\{X_i\}$ is a basis of $D_p^{1,0}$. $d\eta$ is positive since it represents a transversal Kähler form. \mathcal{H} is contractible. This fact can be observed as follow. Given any $\varphi \in \mathcal{H}$ and suppose p is a minimum point of φ , then we have $d\eta_{\varphi}$ is positive at p. Since $\eta_{\varphi} \wedge (d\eta_{\varphi})^n \neq 0$ everywhere, $d\eta_{\varphi}$ must be positive at every point in \mathcal{M} . Therefore \mathcal{H} can be identified with $\{\varphi \in C_B^\infty(\mathcal{M}) : d\eta_{\varphi} > 0\}$. This implies \mathcal{H} is convex.

Let us consider one class of deformations of Sasakian structure those that fix the characteristic foliation but deform the contact form. Let $\varphi \in \mathcal{H}$ and define

(11)
$$\begin{aligned} \Phi_{\varphi} &= \Phi - \xi \otimes (d_B^c \varphi) \circ \Phi, \\ g_{\varphi} &= \frac{1}{2} d\eta_{\varphi} \circ (Id \otimes \Phi_{\varphi}) + \eta_{\varphi} \otimes \eta_{\varphi} \end{aligned}$$

It follows that $(\xi, \eta_{\varphi}, \Phi_{\varphi}, g_{\varphi})$ is a Sasakian structure on M. Furthermore, we have that $(\xi, \eta_{\varphi}, \Phi_{\varphi}, g_{\varphi})$ and (ξ, η, Φ, g) with the same transversely holomorphic structure on $\nu(\mathcal{F}_{\xi})$ and the same holomorphic structure on the cone C(M).

On the other hand, if $(\xi, \tilde{\eta}, \Phi, \tilde{g})$ is another Sasakian structure with the same Reeb field, the same transversely holomorphic structure on $\nu(\mathcal{F}_{\xi})$, then $[d\eta]_B$ and $[d\tilde{\eta}]_B$ are the same cohomology class in $H_B^{1,1}(M)$. By transversal global $\partial\bar{\partial}$ -lemma from [7], there exists a unique basic function $\tilde{\varphi} \in \mathcal{H}$ up to a constant such that

(12)
$$d\tilde{\eta} = d\eta + \sqrt{-1}\partial_B \overline{\partial}_B \tilde{\varphi}.$$

Furthermore, if $(\xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ induce the same holomorphic structure on the cone C(M)as that by (ξ, η, Φ, g) , then there exists a unique function $\varphi \in \mathcal{H}$ up to a constant such that $\tilde{\eta} = \eta_{\varphi}$, $\tilde{\Phi} = \Phi_{\varphi}$ and $\tilde{g} = g_{\varphi}$. Indeed, Let $\overline{\tilde{g}} = d\tilde{r}^2 + \tilde{r}^2 \tilde{g}$ be the Kähler metric on C(M), since with the same holomorphic structure on C(M), then we have $\tilde{r} \frac{\partial}{\partial \tilde{r}} = J(\xi) = r \frac{\partial}{\partial r}$; this implies that $\tilde{r} = r \exp \frac{1}{2}\varphi$ for some basic function φ , then we have $\tilde{\eta} = d^c \log \tilde{r}^2 = \eta + d^c \varphi = \eta_{\varphi}$. In this sense, we also call \mathcal{H} by the space of Sasakian metrics. Let \mathcal{T} be the space of all transverse Kähler form in the basic (1,1) class $[d\eta]_B$, then the natural map

(13)
$$\mathcal{H} \to \mathcal{T}, \quad \varphi \mapsto \frac{1}{2}(d\eta + \sqrt{-1}\partial_B \overline{\partial}_B \varphi)$$

is surjective. Consider functional $\mathcal{I}: \mathcal{H} \to R$ defined by

(14)
$$\mathcal{I}(\varphi) = \sum_{p=0}^{n} \frac{n!}{(p+1)!(n-p)!} \int_{M} \varphi \eta \wedge (d\eta)^{n-p} \wedge (\sqrt{-1}\partial_{B}\overline{\partial}_{B}\varphi)^{p}.$$

In this paper, we normalize $\int_M \eta \wedge (d\eta)^n = 1$. Set

(15)
$$\mathcal{H}_0 = \{\varphi \in \mathcal{H} | \mathcal{I}(\varphi) = 0\}$$

we have the following identifications

(16)
$$\mathcal{H}_0 \cong \mathcal{T}, \quad \varphi \leftrightarrow \frac{1}{2} (d\eta + \sqrt{-1} \partial_B \overline{\partial}_B \varphi).$$

and

(17)
$$\mathcal{H} \cong \mathcal{H}_0 \times R, \quad \varphi \leftrightarrow (\varphi - \mathcal{I}(\varphi), \mathcal{I}(\varphi)).$$

There is a Weil-Peterson metric in the space \mathcal{H} . This metric induce geodesic equation. A natural connection of the metric can be deduced from the geodesic equation. Let (M, g) be a compact Sasakian manifold with the contact structure (ξ, η, Φ) and let \mathcal{H} defined as in (9). One can define a new Sasakian structure by fixing the Reeb vector field ξ and varying η as follows. $\forall \varphi \in \mathcal{H}$, let η_{φ} and Φ_{φ} defined as in (11), then $(\xi, \eta_{\varphi}, \Phi_{\varphi}, g_{\varphi})$ is a Sasakian structure on M. The following lemma indicates that all these Sasakian metrics have the same volume (e.g., see [1]).

Lemma 1. For any $\varphi \in C^{\infty}_B(M)$, we have

(18)
$$\int_M \eta_{\varphi} \wedge (d\eta_{\varphi})^n = \int_M \eta \wedge (d\eta)^n.$$

Clearly, the tangent space $T\mathcal{H}$ is $C_B^{\infty}(M)$. Each Sasaki potential $\varphi \in \mathcal{H}$ define a measure $d\mu_{\varphi} = \eta_{\varphi} \wedge (d\eta_{\varphi})^n$, so we define a Riemannian metric on the infinite dimensional manifold \mathcal{H} using the L^2 norm provided by this measure,

(19)
$$(\psi_1,\psi_2)_{\varphi} = \int_M \psi_1 \cdot \psi_2 d\mu_{\varphi}, \quad \forall \psi_1,\psi_2 \in C_B^{\infty}(M).$$

For a path $\varphi(t) \in \mathcal{H} \ (0 \le t \le 1)$, the length is given by

(20)
$$\int_0^1 \sqrt{\int_M (\frac{\partial \varphi}{\partial t})^2 d\mu_{\varphi(t)}} \, dt.$$

Lemma 2. The corresponding geodesic equation can be written as

(21)
$$\frac{\partial^2 \varphi}{\partial t^2} d\mu_{\varphi} - n\eta_{\varphi} \wedge (\sqrt{-1}\partial_B(\frac{\partial \varphi}{\partial t}) \wedge \overline{\partial}_B(\frac{\partial \varphi}{\partial t})) \wedge (d\eta_{\varphi})^{n-1} = 0,$$

or, equivalently,

(22)
$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{4} |d_B \frac{\partial \varphi}{\partial t}|^2_{g_{\varphi}} = 0.$$

Proof. Indeed, let $\varphi(t, s) : [0, 1] \times (-\epsilon, \epsilon) \to \mathcal{H}$ be a family of curves with the fixed end points, and supposing that t is the path length parameter when s = 0, i.e. $\int_{\mathcal{M}} (\frac{\partial \varphi}{\partial t})^2 d\mu_{\varphi} \equiv C^*$ when s = 0. By direct calculation, we have

$$\begin{split} &= \frac{2\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial t\partial_s}d\mu_{\varphi}}{\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial s}d\mu_{\varphi}\right) - 2\frac{\partial\varphi}{\partial s}\frac{\partial^2\varphi}{\partial t^2}d\mu_{\varphi} \\ &- 2\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial s}\frac{\partial\varphi}{\partial s}d_B^c\left(\frac{\partial\varphi}{\partial t}\right) \wedge (d\eta_{\varphi})^n - 2\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial s}\eta_{\varphi} \wedge (n\sqrt{-1}\partial_B\overline{\partial}_B\frac{\partial\varphi}{\partial t}) \wedge (d\eta_{\varphi})^{n-1}, \\ &= \left(\frac{\partial\varphi}{\partial t}\right)^2 \frac{\partial}{\partial s}(d\mu_{\varphi}) \\ &= \left(\frac{\partial\varphi}{\partial t}\right)^2 \eta_{\varphi} \wedge (n\sqrt{-1}\partial_B\overline{\partial}_B\frac{\partial\varphi}{\partial s}) \wedge (d\eta_{\varphi})^{n-1} + \left(\frac{\partial\varphi}{\partial t}\right)^2 d_B^c\left(\frac{\partial\varphi}{\partial s}\right) \wedge (d\eta_{\varphi})^n \\ &= -d\left(\left(\frac{\partial\varphi}{\partial t}\right)^2 \eta_{\varphi} \wedge (n\sqrt{-1}\partial_B\frac{\partial\varphi}{\partial s}) \wedge (d\eta_{\varphi})^{n-1}\right) + \left(\frac{\partial\varphi}{\partial t}\right)^2 (n\sqrt{-1}\partial_B\frac{\partial\varphi}{\partial s}) \wedge (d\eta_{\varphi})^n \\ &+ d\left(\frac{\partial\varphi}{\partial t}\right)^2 \wedge \eta_{\varphi} \wedge (n\sqrt{-1}\partial_B\frac{\partial\varphi}{\partial s}) \wedge (d\eta_{\varphi})^{n-1} + \left(\frac{\partial\varphi}{\partial t}\right)^2 d_B^c\left(\frac{\partial\varphi}{\partial s}\right) \wedge (d\eta_{\varphi})^n, \\ &= -d\left((n\sqrt{-1}\frac{\partial\varphi}{\partial s})\eta_{\varphi} \wedge \partial_B\left(\frac{\partial\varphi}{\partial t}\right)^2 \wedge (d\eta_{\varphi})^{n-1}\right) + (n\sqrt{-1}\frac{\partial\varphi}{\partial s})\partial_B\left(\frac{\partial\varphi}{\partial t}\right)^2 \wedge (d\eta_{\varphi})^n \\ &+ (n\sqrt{-1}\frac{\partial\varphi}{\partial s})\eta_{\varphi} \wedge \partial_B\left(\frac{\partial\varphi}{\partial t}\right)^2 \wedge (d\eta_{\varphi})^{n-1} \\ &= -d\left((n\sqrt{-1}\frac{\partial\varphi}{\partial s})\eta_{\varphi} \wedge \partial_B\left(\frac{\partial\varphi}{\partial t}\right)^2 \wedge (d\eta_{\varphi})^{n-1}\right) + (n\sqrt{-1}\frac{\partial\varphi}{\partial s})\partial_B\left(\frac{\partial\varphi}{\partial t}\right)^2 \wedge (d\eta_{\varphi})^n \\ &+ (2(n\sqrt{-1}\frac{\partial\varphi}{\partial s})\eta_{\varphi} \wedge \partial_B\left(\frac{\partial\varphi}{\partial t}\right)^2 \wedge (d\eta_{\varphi})^{n-1} \\ &= -d\left((n\sqrt{-1}\frac{\partial\varphi}{\partial s})\eta_{\varphi} \wedge \partial_B\left(\frac{\partial\varphi}{\partial t}\right)^2 \wedge (d\eta_{\varphi})^{n-1}\right) + (n\sqrt{-1}\frac{\partial\varphi}{\partial s})\partial_B\left(\frac{\partial\varphi}{\partial t}\right)^2 \wedge (d\eta_{\varphi})^n \\ &+ (2(n\sqrt{-1}\frac{\partial\varphi}{\partial s})\eta_{\varphi} \wedge \partial_B\left(\frac{\partial\varphi}{\partial t}\right)^2 \wedge (d\eta_{\varphi})^{n-1} \\ &+ (2(n\sqrt{-1}\frac{\partial\varphi}{\partial s})\eta_{\varphi} \wedge \partial_B\left(\frac{\partial\varphi}{\partial t}\right)^2 \wedge (d\eta_{\varphi})^{n-1}. \end{split}$$

$$= \frac{\frac{d}{ds}L(s)|_{s=0}}{\frac{d}{ds}(\int_0^1 \sqrt{\int_M (\frac{\partial\varphi}{\partial t})^2 d\mu_{\varphi(t)}} dt)|_{s=0}}$$

$$= \frac{1}{2} \int_0^1 (\int_M (\frac{\partial \varphi}{\partial t})^2 d\mu_{\varphi})^{-\frac{1}{2}} |_{s=0} (\int_M 2 \frac{\partial \varphi}{\partial t} \frac{\partial^2 \varphi}{\partial t \partial s} d\mu_{\varphi} + (\frac{\partial \varphi}{\partial t})^2 \frac{\partial}{\partial s} (d\mu_{\varphi}))|_{s=0} dt$$

$$= \int_0^1 (C^*)^{-\frac{1}{2}} (\int_M 2 \frac{\partial \varphi}{\partial s} \{\eta_{\varphi} \wedge (n\sqrt{-1}\partial_B \frac{\partial \varphi}{\partial t} \wedge \overline{\partial}_B \frac{\partial \varphi}{\partial t}) \wedge (d\eta_{\varphi})^{n-1} - \frac{\partial^2 \varphi}{\partial t^2} d\mu_{\varphi}\})|_{s=0} dt$$

The geodesic equation on the infinite dimensional Riemannian manifold ${\mathcal H}$ follows from above formulas. \Box

In [9], it has been proved that every Sasakian manifold can locally be generated by a free real function of 2n variables. This function is a Sasakian analogue of the Kähler potential for the Kähler geometry. More precisely, for any point P in M, one can choose local coordinates $(x, z^1, z^2, \dots, z^n) \in R \times C^n$ on a small neighborhood U around P, such that

(23)
$$\begin{cases} \xi &= \frac{\partial}{\partial x}; \\ \eta &= dx - \sqrt{-1}(h_j dz^j - h_{\overline{j}} d\overline{z}^j); \\ \Phi &= \sqrt{-1}\{X_j \otimes dz^j - \overline{X}_j \otimes d\overline{z}^j\}; \\ g &= \eta \otimes \eta + 2h_{i\overline{j}} dz^i d\overline{z}^l, \end{cases}$$

where $h: U \to R$ is a local **basic** function, i.e. $\frac{\partial h}{\partial x} = 0$, $h_i = \frac{\partial h}{\partial z^i}$, $h_{i\overline{j}} = \frac{\partial^2 h}{\partial z^i \partial \overline{z}^j}$, and $X_j = \frac{\partial}{\partial z^j} + \sqrt{-1}h_j\frac{\partial}{\partial x}$, $\overline{X}_j = \frac{\partial}{\partial \overline{z}^j} - \sqrt{-1}h_{\overline{j}}\frac{\partial}{\partial x}$. In the above, we setting $2dz^i d\overline{z}^j = dz^i \otimes d\overline{z}^j + d\overline{z}^j \otimes dz^i$. In such local coordinates, $D \otimes C$ is spanned by X_i and \overline{X}_i , it is clear that

(24)

$$\begin{aligned}
\Phi X_i &= \sqrt{-1}X_i, \quad \Phi X_i = -\sqrt{-1}X_i, \\
[X_i, X_j] &= [\overline{X}_i, \overline{X}_j] = [\xi, X_i] = [\xi, \overline{X}_i] = 0, \\
[X_i, \overline{X}_j] &= -2\sqrt{-1}h_{i\overline{j}}\xi.
\end{aligned}$$

Obviously, $\{\eta, dz^i, d\overline{z}^j\}$ is the dual basis of $\{\frac{\partial}{\partial x}, X_i, \overline{X}_j\}$, and

(25)
$$d\eta = 2\sqrt{-1}h_{i\bar{j}}dz^i \wedge d\bar{z}^j.$$

Remark 1. For a fixed point $P \in M$, one always can choose the above local coordinates (x, z^1, \dots, z^n) centered at P satisfying additionally that $\{\frac{\partial}{\partial z^i}|_P\} \in D^c$ or equivalently $h_i(P) = 0$ for all j. Indeed, one can only change local coordinates by (y, u^1, \dots, u^n) , where $y = x - \sqrt{-1}h_i(P)z^i + \sqrt{-1}h_{\overline{j}}(P)\overline{z}^j$ and $u^k = z^k$ for all $k = 1, \dots, n$, and change potential function by $h^* = h - h_i(P)u^i - h_{\overline{j}}(P)\overline{u}^j$. Furthermore, in the same way as that in Kähler case, one can choose a normal coordinates (x, z^1, \dots, z^n) such that $h_i(P) = 0$, $h_{i\overline{j}}(P) = \delta_j^i$, and $d(h_{i\overline{j}})|_P = 0$. This local coordinates also be called by a normal coordinates on Sasakian manifold.

In the above local coordinates (x, z^1, \cdots, z^n) , we have

(26)
$$\begin{aligned} \eta_{\varphi} &= dx - \sqrt{-1} \left((h_j + \frac{1}{2} \varphi_j) dz^j - (h_{\overline{j}} + \frac{1}{2} \varphi_{\overline{j}}) d\overline{z}^j \right); \\ \Phi_{\varphi} &= \sqrt{-1} \{ Y_j \otimes dz^j - \overline{Y}_j \otimes d\overline{z}^j \}; \\ g_{\varphi} &= \eta \otimes \eta + 2(h + \frac{1}{2} \varphi)_{i\overline{j}} dz^i d\overline{z}^j \\ d\eta_{\varphi} &= 2\sqrt{-1}(h + \frac{1}{2} \varphi)_{i\overline{j}} dz^i \wedge d\overline{z}^j, \end{aligned}$$

where $Y_j = \frac{\partial}{\partial z^j} + \sqrt{-1}((h_j + \frac{1}{2}\varphi_j)\frac{\partial}{\partial x} \text{ and } \overline{Y}_j = \frac{\partial}{\partial \overline{z}^j} - \sqrt{-1}((h_{\overline{j}} + \frac{1}{2}\varphi_{\overline{j}})\frac{\partial}{\partial x}$. Furthermore,

$$\eta \wedge (d\eta)^n = (2\sqrt{-1})^n n! det(h_{i\bar{j}}) dx \wedge dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^n$$

$$\eta_{\varphi} \wedge (d\eta_{\varphi})^n = (2\sqrt{-1})^n n! det(h_{i\bar{j}} + \frac{1}{2}\varphi_{i\bar{j}}) dx \wedge dz^1 \wedge d\overline{z}^1 \wedge \dots \wedge dz^n \wedge d\overline{z}^n,$$

$$\eta_{\varphi} \wedge (d\eta_{\varphi})^n = \frac{\det(h_{i\bar{j}} + \frac{1}{2}\varphi_{i\bar{j}})}{\det(h_{i\bar{j}})} \eta \wedge (d\eta)^n,$$

$$\begin{array}{rcl} & n\eta_{\varphi} \wedge (\sqrt{-1}\partial_{B}(\frac{\partial\varphi}{\partial t}) \wedge \overline{\partial}_{B}(\frac{\partial\varphi}{\partial t})) \wedge (d\eta_{\varphi})^{n-1} \\ = & (\sqrt{-1})^{n}2^{n-1}n! det(h_{i\bar{j}} + \frac{1}{2}\varphi_{i\bar{j}})h_{\varphi}^{i\bar{j}}\partial_{i}(\frac{\partial\varphi}{\partial t})\partial_{\bar{j}}(\frac{\partial\varphi}{\partial t}) dx \wedge dz^{1} \wedge d\bar{z}^{1} \wedge \dots \wedge dz^{n} \wedge d\bar{z}^{n} \\ = & \frac{1}{2}h_{\varphi}^{i\bar{j}}\partial_{i}(\frac{\partial\varphi}{\partial t})\partial_{\bar{j}}(\frac{\partial\varphi}{\partial t})\eta_{\varphi} \wedge (d\eta_{\varphi})^{n} \\ = & \frac{1}{4}|d\frac{\partial\varphi}{\partial t}|_{g_{\varphi}}^{2}\eta_{\varphi} \wedge (d\eta_{\varphi})^{n} \end{array}$$

where $h^{i\bar{l}}_{\varphi}(h+\frac{1}{2}\varphi)_{j\bar{l}}=\delta^i_j$. In turn, the geodesic equation can be written as

(27)
$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} h_{\varphi}^{i\bar{j}} \partial_i (\frac{\partial \varphi}{\partial t}) \partial_{\bar{j}} (\frac{\partial \varphi}{\partial t}) = 0.$$

Next, we define the corresponding affine connection of ${\mathcal H}$ from the geodesic equation.

Let $\varphi(t) : [0,1] \to \mathcal{H}$ be any path in \mathcal{H} and $\psi(t)$ be another **basic** function on $M \times [0,1]$, which we regard as a vector field along the path $\varphi(t)$. Then we define the covariant derivative of ψ along the path φ by

(28)
$$D_{\dot{\varphi}}\psi = \frac{\partial\psi}{\partial t} - \frac{1}{4} < d_B\psi, d_B\dot{\varphi} >_{g_{\varphi}},$$

where $\langle \rangle_{g_{\varphi}}$ is the Riemannian inner product on co-tangent vectors to (M, g_{φ}) , and $\dot{\varphi} = \frac{\partial \varphi}{\partial t}$.

Let $\varphi : [0,1] \to \mathcal{H}$ be a smooth path in \mathcal{H} . Then φ is called a geodesic in \mathcal{H} if one of the following equivalent conditions is satisfied:

(1) $D_{\dot{\varphi}}\dot{\varphi} \equiv 0$, i.e. $\dot{\varphi}$ is parallel along φ ;

(2) $\ddot{\varphi} = \frac{1}{4} |d_B \dot{\varphi}|^2_{g_{\varphi}}$ on $M \times [0, 1]$.

The following is a Sasakian counterpart of Donaldson's geodesic conjecture for the space of Kähler metrics.

Question: Are any two Sasakian metrics in the space \mathcal{H} connected by a smooth geodesic?

We shall now show that the above connection is compatible with the Riemannian structure defined in (19).

Proposition 1. Let $\varphi : [0,1] \to \mathcal{H}$ be a smooth path in \mathcal{H} , and ψ_1, ψ_2 are two **basic** function on $M \times [0,1]$. Then, we have

(29)
$$\frac{d}{dt}(\psi_1,\psi_2)_{\varphi} = (D_{\dot{\varphi}}\psi_1,\psi_2)_{\varphi} + (\psi_1,D_{\dot{\varphi}}\psi_2)_{\varphi}.$$

Proof. By direct calculation, we have

$$\begin{split} \frac{d}{dt}(\psi_1,\psi_2)_{\varphi} &= \int_M \frac{\partial \psi_1}{\partial t}\psi_2 + \psi_1 \frac{\partial \psi_2}{\partial t} d\mu_{\varphi} \\ &+ \int_M \psi_1 \psi_2 \frac{\partial}{\partial t} (d\mu_{\varphi}) \\ &= \int_M \frac{\partial \psi_1}{\partial t} \psi_2 + \psi_1 \frac{\partial \psi_2}{\partial t} d\mu_{\varphi} \\ &- \int_M n \sqrt{-1} \eta_{\varphi} \wedge \partial_B(\psi_1 \psi_2) \wedge \overline{\partial}_B(\dot{\varphi}) \wedge (d\eta_{\varphi})^{n-1} \\ &= \int_M \frac{\partial \psi_1}{\partial t} \psi_2 + \psi_1 \frac{\partial \psi_2}{\partial t} d\mu_{\varphi} \\ &- \int_M n \sqrt{-1} \psi_2 \eta_{\varphi} \wedge \partial_B(\psi_1) \wedge \overline{\partial}_B(\dot{\varphi}) \wedge (d\eta_{\varphi})^{n-1} \\ &- \int_M n \sqrt{-1} \psi_1 \eta_{\varphi} \wedge \partial_B(\psi_2) \wedge \overline{\partial}_B(\dot{\varphi}) \wedge (d\eta_{\varphi})^{n-1} \\ &= (D_{\dot{\varphi}} \psi_1, \psi_2)_{\varphi} + (\psi_1, D_{\dot{\varphi}} \psi_2)_{\varphi}. \end{split}$$

For any $\varphi_0 \in \mathcal{H}$, and any $\psi_1, \psi_2, \psi_3 \in C_B^{\infty}(M)$. We can consider a smooth function $\varphi(s,t) : [-\epsilon,\epsilon] \times [-\epsilon,\epsilon] \to \mathcal{H}$ such that: (1) $\varphi(0,0) = \varphi_0$; (2) $\frac{\partial \varphi}{\partial s}|_{(0,0)} = \psi_1$, and $\frac{\partial \varphi}{\partial t}|_{(0,0)} = \psi_2$. We define the torsion $\tau : T\mathcal{H} \otimes T\mathcal{H} \to T\mathcal{H}$ and the curvature tensor $R : T\mathcal{H} \otimes T\mathcal{H} \to End(T\mathcal{H})$ of the above connection by

(30)
$$\tau_{\varphi_0}(\psi_1,\psi_2) = \left(D_{\frac{\partial\varphi}{\partial s}}\frac{\partial\varphi}{\partial t} - D_{\frac{\partial\varphi}{\partial t}}\frac{\partial\varphi}{\partial s}\right)|_{(0,0)},$$

(31)
$$R_{\varphi_0}(\psi_1,\psi_2)\psi_3 = \left(D_{\frac{\partial\varphi}{\partial s}}D_{\frac{\partial\varphi}{\partial t}} - D_{\frac{\partial\varphi}{\partial t}}D_{\frac{\partial\varphi}{\partial s}}\right)\psi_3|_{(0,0)},$$

at the point φ_0 .

By the definition, it is easy to check that $\tau_{\varphi_0}(\psi_1, \psi_2) = 0$ for all $\psi_1, \psi_2 \in C_B^{\infty}$ and $\varphi_0 \in \mathcal{H}$.

Proposition 2. $\tau \equiv 0$, that is, the connection is torsion free.

Fixing a Sasakian metric $\varphi_0 \in \mathcal{H}$, and $p \in M$ arbitrarily. By Remark 1, there exist local normal coordinates (x, z^1, \dots, z^n) centered at p on the Sasakian manifold (M, g_{φ_0}) such that $(h_{\varphi_0})_{i\overline{j}}(p) = \delta_{ij}$ and $d(h_{\varphi_0})_{i\overline{j}}(p) = 0$. By direct calculation,

$$(32) \begin{array}{l} D_{\frac{\partial\varphi}{\partial s}} D_{\frac{\partial\varphi}{\partial t}} \psi_{3} = D_{\frac{\partial\varphi}{\partial s}} \left(\frac{\partial\psi_{3}}{\partial t} - \frac{1}{4} < d_{B}\psi_{3}, d_{B}\frac{\partial\varphi}{\partial t} > \right) \\ = \frac{\partial^{2}\psi_{3}}{\partial t\partial s} - \frac{1}{4} < d_{B} \left(\frac{\partial\psi_{3}}{\partial t} \right), d_{B} \left(\frac{\partial\varphi}{\partial s} \right) >_{\varphi} - \frac{1}{4} < d_{B} \left(\frac{\partial\psi_{3}}{\partial s} \right), d_{B} \left(\frac{\partial\varphi}{\partial t} \right) >_{\varphi} \\ - \frac{1}{4} < d_{B}(\psi_{3}), d_{B} \left(\frac{\partial^{2}\varphi}{\partial s\partial t} \right) >_{\varphi} \\ + \frac{1}{8}h_{\varphi}^{k\bar{j}} \left(\frac{\partial\varphi}{\partial t} \right)_{\bar{j}} \left(\frac{\partial\varphi}{\partial s} \right)_{k\bar{l}} h_{\psi}^{k\bar{l}} (\psi_{3})_{i} + \frac{1}{8}h_{\varphi}^{k\bar{j}} \left(\frac{\partial\varphi}{\partial t} \right)_{i} \left(\frac{\partial\varphi}{\partial s} \right)_{k\bar{l}} h_{\varphi}^{k\bar{l}} (\psi_{3})_{\bar{j}} \\ + \frac{1}{16} \left(h_{\varphi}^{k\bar{l}} \left(\frac{\partial\varphi}{\partial s} \right)_{\bar{l}} \partial_{k} \right) \left(h_{\varphi}^{i\bar{j}} \left(\frac{\partial\varphi}{\partial t} \right)_{\bar{j}} \partial_{i} + h_{\varphi}^{i\bar{j}} \left(\frac{\partial\varphi}{\partial t} \right)_{i} \partial_{\bar{j}} \right) \psi_{3} \\ + \frac{1}{16} \left(h_{\varphi}^{k\bar{l}} \left(\frac{\partial\varphi}{\partial s} \right)_{k} \partial_{\bar{l}} \right) \left(h_{\varphi}^{i\bar{j}} \left(\frac{\partial\varphi}{\partial t} \right)_{\bar{j}} \partial_{i} + h_{\varphi}^{i\bar{j}} \left(\frac{\partial\varphi}{\partial t} \right)_{i} \partial_{\bar{j}} \right) \psi_{3}, \end{array}$$

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$$(33) \begin{array}{rcl} D_{\frac{\partial\varphi}{\partial t}} D_{\frac{\partial\varphi}{\partial s}} \psi_{3} &= D_{\frac{\partial\varphi}{\partial t}} \left(\frac{\partial\psi_{3}}{\partial s} - \frac{1}{4} < d_{B}\psi_{3}, d_{B}\frac{\partial\varphi}{\partial s} > \right) \\ &= \frac{\partial^{2}\psi_{3}}{\partial s\partial t} - \frac{1}{4} < d_{B} \left(\frac{\partial\psi_{3}}{\partial s} \right), d_{B} \left(\frac{\partial\varphi}{\partial t} \right) >_{\varphi} - \frac{1}{4} < d_{B} \left(\frac{\partial\psi_{3}}{\partial t} \right), d_{B} \left(\frac{\partial\varphi}{\partial t} \right) >_{\varphi} \\ &- \frac{1}{4} < d_{B}(\psi_{3}), d_{B} \left(\frac{\partial^{2}\varphi}{\partial t\partial s} \right) >_{\varphi} \\ &+ \frac{1}{8}h_{\varphi}^{k\bar{j}} \left(\frac{\partial\varphi}{\partial s} \right)_{\bar{j}} \left(\frac{\partial\varphi}{\partial t} \right)_{k\bar{l}} h_{\psi}^{k\bar{l}} (\psi_{3})_{i} + \frac{1}{8}h_{\varphi}^{k\bar{j}} \left(\frac{\partial\varphi}{\partial s} \right)_{i} \left(\frac{\partial\varphi}{\partial t} \right)_{k\bar{l}} h_{\psi}^{i\bar{l}} (\psi_{3})_{\bar{j}} \\ &+ \frac{1}{16} (h_{\varphi}^{k\bar{l}} \left(\frac{\partial\varphi}{\partial t} \right)_{\bar{l}} \partial_{k}) (h_{\varphi}^{i\bar{l}} \left(\frac{\partial\varphi}{\partial s} \right)_{\bar{j}} \partial_{i} + h_{\varphi}^{i\bar{j}} \left(\frac{\partial\varphi}{\partial s} \right)_{i} \partial_{\bar{j}}) \psi_{3} \\ &+ \frac{1}{16} (h_{\varphi}^{k\bar{l}} \left(\frac{\partial\varphi}{\partial t} \right)_{k} \partial_{\bar{l}}) (h_{\varphi}^{i\bar{l}} \left(\frac{\partial\varphi}{\partial s} \right)_{\bar{j}} \partial_{i} + h_{\varphi}^{i\bar{j}} \left(\frac{\partial\varphi}{\partial s} \right)_{i} \partial_{\bar{j}}) \psi_{3} \end{array}$$

and

$$\begin{array}{rcl} & R_{\varphi_{0}}(\psi_{1},\psi_{2})\psi_{3} \\ = & \frac{1}{8}\{h_{\varphi}^{k\bar{j}}(\psi_{2})_{\bar{j}}(\psi_{1})_{k\bar{l}}h_{\varphi}^{i\bar{l}}(\psi_{3})_{i} + h_{\varphi}^{k\bar{j}}(\psi_{2})_{i}(\psi_{1})_{k\bar{l}}h_{\varphi}^{i\bar{l}}(\psi_{3})_{\bar{j}}\} \\ & -\frac{1}{8}\{h_{\varphi}^{k\bar{j}}(\psi_{1})_{\bar{j}}(\psi_{2})_{k\bar{l}}h_{\varphi}^{i\bar{l}}(\psi_{3})_{i} + h_{\varphi}^{k\bar{j}}(\psi_{1})_{i}(\psi_{2})_{k\bar{l}}h_{\varphi}^{i\bar{l}}(\psi_{3})_{\bar{j}}\} \\ (34) & +\frac{1}{16}[h_{\varphi}^{k\bar{l}}(\psi_{1})_{\bar{l}}\partial_{k}, h_{\varphi}^{i\bar{j}}(\psi_{2})_{\bar{j}}\partial_{i} + h_{\varphi}^{i\bar{j}}(\psi_{2})_{i}\partial_{\bar{j}}]\psi_{3} \\ & +\frac{1}{16}[h_{\varphi}^{k\bar{l}}(\psi_{1})_{k}\partial_{\bar{l}}, h_{\varphi}^{i\bar{j}}(\psi_{2})_{\bar{j}}\partial_{i} + h_{\varphi}^{i\bar{j}}(\psi_{2})_{i}\partial_{\bar{j}}]\psi_{3} \\ = & \frac{1}{16} < \partial_{B}\psi_{3}, \overline{\partial}_{B}(<\partial_{B}\psi_{1}, \overline{\partial}_{B}\psi_{2} > \varphi_{0} - <\partial_{B}\psi_{2}, \overline{\partial}_{B}\psi_{1} > \varphi_{0}) > \varphi_{0} \\ & -\frac{1}{16} < \overline{\partial}_{B}\psi_{3}, \partial_{B}(<\partial_{B}\psi_{1}, \overline{\partial}_{B}\psi_{2} > \varphi_{0} - <\partial_{B}\psi_{2}, \overline{\partial}_{B}\psi_{1} > \varphi_{0}) > \varphi_{0} \end{array}$$

Let $\sigma \subset T\mathcal{H}$ be a plane spanned by two *R*-linearly independent vectors $\psi_1, \psi_2 \in C_B^{\infty}(M)$, then the sectional curvature $K_{\varphi_0}(\sigma)$ of \mathcal{H} at point φ_0 along the plane σ by

(35)
$$K_{\varphi_0}(\sigma) = \frac{(R_{\varphi_0}(\psi_1, \psi_2)\psi_2, \psi_1)_{\varphi_0}}{(\psi_1, \psi_1)_{\varphi_0}(\psi_2, \psi_2)_{\varphi_0} - (\psi_1, \psi_2)_{\varphi_0}^2}.$$

Theorem 1. The sectional curvature of \mathcal{H} is non-positive, i.e. $K_{\varphi_0}(\sigma) \leq 0$ for every plane in $T\mathcal{H}$.

Proof. Suppose that 2-plane σ spanned by $\psi_1, \psi_2 \in C^{\infty}_B(M)$, and denote $f = \langle \partial_B \psi_1, \overline{\partial}_B \psi_2 \rangle_{\varphi_0} - \langle \partial_B \psi_2, \overline{\partial}_B \psi_1 \rangle_{\varphi_0}$ then

$$(R_{\varphi_{0}}(\psi_{1},\psi_{2})\psi_{2},\psi_{1})_{\varphi_{0}} = \int_{M} \frac{1}{16}\psi_{1} < \partial_{B}\psi_{2} - \overline{\partial}_{B}\psi_{2}, d_{B}(f) >_{\varphi_{0}} s\mu_{\varphi_{0}}$$

$$= \int_{M} \frac{n}{4}\psi_{1}(\sqrt{-1}\partial_{B}\psi_{2}\wedge\overline{\partial}_{B}f)\wedge\eta_{\varphi_{0}}\wedge(d\eta_{\varphi_{0}})^{n-1}$$

$$+ \int_{M} \frac{n}{4}\psi_{1}(\sqrt{-1}\partial_{B}\psi_{2}\wedge\partial_{B}f)\wedge\eta_{\varphi_{0}}\wedge(d\eta_{\varphi_{0}})^{n-1}$$

$$= \int_{M} \frac{n}{4}(\sqrt{-1}\partial_{B}\psi_{2}\wedge\overline{\partial}_{B}(f\psi_{1}))\wedge\eta_{\varphi_{0}}\wedge(d\eta_{\varphi_{0}})^{n-1}$$

$$+ \int_{M} \frac{n}{4}(\sqrt{-1}\partial_{B}\psi_{2}\wedge\partial_{B}(\psi_{1}f))\wedge\eta_{\varphi_{0}}\wedge(d\eta_{\varphi_{0}})^{n-1}$$

$$+ \int_{M} \frac{n}{4}f(\sqrt{-1}\partial_{B}\psi_{2}\wedge\partial_{B}(\psi_{1}f))\wedge\eta_{\varphi_{0}}\wedge(d\eta_{\varphi_{0}})^{n-1}$$

$$= \frac{1}{16}\int_{M} f(<\partial_{B}\psi_{1},\overline{\partial}_{B}\psi_{2}>_{\varphi_{0}} - <\partial_{B}\psi_{2},\overline{\partial}_{B}\psi_{1}>_{\varphi_{0}})d\mu_{\varphi_{0}}$$

$$= -\frac{1}{16}\int_{M} |f|^{2}d\mu_{\varphi_{0}} \leq 0.$$

In the following, we consider the subset $\mathcal{H}_0 \subset \mathcal{H}$ which identities with the space of transverse Kähler metric \mathcal{T} . The function \mathcal{I} gives a 1-form α on \mathcal{H} with

(37)
$$\alpha_{\varphi}(\psi) = d\mathcal{I}_{\varphi}(\psi) = \int_{M} \psi \eta_{\varphi} \wedge (\eta_{\varphi})^{n}$$

for every $\varphi \in \mathcal{H}$ and $\psi \in C^{\infty}_B(M)$. Then, there is obviously a decomposition of the tangent space:

(38)
$$T_{\varphi}\mathcal{H} = T_{\varphi}\mathcal{H}_0 \oplus R = \{\psi \in C_B^{\infty}(M) | \alpha_{\varphi}(\psi) = 0\} \oplus R$$

for every $\varphi \in \mathcal{H}_0$.

A subset \mathcal{B} is said to be **totally geodesic** in \mathcal{H} if for every smooth path $\varphi_t, a \leq \varphi_t$ $t \leq b$ in \mathcal{B} , the operator $D_{\dot{\varphi}}$ preserves the subset $\Gamma([a, b], \varphi^*T\mathcal{B})$ of $\Gamma([a, b], \varphi^*T\mathcal{H})$, where Γ means the space of smooth section.

A subset \mathcal{B} is said to be **totally convex** in \mathcal{H} if for every geodesic $\varphi_t, a \leq t \leq b$ in \mathcal{H} with $\varphi_a, \varphi_b \in \mathcal{B}$ always lies in \mathcal{B} .

Theorem 2. $\mathcal{H}_0 \subset \mathcal{H}$ is totally geodesic and totally convex.

Proof. To prove \mathcal{H}_0 is totally geodesic. Let $\varphi_t, a \leq t \leq b$ is a smooth path in \mathcal{H}_0 and suppose that $\{\psi_t | a \leq t \leq b\} \in \Gamma([a, b], \varphi^*T\mathcal{H}_0)$. It suffices to show that $D_{\dot{\varphi}}\psi \in T_{\varphi_t}\mathcal{H}_0$ or equivalently $(D_{\dot{\varphi}}\psi,1)_{\varphi} = 0$ for every $t \in [a,b]$. Since $D_{\dot{\varphi}}1 = 0$ and $(\psi, 1)_{\varphi} = \alpha_{\varphi}(\psi) = 0$, we have $(D_{\varphi}\psi, 1)_{\varphi} = \frac{d}{dt}(\psi, 1)_{\varphi} = 0$. Let $\varphi_t, a \leq t \leq b$ be a geodesic in \mathcal{H} with $\varphi_a, \varphi_b \in \mathcal{H}_0$. Then we have

(39)
$$\frac{d^2}{dt^2}\mathcal{I}(\varphi_t) = \frac{d}{dt}(\alpha_{\varphi}(\dot{\varphi})) = (D_{\dot{\varphi}}\dot{\varphi}, 1)_{\varphi} = 0$$

for every $t \in [a, b]$. Furthermore $\mathcal{I}(\varphi_a) = \mathcal{I}(\varphi_b) = 0$, then $\mathcal{I}(\varphi_t) = 0$ for all $t \in [a, b]$, i.e. the geodesic $\varphi_t, a \leq t \leq b$ lies in \mathcal{H}_0 .

When (M, q) is a 3-dimensional Sasakian manifold, there is a simple connection of the geodesic equation in Lemma 2 with the geodesic equation discussed by Donaldson [6] for the space of volume forms on Riemannian manifold.

In [6], Donaldson introduced a Weil-Peterson type metric on the space of volume forms on any Riemannian manifold (M, g) with fixed volume. This infinite dimensional space can be parameterized by smooth functions such that

(40)
$$H^* = \{ f \in C^{\infty}(M) : 1 + \triangle_g f > 0 \}$$

The tangent space is exactly $C^{\infty}(M)$. A Riemannian structure was introduced by Donaldson,

(41)
$$\langle \psi_1, \psi_2 \rangle_f = \int_M \psi_1 \cdot \psi_2 (1 + \triangle_g f) dv_g.$$

The energy on a path $\psi(t): [0,1] \to \mathcal{H}^*$ can be defined as

(42)
$$\int_0^1 \int_M (\frac{\partial \psi}{\partial t})^2 (1 + \Delta_g \psi) dv_g.$$

From this, one may induce the geodesic equation

(43)
$$\frac{\partial^2 \psi}{\partial t^2} (1 + \Delta_g \psi) - |d \frac{\partial \psi}{\partial t}|_g^2 = 0.$$

Again, this equation is degenerate. To approach the above degenerated elliptic equation, Donaldson introduced a perturbation of the geodesic equation

(44)
$$\frac{\partial^2 \psi}{\partial t^2} (1 + \Delta_g \psi) - |d \frac{\partial \psi}{\partial t}|_g^2 = \epsilon.$$

for any $\epsilon > 0$. The idea is to find solution of equation (44) at each $\epsilon > 0$ level and hope there is certain control of these solutions so that one may pass $\epsilon \to 0$ to obtain solution of the original equation (43).

Indeed, (44) is an elliptic fully nonlinear equation. Very recently, Chen and He [4] established uniform L^{∞} estimates of $\Delta \psi$ for solutions of (44).

Back to geodesic equation in Lemma 2. When n = 1, in a local coordinate (x, z^1) , the geodesic equation on the infinite dimensional Riemannian manifold \mathcal{H} can be written as following,

(45)
$$0 = \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} h_{\varphi}^{1\bar{1}} \partial_1(\frac{\partial \varphi}{\partial t}) \partial_{\bar{1}}(\frac{\partial \varphi}{\partial t}) \\ = \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} h_{\varphi}^{1\bar{1}} h_{1\bar{1}} h^{1\bar{1}} \partial_1(\frac{\partial \varphi}{\partial t}) \partial_{\bar{1}}(\frac{\partial \varphi}{\partial t}) \\ = \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{4} (1 + \frac{1}{4} \triangle_g \varphi)^{-1} |d\frac{\partial \varphi}{\partial t}|_g^2,$$

where we used $\triangle_g \varphi = 2h^{1\bar{1}} \varphi_{1\bar{1}}$ since φ is a **basic** function. Therefore, in 3-dimensional case, the geodesic equation is equivalent to the following equation,

(46)
$$\frac{\partial^2 \varphi}{\partial t^2} \left(1 + \frac{1}{4} \Delta_g \varphi\right) - \frac{1}{4} \left| d \frac{\partial \varphi}{\partial t} \right|_g^2 = 0$$

Any path φ_t in \mathcal{H} , parameterized by $t \in [0, 1]$, can be seen as a function on the product manifold $M \times [0, 1]$. Set

(47)
$$\phi(\cdot, t) = \varphi_t(\cdot).$$

If $\varphi_t \in \mathcal{H}$ is a geodesic connecting two fixed Sasakian metrics $\varphi_0, \varphi_1 \in \mathcal{H}$, the corresponding function ϕ must solve the following Dirichlet problem on $M \times [0, 1]$,

(48)
$$\begin{cases} \frac{\partial^2 \phi}{\partial t^2} (1 + \frac{1}{4} \triangle_g \phi) - \frac{1}{4} |d \frac{\partial \phi}{\partial t}|_g^2 = 0, \\ \phi(\cdot, 0) = \varphi_0(\cdot), \\ \phi(\cdot, 1) = \varphi_1(\cdot), \end{cases}$$

with $1 + \frac{1}{4} \triangle_g \phi > 0$.

Replacing φ by $\psi = \frac{1}{4}\varphi$, equation (46) is the same as equation (44). To this end, we consider the following Dirichlet problem in $\epsilon > 0$ level.

(49)
$$\begin{cases} \frac{\partial^2 \phi}{\partial t^2} (1 + \frac{1}{4} \triangle_g \phi) - \frac{1}{4} |d \frac{\partial \phi}{\partial t}|_g^2 = \epsilon, \\ \phi(\cdot, 0) = \varphi_0(\cdot), \\ \phi(\cdot, 1) = \varphi_1(\cdot), \end{cases}$$

In order to use Chen-He's result, we need to check that the solution to the Dirichlet problem (49) with **basic** boundary data is **basic**.

Lemma 3. Suppose $\phi \in C^3(M \times [0,1])$ is a solution of the Dirichlet problem (49) with basic boundary data φ_0, φ_1 , then $\phi(\cdot, t)$ is basic for each $t \in [0,1]$.

Proof. Choose a local coordinates (x, z^1) around the considered point, and denoting that $X_1 = \frac{\partial}{\partial z^1} + \sqrt{-1}h_1\frac{\partial}{\partial x}, \ \overline{X}_1 = \frac{\partial}{\partial \overline{z}^1} - \sqrt{-1}h_1\frac{\partial}{\partial x}$. Then,

$$\begin{split} &\xi \triangle_g \varphi = \frac{\partial}{\partial x} (\nabla_g d\varphi(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) + 2h^{1\bar{1}} \nabla_g d\varphi(X_1, \overline{X}_1)) \\ &= \frac{\partial}{\partial x} \{ \frac{\partial^2 \varphi}{\partial x^2} + 2h^{1\bar{1}} (X_1 \overline{X}_1 \varphi + \sqrt{-1} h_{1\bar{1}} \frac{\partial \varphi}{\partial x}) \} \\ &= \frac{\partial}{\partial x} \{ (1 + 2h^{1\bar{1}} h_1 h_{\bar{1}}) \frac{\partial^2 \varphi}{\partial x^2} + 2h^{1\bar{1}} \frac{\partial^2 \varphi}{\partial z^1 \partial \bar{z}^1} - 2\sqrt{-1} h^{1\bar{1}} h_{\bar{1}} \frac{\partial^2 \varphi}{\partial x \partial z^1} + 2\sqrt{-1} h^{1\bar{1}} h_1 \frac{\partial^2 \varphi}{\partial x \partial \bar{z}^1} \} \\ &= \Delta_g(\xi \varphi), \end{split}$$

and

$$\begin{split} & \xi | d\frac{\partial \varphi}{\partial t} |_{g}^{2} = \frac{\partial}{\partial x} \{ (\frac{\partial}{\partial x} \frac{\partial \varphi}{\partial t})^{2} + 2h^{11} (X_{1} \frac{\partial \varphi}{\partial t}) (\overline{X}_{1} \frac{\partial \varphi}{\partial t}) \} \\ &= \frac{\partial}{\partial x} \{ (\frac{\partial}{\partial x} \frac{\partial \varphi}{\partial t})^{2} + 2h^{1\overline{1}} \frac{\partial^{2} \varphi}{\partial t \partial z^{1}} \frac{\partial^{2} \varphi}{\partial t \partial \overline{z}^{1}} + 2h^{1\overline{1}} h_{1} h_{1} h_{1} (\frac{\partial^{2} \varphi}{\partial t \partial x})^{2} \\ &- 2\sqrt{-\overline{1}} h^{1\overline{1}} h_{1} \frac{\partial^{2} \varphi}{\partial t \partial z^{1}} \frac{\partial^{2} \varphi}{\partial t \partial z^{1}} + 2\sqrt{-\overline{1}} h^{1\overline{1}} h_{1} \frac{\partial^{2} \varphi}{\partial t \partial \overline{z}^{1}} \frac{\partial^{2} \varphi}{\partial t \partial x} \} \\ &= 2\frac{\partial^{2} \varphi}{\partial t \partial x} (\frac{\partial^{2}}{\partial t \partial x} \frac{\partial \varphi}{\partial x}) + 2h^{1\overline{1}} \{2h_{1} h_{\overline{1}} \frac{\partial^{2} \varphi}{\partial t \partial x} (\frac{\partial^{2}}{\partial t \partial x} \frac{\partial \varphi}{\partial x}) \\ &+ \frac{\partial^{2} \varphi}{\partial t \partial z^{1}} (\frac{\partial^{2}}{\partial t \partial \overline{z}^{1}} \frac{\partial \varphi}{\partial x}) + (\frac{\partial^{2}}{\partial t \partial z} \frac{\partial \varphi}{\partial x}) \frac{\partial^{2} \varphi}{\partial t \partial \overline{z}^{1}} \\ &- \sqrt{-\overline{1}} h_{\overline{1}} \frac{\partial \varphi}{\partial t \partial \overline{z}^{1}} (\frac{\partial^{2}}{\partial x} \frac{\partial \varphi}{\partial x}) - \sqrt{-\overline{1}} h_{\overline{1}} (\frac{\partial^{2}}{\partial t \partial \overline{z}^{1}} \frac{\partial \varphi}{\partial t \partial x}) \frac{\partial^{2} \varphi}{\partial t \partial x} \\ &+ \sqrt{-\overline{1}} h_{1} (\frac{\partial^{2}}{\partial t \partial \overline{z}^{1}} \frac{\partial \varphi}{\partial x}) \frac{\partial^{2} \varphi}{\partial t \partial \overline{x}} + \sqrt{-\overline{1}} h_{1} (\frac{\partial^{2} \varphi}{\partial t \partial \overline{z}^{1}} (\frac{\partial^{2} \varphi}{\partial t \partial x} \frac{\partial \varphi}{\partial x}) \} \\ &= 2 < d\frac{\partial \varphi}{\partial t}, d\frac{\partial \psi}{\partial t} (\frac{\partial \varphi}{\partial x}) >_{g} = 2 < d\frac{\partial \varphi}{\partial t}, d\frac{\partial \psi}{\partial t} (\xi \varphi) >_{g} . \end{split}$$

From equation (49) and above formula,

(50)
$$(1 + \frac{1}{4} \triangle_g \varphi) \frac{\partial^2}{\partial t^2} (\xi \varphi) + \frac{1}{4} \frac{\partial^2 \varphi}{\partial t^2} \triangle_g (\xi \varphi) - \frac{1}{2} < d \frac{\partial \varphi}{\partial t}, d \frac{\partial}{\partial t} (\xi \varphi) >_g = 0.$$

Since $\epsilon > 0$, the equation is a strictly elliptic equation for $\xi \varphi$ on $M \times [0, 1]$. The Maximum principle for elliptic equation yields that $\xi \varphi \equiv 0$ since it is identical to zero on the boundary. \Box

Definition 1. For any two points $\varphi_0, \varphi_1 \in \mathcal{H}$, suppose $\varphi(t) : [0,1] \to \overline{\mathcal{H}}$ with φ satisfying geodesic equation (27) in the viscosity sense. We say φ is a C_w^2 geodesic segment which connects ψ_0, ψ_1 , if $\Delta \varphi$ is in L^{∞} .

In the case n = 1, Chen-He's result [4] and Lemma 3 yield that, for any two points $\varphi_0, \varphi_1 \in \mathcal{H}$, there exists a smooth solution of (49), $\varphi(t) : [0,1] \to \mathcal{H}^0$ which connects φ_0, φ_1 for any $\epsilon > 0$ with $\Delta \varphi$ bounded uniformly. In turn, for any two points $\varphi_0, \varphi_1 \in \mathcal{H}$, there exists a C_w^2 geodesic segment.

When n > 1, the Dirichlet problem for the geodesic equation in Lemma 2 is more complicated. It can be reduced to a Dirichlet boundary value problem of a non-standard complex Monge-Ampère type equation. The existence and regularity of this equation will be studied in our forthcoming paper [10].

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