3rd Assignment, due on February 29, 2016.

Problem 1 [10], Let $F : M \to N$ is a diffeomorphism of M into N, let X_1, X_2 are vector fields on M and Y_1, Y_2 are vector fields on N. Suppose $F_* X_i = Y_i, i = 1, 2$, prove that $F_*[X_1, X_2] = [Y_1, Y_2]$.

Problem 2 [10], Let Δ be a k-dimensional involutive distribution on M. Let $p \in M$, let Γ_m be the set of all those point $q \in M$ such that there is a piecewise smooth curve joining p to q whose smooth portions are 1-dimensional integral curves of Δ. Show that Γ_p is the unique maximal integral manifold of Δ passing through p. (Hint: key step is to show Γ_p is a manifold under appropriate differential structure and it is a immersed submanifold of M).

Problem 3 [10], Suppose $X = \sum_{i=1}^n a_i(x) e_i$ is a vector field on $\mathbb{R}^n$ with $\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}, \forall i, j = 1, \cdots, n$, show that there exists $f \in C^\infty(\mathbb{R}^n)$ such that $\nabla f = X$.

Problem 4 [10], Let X as in Problem 3, assume in addition that $X \neq 0$ everywhere. Let Δ be the distribution on $\mathbb{R}^n$ define as: $\forall x \in \mathbb{R}^n$, $\Delta_x = \{Y = \sum_{i=1}^n b_i(x) e_i | \sum_{i=1}^n a_i(x) b_i(x) = 0\}$. Show that Δ is completely integrable.

Problem 5 [10], Let G be a Lie group and H a closed Lie subgroup which is normal in G. Show that G/H is a Lie group with appropriate differential and $\pi : G \in G/H$ is a Lie group homomorphism.

Problem 6 [10], Let $G = Gl(n, \mathbb{R})$ and define n^2 covector fields $\sigma_{ij}, 1 \leq i, j \leq n$, on G by $\sigma_{ij} = \sum_{k=1}^n y_{ik} dx_{kj}$, where $Y = (y_{ij})$ is the inverse of $X = (x_{ij})$. Show that these forms are invariant under $R_A : G \to G$, right translation by A. Further show that $\{\sigma_{ij}\}$ is a field of frames on G.

Problem 7 [10], Suppose N is a closed regular submanifold of M , then show that any C^∞ vector field X on N is a restriction of a C^∞ vector field $\tilde{X}$ on M (i.e., $\tilde{X}(p) = X(p), \forall p \in N$).

Problem 8 [10], Assume $\phi \in \bigwedge^r(V)$ and $v \in V$. Define an element $i_v \phi$ of $\bigwedge^{r-1}(V)$ by

$$i_v \phi(v_1, \cdots, v_{r-1}) = \phi(v, v_1, \cdots, v_{r-1}).$$

Show that i_v determines a linear mapping of $\bigwedge^r(V)$ into $\bigwedge^{r-1}(V)$ and that if $\phi \in \bigwedge^r(V), \psi \in \bigwedge^s(V)$, then $i_v(\phi \wedge \psi) = (i_v(\phi)) \wedge \psi + (-1)^r \phi \wedge (i_v(\psi))$. Suppose M is a manifold, extend this definition and these properties to $\bigwedge^r(M) \to \bigwedge^{r-1}(M)$ (with v replaced by a vector field X).

Problem 9 [10], Let X be a vector field on M, by Problem 8, we may define $i_X : \bigwedge^r(M) \to \bigwedge^{r-1}(M)$. Show that i_X is not only $\mathbb{R}$-linear, but $C^\infty(M)$-linear and that the operator $L_X = i_X d + d i_X$ is an $\mathbb{R}$-linear mapping of $\bigwedge(M) \to \bigwedge(M)$ with the following properties: (i) $L_X(\bigwedge^r(M)) \subset \bigwedge^r(M)$; (ii) if $\phi \in \bigwedge^r(M)$ and $\psi \in \bigwedge^s(M)$, then $L_X(\phi \wedge \psi) = (L_X \phi)\psi + \phi \wedge L_X \psi$; and (iii) $L_X d = d L_X$.

Problem 10. [10] Suppose $\omega_1, \cdots, \omega_k$ are linearly independent (pointwise) 1-forms on M. Let $\theta_1, \cdots, \theta_k$ be 1-forms on M such that,

$$\sum_{i=1}^k \theta_i \wedge \omega_i \equiv 0.$$

Show that there exist smooth functions f_{ij} on M with $f_{ij} = f_{ji}$ such that

$$\theta_i = \sum_{j=1}^k f_{ij} \omega_j, \forall i = 1, \cdots, k.$$