2nd Assignment, due on February 8, 2016.

Problem 1 [10], Let G consist of all 3×3 matrices which have 1 along the diagonal and zero below and Γ the matrices in G with integer entries. Show that Γ is a closed discrete subgroup and G/Γ is a compact Hausdorff space.

Problem 2 [10], Let U be an open set in $\mathbb{R}^n$, suppose a vector $X_p \in T_p(\mathbb{R}^n)$ is given at each $p \in U$. Show that this defines a C^∞ vector field if and only if for each $f \in C^\infty(U)$, $Xf \in C^\infty(U)$.

Problem 3 [10], Define a C^∞ structure of a manifold on $T(M)$ in such a manner that for each coordinate system (U,ϕ) on M , with local coordinates $(x_1,\cdots,x_n)$ and frames $E_1,\cdots,E_n$, the set $\tilde{U}=\pi^{-1}(U)$ with mapping $\tilde{\phi}:\mathbb{R}^n\times\mathbb{R}^n\to\tilde{U}$ defined as follows is a coordinate neighborhood: For $p\in U, X_p\in\tilde{U}$, we suppose $X_p=\sum_{i=1}^n a_iE_i(p)$ and define

$$\tilde{\phi}(X_p)=(x_1(p),\cdots,x_n(p);a_1,\cdots,a_n)=(\phi(p);a_1,\cdots,a_n).$$

Problem 4 [10], Using problem 3, show that the C^∞-vector fields on M correspond precisely to the C^∞ mappings $X: M\to T(M)$ satisfying $\pi\circ X=i_M$, the identity map on M.

Problem 5 [10], Let $N\subset M$ be a regular submanifold and (U,ϕ) be a preferred coordinate neighborhood relative to N with local coordinates $(x_1,\cdots,x_m)$ and frames $(E_l,\cdots,E_m)$. If $N\cap U$ is given by $x^{n+1}=\cdots=x^m=0$, show that $E_l,,...,E_n$, is a basis of $T_p(N)$ for every $P\in N\cap U$.

Problem 6 [10], Suppose that a C^∞ action θ of $\mathbb{R}\times M$, $M=\mathbb{R}^2$ has infinitesimal generator $X=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ on M. Determine θ.

Problem 7 [10], Let $M=Gl(2,\mathbb{R})$ and define an action of $\mathbb{R}$ on M by the formula

$$\theta(t,A)=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\cdot A,\quad \forall A\in Gl(2,\mathbb{R}),$$

with the dot denoting matrix multiplication. Find the infinitesimal generator.

Problem 8 [10], Prove that if A is a nonsingular $n\times n$ matrix and X is a $n\times n$ matrix, then $Ae^XA^{-1}=e^{AXA^{-1}}$. From this deduce that $\det e^X=e^{tr(X)}$. Use this to determine those matrices A such that e^{tA} is a one-parameter subgroup of $Sl(n,\mathbb{R})$.

Problem 9 [10], Let $M=\mathbb{R}^2$, the xy-plane, and $X=y(\frac{\partial}{\partial x})-x(\frac{\partial}{\partial y})$. Find the maximal domain W and the one-parameter group $\theta: W\to M$.

Problem 10. [10] Suppose θ is a local one-parameter group with maximal domain W and infinitesmal generator X acting on C^∞ manifold M. For $p\in M$, denote $I(p)$ the set $\alpha(p)<t<\beta(p)$ of all real numbers t such that (t,p) is in W. Show that if $I(p)$ is a bounded interval, then the integral curve is a closed subset of M. And show that $t\to\theta(t,p)$ is an imbedding of $I(p)$ in M.