First Assignment, due on January 22, 2016.

Problem 1 [10], Prove the following Implicit Function Theorem: Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set, and let $F = (F_1, \cdots, F_m) : \Omega \to \mathbb{R}^m$ be a C^k map. Suppose $(x_0, y_0) \in \Omega$ and $F(x_0, y_0) = 0$ and the partial Jacobi $m \times m$ matrix $\frac{\partial F}{\partial y} = (\frac{\partial F_i(x,y)}{\partial y_j})_{i,j=1}^m$ is non-singular at (x_0, y_0). Then there are an open neighborhood $U \subset \mathbb{R}^n$ of x_0, a neighborhood V of y_0 in $\mathbb{R}^m$, and a unique C^k map $f : U \to V$ such that $F(x, f(x)) = 0, \forall x \in U$.

Problem 2 [10], Using stereographic projection ϕ_N from north pole N of $\mathbb{S}^n \setminus N \subset \mathbb{R}^{n+1}$ to $\mathbb{R}^n \times \{0\}$ determine a coordinate chart (U_N, ϕ_N). Perform the same for south pole S to get (U_S, ϕ_S). Show tht these two neighborhood determine a C^∞ structure on $\mathbb{S}^n$.

Problem 3 [10], Let A be a closed subset and K be a compact subset of a C^∞ manifold M with $A \bigcap K = \emptyset$. Prove that there is a C^∞ function f defined on M with values in $[0, 1]$ such that $f = 1$ on K and $f = 0$ on A.

Problem 4 [10], Let $F : M \to N$ be a C^∞ mapping of manifolds, let A be an immersed submanifold of M, show that $F|A$ is a C^∞ mapping into N.

Problem 5 [10], Let $F : N \to M$ be a one-to-one immersion which is proper (i.e., the inverse image of any comact set is compact). Show that F is an imbedding and that its image is a closed regular submanifold of M and conversely.

Problem 6 [10], Show by example that there may be functions that are C^∞ on a submanifold N of M that cannot be obtained by restriction of C^∞ function on M.

Problem 7 [10], Show that if G is a Lie group, $a \in G$, then the map $I_a : G \to G$, defined by $I_a(x) = axa^{-1}$, is an automorphism of G.

Problem 8 [10], Show that $O(n + 1)$ acts transitively on $\mathbb{S}^n$ in a natural way and determine the isotropy subgroup of $(0, \cdots, 0, 1)$.

Problem 9 [10], Prove that a discrete normal subgroup of a connected Lie group is in the center of G.

Problem 10. [10] Suppose that a C^∞ manifold M is a metric space and that Γ is a discrete group of C^∞ isometries acting discountinuously on M. Show that the action is necessarily properly discontinuous.