MATH380 MIDDLE TERM TEST (2011)

Please complete 4 *problems.*

Problem 1. Let γ be a regular curve with arc-length parametrization. Suppose $|\gamma(s)| = 1$ for all s (i.e., the image of γ is in the unit sphere), show that the curvature of $k(s) \geq 1$ for $\forall s$. If further assume that $k(s) \equiv 1$, prove that the torsion $\tau \equiv 0$ so that γ is a portion of a great circle.

Problem 2. Suppose S is a regular surface, prove that for each $p \in S$, after proper choice of (x, y, z) in $\mathbb{R}^3$, such that S is locally a graph near $p = (0, 0, 0)$: $z = h(x, y)$, $\nabla h(0, 0) = 0$ and $h_{xy}(0, 0) = 0$. Compute the principal curvatures of S at p using this coordinates.

Problem 3. Suppose S is a regular surface, suppose $p \in S$ is a point $\|p\| = max_{q \in S}\|q\|$. Show that at p the principal curvatures k_1, k_2 of S have the same sign and they satisfy

$$|k_1(p)| \geq \frac{1}{\|p\|}, |k_2(p)| \geq \frac{1}{\|p\|}.$$

Problem 4. Let S be a regular surface such that its Gauss curvature $K \equiv 0$ and its mean curvature $H \neq 0$ everywhere. Suppose $\mathbf{x}(u, v)$ $((u, v) \in (-\delta, \delta) \times (-\delta, \delta),\ \delta > 0)$ is chosen such that $v = c$ is the line of curvature with respect to $k_1 = 0$ for each $c \in (-\delta, \delta)$. Prove that

$$N_u \equiv 0, N_v \neq 0, < \mathbf{x}, N > \text{ and } < \mathbf{x}, N_v > \text{ are independent of } u.$$

Finally, conclude that $v = c$ is a straight line segment for each $c \in (-\delta, \delta)$.

Problem 5. Show that if a curve $\gamma \subset S$ (where S is a regular surface) is both a line of curvature and a geodesic, then it is a plane curve.

Problem 6. Let $\mathbf{x}(u, v), (u, v) \in B_r(0)$ be a local parametrization of a regular surface S, where $B_r(0)$ is a ball of radius $r > 0$ centered at 0 in $\mathbb{R}^2$. Suppose that any real numbers a, b with $a^2 + b^2 = 1$, $\mathbf{x}(at, bt)$ is an arc length parameterized geodesic in S for $-r < t < r$. Show that $E(0) = G(0) = 1, F(0) = 0$, and the Christoffel symbols $\Gamma^k_{ij}(0) = 0, \forall i, j, k$, and $K(0) = -\frac{1}{2}\big(E_{vv} + G_{uu}\big) + F_{uv}$.

Frenet formulas:

$$\mathbf{t}^{'} = k\mathbf{n},$$
$$\mathbf{n}^{'} = -k\mathbf{t} - \tau\mathbf{b},$$
$$\mathbf{b}^{'} = \tau\mathbf{n}.$$

Rodrigues formula for line of curvature:

$$N^{'}(t) = \lambda(t)\gamma^{'}(t).$$

Coefficients of the Second Fundamental form:

$$e = - < N_u, \mathbf{X}_u > = < N, \mathbf{X}_{uu} >,$$
$$f = - < N_u, \mathbf{X}_v > = - < N_v, \mathbf{X}_u > = < N, \mathbf{X}_{uv} >,$$
$$g = - < N_v, \mathbf{X}_v > = < N, \mathbf{X}_{vv} > .$$

Weingarten equations:

$$a_{11} = \frac{fF - eG}{EG - F^2},$$
$$a_{12} = \frac{gF - fG}{EG - F^2},$$
$$a_{21} = \frac{eF - fE}{EG - F^2},$$
$$a_{22} = \frac{fF - gE}{EG - F^2}.$$

Gauss curvature: $K = \frac{eg - f^2}{EG - F^2}$, mean curvature: $H = \frac{1}{2}\frac{eG - efF + gE}{EG - F^2}$.