Solutions of Assignment #3

P294, #2. In view of geodesic equations (4) in page 254, simply compute the Christoffel symbols in geodesic coordinates ($E = 1, F = 0$) using (2) in page 232.

P307, #6a. Direct calculation shows that in the coordinate (ρ, θ), the first fundamental form $E = 1, F = 0, G = \rho^2$. Therefore, it's a local isometry.

P307, #7a Under the assumption that $\alpha^{'}$ is not an asymptotic direction, the statements in (a) is prove in example 4 in page 195 (together with page 194). To check that assumption, we note that $\alpha^{"} = kn$. As $k \neq 0$, n is defined. As α is a geodesic of arc-length parameterized, n is parallel to N along α (we may take $n = N$ along α) and $k_n = k \neq 0$. Therefore, $N^{'} = n^{'} = -k\alpha^{'} - \tau b \neq 0$.

P335, #1. Set $S_1 = S - F$. Since F is closed, S_1 is an open subset of S. Therefore S_1 is a connected regular surface. Clearly S is a non-trivial extension of S_1, by Proposition 1 in page 326, S_1 can not be a complete surface.

P335, #4 Pick a nbhd of p_0 such that S is locally a graph over a plane, say locally $S = \{(x, y, h(x, y)\}$ with $h(0, 0) = 0, |\nabla h(x, y)| \leq C$ for some constant $C > 0$ $\forall (x, y)$ close to 0. It's easy to see d and $\bar{d}$ are equivalent in a small nbhd of 0.

P335, #6. Since S is non-compact and complete, by Corollary 2 in page 335, there is a sequence of p_n such that the distance $d(p, p_n) = l_n$ tends to ∞. By Hopf-Rinow Theorem, there is a geodesic γ_n such that $\gamma_n(0) = p, \gamma_n(l_n) = p_n$. Note that for any two point in $\gamma_n[0, l_n]$, γ_n realizes the intrinsic distance. Let $w_n = \gamma_n^{'}(0)$, then $|w_n| = 1$. Since $S^1 \subset \mathbb{R}^2 = T_pS$ is compact, there is subsequence $w_{n_k} \to w_0$. Claim: the geodesic γ starting from p with direction w_0 is a ray issuing from p. This can be checked as follow: $\forall q_1, q_2 \in \gamma$, approximate it by γ_n. There exist $q_{1,n}, q_{2,n} \in \gamma_n$, $q_{1,n} \to q_1, q_{2,n} \to q_2$. Take the limit, we have the conclusion.

P354, #1. No, e.g., a torus. It's compact, but there are even some points with $K < 0$.

P354, #5 (a), it's pretty obvious from the explicit solution given (using problem #4 in page355).
(b), again, it follows directly the equation

$$c = \frac{u^{'} f^2}{\sqrt{f^2(u^{'})^2 + (f^{'})^2 + (g^{'})^2}},$$

solving for $u^{'}$ and integrate it.

P368, #1. (a), As we have proved that $< J^{'}(s), \gamma^{'}(s) > \equiv const$. Since it is 0 at $s = 0$, so $\frac{d}{ds} < J(s), \gamma(s) > = < J^{'}(s), \gamma^{'}(s) > \equiv 0$. $< J(s), \gamma(s) > \equiv const$. But again, it's vanishing at $s = 0$, so it vanishing identically.

(b), by (a), $J(s) = u(s)e_2(s)$. Now, $(\gamma^{'} \wedge J) \wedge \gamma^{'} = u(e_1 \wedge e_2) \wedge e_1 = e_2$, by the equation for J, as $\frac{D}{ds}e_2 = 0$, we get

$$u^{"}(s) + K(s)u(s) = 0. \tag{1}$$

The initial conditions are clear.

P368, #2. Since any meridian is a geodesic for surface of revolution, since geodesic is uniquely determined by the initial point and its direction, any geodesic starting from the origin must be a meridian for the paraboloid $z = x^2 + y^2$. Using the polar coordinates (ρ, θ) for T_0S, then the exponential map exp_0 can be expressed as (for $V = (\rho\cos\theta, \rho\sin\theta)$):

$$exp_0(V) = \Phi(\rho, \theta) = (t(\rho)\cos\theta, t(\rho)\sin\theta, t^2(\rho)),$$

where $t(\rho)$ satisfies $\frac{dt(\rho)}{d\rho} = \frac{1}{\sqrt{1+4t^2(\rho)}}$ with $t(0) = 0$ (or the other way around: $\rho = \int_0^t \sqrt{1+4\mu^2}d\mu$). We need to show $\exp_0$ is a local diffeomorphism when $\rho > 0$. Note that

$$\Phi_\theta = (-t(\rho)\sin\theta, t(\rho)\cos\theta, 0), \Phi_\rho = \frac{1}{\sqrt{1+4\rho^2}}(\cos\theta, \sin\theta, 2t(\rho)).$$

Therefore, $\Phi_\theta \perp \Phi_\rho$ and for $\rho > 0$, $\Phi_\theta \wedge \Phi_\rho \neq 0$.