Solutions of Assignment #4

P261, #8. We want to prove that every point in S is an umbilical point. This is equivalent to show that a any point p, any $v \in T_pS$ is an eigenvector of dN_p. For such p, v, there is a (local) geodesic γ such that $\gamma(0) = p, \gamma^{'}(0) = v$. Since $k_g = 0$, $k_n^2 = k^2$, and N and n are parallel. by the assumption, γ is a plane curve, by Frenet formula, $N^{'}(t) = n^{'}(t) = -k\gamma^{'}$. It follows from Rodrigus' formula, γ is a line of curvature. in particular, v is an eigenvector of dN_p.

P282, #1. Since $S \subset \mathbb{R}^3$ is compact, there exist $r > 0$ $p \in S$ such that $S \subset \bar{B}_r(0)$ and $|p| = r$. As we know, $K(p) > 0$. By Gauss-Bonnet, $\int_S K < 0$. So there is $q \in S$, $K(q) < 0$. Now, as S (should be assumed) is connected, by the continuity of K, there is a point $a \in S$, $K(a) = 0$.

P282, #3 It should be algebraic values of volumes $N(A), N(B)$. $V(N(A)) = \int_{N(A)\subset\mathbb{S}^2}$, and similar expression for $V(N(B))$. Since the Jacobi of the Gauss map is K, $V(N(A)) = \int_A K d\sigma$. Now the statement follows Gauss-Bonnet as Γ is geodesic, and $\chi(A) = \chi(B)$.

P294, #1. This follows from the proof of Minding's theorem in P289, i.e., the first fundamental form is independent of θ in geodesic polar coordinates.

P294, #4 This is one of the questions in the midterm exam, we did it in the class.

P294, #6. We may assume the pre-image $\tilde{C}$ of C is $\{(\rho_0, \theta) | \theta_1 \le \theta \le \theta_2\}$. So the length of C is $\int_{\theta_1}^{\theta_2} \sqrt{G} d\theta$. Using the Gauss lemma to expend

$$\sqrt{G} = 0 + \rho_0 + \sqrt{G}_{\rho\rho}(\rho^*, \theta^*)\frac{\rho_0^2}{2}.$$

If ρ_0 is sufficient small, $\sqrt{G}_{\rho\rho} = -K\sqrt{G}_{\rho\rho}$. Now the statements of (a) and (b) follow accordingly.

P294, #7. We may assume γ is arc-length parameterized. Let $v = X_\rho, w = \gamma^{'}$. Use Lemma2 in page 257, as γ is a geodesic, $\frac{d\phi}{ds} = -[\frac{Dv}{ds}]$. Use formula (1) in page 239, $a = 1, b = 0, u = \rho, v = \theta$, as $\Gamma^2 11 = 0, \Gamma^2 12 = \frac{G_\rho}{2G}$, $\frac{Dv}{ds} = \frac{G_\rho}{2G} X_\theta$ (the coefficient in front of X_ρ must be 0 since X_ρ is a unit vector field). As v is a unit vector filed, $\frac{Dv}{ds} \perp v$. by the Gauss lemma, $\frac{X_\theta}{G^{frac12}}$ is a unit vector field $\perp v$, from the orientation, $N \wedge v = \frac{X_\theta}{G^{frac12}}$. It follows from Definition 9 in page 248 $[\frac{Dv}{ds}]\frac{X_\theta}{G^{frac12}} = \frac{Dv}{ds} = \frac{G_\rho}{2G} X_\theta$, so $[\frac{Dv}{ds}] = (\sqrt{G})_\rho \theta^{'}(s)$. the equation is verified.

P294, #10 Set $Q = \varphi \circ \psi^{-1}$, we want to show $Q = id$ on S. Set $A = \{q \in S | Q(q) = q, dQ_p = ID\}$. It's obvious A is closed and non-empty since $p \in A$. We verify A is open. Suppose $q \in SA$, pick a geodesic normal coordinates at p, consider exp_q which is a local parametrization around q. For any geodesic ray γ starting from q, $\tilde{\gamma} = Q(\gamma)$ is also a geodesic starting from q since Q is an isometry and $Q(q) = q$. Since $\tilde{\gamma}^{'}(0) = dQ_q \gamma^{'}(0) = \gamma^{'}(0)$. γ and

$\tilde{\gamma}$ are two geodesic with the same initial point and same initial velocity, they must be the same. From this, we conclude $Q = id$ near q. Some A is open. By connectedness, $A = S$.

P306, #3c. From (a) and (b), $\mathbf{x}$ is a parametrization of S with coordinate nbhd contains $\alpha(0,l)$. In this coordinates, $\mathbf{x}_s = d(exp_{\alpha(t)})(v(t))$ from the definition of exp. By chain rule and the Gauss lemma, $\mathbf{x}_t = \alpha^{'}(t) + s\frac{D}{dt}v(t)$. $v \perp \frac{D}{dt}v$ (as v is a unit vector field), together with the assumption $\alpha^{'} \perp v$, we conclude $E = 1, F = 0$. If α is an arc-length parameterized, $\mathbf{x}_s(0,t) = \alpha^{'}(t)$, so $G(0,t) = 1$. If in addition α is also a geodesic,

$$G_s(s,t) = \frac{d}{ds} < \mathbf{x}_t, \mathbf{x}_t >= 2 < \frac{D}{ds}\mathbf{x}_t, \mathbf{x}_t >= 2 < \frac{D}{dt}\mathbf{x}_s, \mathbf{x}_t >= 2\frac{d}{dt} < \mathbf{x}_s, \mathbf{x}_t > -2 < \mathbf{x}_s, \frac{D}{dt}\mathbf{x}_t > .$$

Since $F = 0$ and $\frac{D}{dt}\mathbf{x}_t|_{s=0} = 0$, $G_s(0,t) = 0$.

P306, #5. We will use 3(c). Some preparation. In Fermi coordinates (u,v), $E = 1, F = 0$ and $\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{12} = 0$, $\Gamma^2_{12} = \frac{G_u}{2G}, \Gamma^1_{22} = -\frac{G_u}{2G}, \Gamma^2_{22} = \frac{G_v}{2G}$. Write $\gamma(v,t) = X(u(v,t),v)$. Use these and formula (1) in page 239,

$$\gamma_t = u_t X_u, \gamma_v = X_v + u_v X_u, \frac{D}{\partial t}X_u = 0, \frac{D}{\partial t}X_v = \frac{G_u}{2G}u_t X_v.$$

Note by the assumption, $u_v(v,0) = 0$. In particular, at $t = 0$, $\gamma_v = X_v$. We also note that in Fermi coordinates, at $u = 0$, X_u, X_v are orthonormal along $\gamma(v,0)$. Also,

$$\frac{D}{\partial t}\gamma_v = u_{vt}X_u + \frac{G_u}{2G}X_v.$$

Now for (a),

$$E^{'}(0) = 2\int_0^l < \frac{D}{\partial t}\gamma_v, \gamma_v > |_{t=0} dv = 2\int_0^l < \frac{G_u}{2G}X_v, X_v > |_{u=0} dv = 0,$$

as $G_u(0,v) = 0$ by 3(c). Use the fact that $G_u(0,v) = 0, G(0,v) = 1$, and the Gauss equation for $E = 1, F = 0$, $K = -\frac{1}{2}\frac{G_{uu}}{G}$,

$$\begin{aligned}
\frac{D}{\partial t}\frac{D}{\partial t}\gamma_v|_{t=0} &= (u_{vtt}X_u + \frac{1}{2}(\frac{G_u u_t}{G})_t X_v + (\frac{1}{2}\frac{G_u u_t}{G})^2 X_v)_{t=0} \\
&= u_{vtt}X_u|_{t=0} + \frac{1}{2}G_{uu}(0,v)u_t^2(0,v)X_v \\
&= u_{vtt}X_u|_{t=0} - Ku_t^2(0,v)X_v.
\end{aligned}$$

Note that $u_t(0,v) = \eta(v)$. We have

$$\begin{aligned}
\frac{1}{2}E^{''}(0) &= \int_0^l < \frac{D}{\partial t}\gamma_v, \frac{D}{\partial t}\gamma_v > |_{t=0} dv + \int_0^l < \frac{D}{\partial t}\frac{D}{\partial t}\gamma_v|_{t=0}, \gamma_v|_{t=0} > dv \\
&= \int_0^l < \frac{D}{\partial v}\gamma_t, \frac{D}{\partial v}\gamma_t > |_{t=0} dv + \int_0^l < u_{vtt}X_u|_{t=0} - Ku_t^2(0,v)X_v, X_v|_{t=0} > dv \\
&= \int_0^l (\frac{d\eta}{dv})^2 dv - \int_0^l K\eta^2 dv.
\end{aligned}$$

For (b), Suppose there is a curve α with $\alpha_(0) = \gamma(0), \alpha(l) = \gamma(l)$, and $l(\alpha) \leq l - \delta$ for some $\delta > 0$, and α is sufficient close to γ. By (a), $E^{''}(0) > 0$ unless $\frac{d\eta}{dv} \equiv 0$. Since $\eta(0) = \eta(l) = 0$, this implies $\eta \equiv 0$. That is, is α is not the same as γ, η can not vanish identically, so we have $E^{''}(0) > 0$. So the energy $E_\alpha > E_\gamma$.