Solutions of Assignment #3

P174, #24a, b. There are different proofs. In spirit, all are similar. Here we use Problem 26 in page 91 (2nd assignment). Locally, express S as a graph of $z = u(x, y)$ over xy-plane near 0, with $u(0) = \nabla_x u(0) = \nabla_y u(0) = 0$. It easy to compute that the 1st FF at 0 is the identity matrix, the coefficients second FF at 0: $e = u_{xx}, f = u_{xy}, g = u_{yy}$. Moreover, after a rotation, we may assume $u_{xy}(0) = 0$. So $K(0) = u_{xx}(0)u_{yy}(0)$. Expand u near 0

$$u(x, y) = \frac{1}{2}u_{xx}(0)x^2 + \frac{1}{2}u_{yy}(0)y^2 + R,$$

where R is a term of order ≥ 3. (a) $K(0) > 0$ implies $u_{xx}(0)$ and $u_{yy}(0)$ are non-vanishing and they have the same sign, say positive. That implies $u(x, y) > 0$ if $(x, y) \neq (0, 0)$ near 0. (b), on the other hand, if S is locally convex near 0, say $u(x, y) \geq 0$ near 0, we must have $u_{xx}(0) \geq 0, u_{yy}(0) \geq 0$.

P212, #11. (a) As $Y_u = X_u + aN_u, Y_v = X_v + aN_v$,

$$Y_u \wedge Y_v = X_u \wedge X_v + aN_u \wedge X_v + aX_u \wedge N_v + a^2 N_u \wedge N_v.$$

By Weingarten equation, $N_u \wedge X_v = a_{11} X_u \wedge X_v, X_u \wedge N_v = a_{22} X_u \wedge X_v, N_u \wedge N_v = K X_u \wedge X_v$,

$$Y_u \wedge Y_v = (1 + a^2 K + a(a_{11} + a_{22}))X_u \wedge X_v = (1 - 2Ha + Ka^2)X_u \wedge X_v. \tag{1}$$

(b) From (1), $N = \bar{N}$, so $N_u = \bar{N}_u, N_v = \bar{N}_v$ and

$$Y_u = X_u + aN_u = (1 + aa_{11})X_u + aa_{21}X_v, Y_v = X_v + aN_v = (1 + aa_{22})X_v + aa_{12}X_u.$$

Let A be the Weingarten matrix for X and $\bar{A}$ the corresponding Weingarten matrix for Y. From above, relation, we have $\bar{A} = A(I + aA^t)^{-1}$ where A^t the transport of A. Now at each point p, we may pick a local coordinate such that at p $E = G = 1, F = 0$ and A is diagonal. Then it will be easy to verify the formulas for $\bar{K}, \bar{H}$. (c), let $a = \frac{1}{2c}$ the formula for $\bar{K}$ in (b).

P212, #12 As we have proved that every compact surface has an elliptic point p such that $K(p) > 0$. For minimal surface, as $H = 0$, must have $K \leq 0$.

P.228, #3. Let $\gamma \subset S$ be any curve, $\bar{\gamma} = \phi \circ \gamma$. As $l = \int_{t_0}^t \|\gamma'(t)\|dt$ and $\bar{l} = \int_{t_0}^t \|\bar{\gamma}'(t)\|dt = \int_{t_0}^t \|d\phi(\gamma'(t))\|dt$. If ϕ is an isometry, $l = \bar{l}$. If $l = \bar{l}$ for any $t > t_0$, differentiating t, we get $\|\bar{\gamma}'(t)\| = \|d\phi(\gamma'(t))\|$ for any t and any γ. So $< d\phi_p(w), d\phi_p(w) >=< w, w > \forall w \in T_pS$. $\forall w_1, w_2 \in Tp, \forall \alpha, \beta \in \mathbb{R}$, set $w = \alpha w_1 + \beta w_2$, $< d\phi_p(w), d\phi_p(w) >=< w, w >$ implies $< d\phi_p(w_1), d\phi_p(w_2) >=< w_1, w_2 > \forall w_1, w_2 \in T_pS$.

P.228, #18 Pick any parametrization (u, v), ϕ conformal implies

$$d\phi(X_u) \wedge d\phi(X_v) = \lambda^2 X_u \wedge X_v$$

for some function $\lambda \neq 0$. We only need to show $\lambda^2 \equiv 1$. Area preserving means

$$\int_U X_u \wedge X_v dudv = \int_U d\phi(X_u) \wedge d\phi(X_v) dudv$$

for open set U. Take $U = B_r(p) \subset \mathbb{R}^2$, divide above identity by r^2 and let $r \to 0$, we get $X_u \wedge X_v = d\phi(X_u) \wedge d\phi(X_v)$. then by the first identity, $\lambda^2 = 1$.

P.237, #2. Direct computation, use formulas in P236 for Γ_{ij}^k and the Gauss formula (5)in 234.

P.237, #7. No, use Codazzi equation (7a) in page 236.

P.260, #2 $k^2 = k_n^2 + k_g^2$. C straight line segment if and only if $k \equiv 0$. This is equivalent to $k_g \equiv 0$ and $k_n \equiv 0$. That's C is geodesic and asymptotic.

P.260, #4.

$$\begin{aligned} \frac{d}{dt} < v(t), w(t) > &= < v'(t), w(t) > + < v(t), w'(t) > \\ &= < \frac{D}{dt} v(t), w(t) > + < v(t), \frac{D}{dt} w(t) >, \end{aligned}$$

since $v'(t) - \frac{D}{dt}v(t), w'(t) - \frac{D}{dt}w(t)$ are parallel to N and $< N, v > = < N, w > = 0$.