Sketches of Solutions of assignment 6

P480, #36, We use a contour like Fig.7.1-21 on page 479 with indentations at $s = k\pi i, k \in \mathbb{Z}$ and C_R is the left half circle centered at the origin of radius $R = n\pi + \frac{\pi}{2}$. Write $s = \tau + i\omega$, $|\sinh s|^2 = \cosh^2\tau - \cos^2\omega$ and $\cosh^2\tau \geq 1+\tau^2$, $|\cos\omega| \leq |(n+1/2)\pi - \omega|$, we have $\int_{C_R} \frac{e^{st}}{s\sinh s} ds \to 0$ as $n \to \infty$. We arrive at the formula $\mathcal{L}^{-1}\frac{1}{s\sinh s} = \sum_k Res(\frac{e^{st}}{s\sinh s}, k\pi i)$. We now calculate the residues. At $k = 0$, $s\sinh s = s^2(1 - \frac{s^2}{6} + ...)$, so $\frac{1}{s\sinh s} = S^{-2}(1 + \frac{s^2}{6} + ...$ As $e^{st} = 1 + st + \frac{s^2t^2}{2} + ...$, we get $\frac{e^{st}}{s\sinh s} = \frac{1}{s^2} + \frac{t}{s} + analyticpart$. So $Res(\frac{e^{st}}{s\sinh s}, 0) = t$. At $s = k\pi i$, $k \neq 0$, it is a simple pole, we get $Res(\frac{e^{st}}{s\sinh s}, k\pi i) = \frac{e^{tk\pi i}}{k\pi i\cosh k\pi i}$. Now group k and $-k$ together, $Res(\frac{e^{st}}{s\sinh s}, k\pi i) + Res(\frac{e^{st}}{s\sinh s}, -k\pi i) = \frac{(-1)^k 2\sin k\pi t}{k\pi}$. Finally, we have $\mathcal{L}^{-1}\frac{1}{s\sinh s} = t + \sum_{k=1}^{\infty} \frac{(-1)^k 2\sin k\pi t}{k\pi}$.

P536, #2, (a), Write $z = x + iy$, we have $w = \sin x\cosh y + i\cos x\sinh y$. The image of the line $Rez = \alpha$ under $\sin z$ is $w = \sin\alpha\cosh y + i\cos\alpha\sinh y$ which is the hyperbola $\frac{u^2}{\sin^2\alpha} - \frac{v^2}{\cos^2\alpha} = 1$ on the right side of the imaginary axis $\alpha > 0$ (and on the left side of the imaginary axis when $\alpha < 0$) for each $0 < |\alpha| < \frac{k\pi}{2}$. When $\alpha = 0$, $w = i\sinh y$ which is the imaginary axis. So the image of the infinite strip $|Rez| \leq a < \frac{pi}{2}$ is the region in the middle bounded by two hyperbolas $\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1$. (b). The map is 1-1. (c). it is not 1-1 on $Rez = \pi/2$, e.g., $T(\pi/2+i) = T(\pi/2-i)$.

P536, #4, The image of the line $Rez = \alpha$ under $\cos z$ is $w = \cos\alpha\cosh y + i\sin\alpha\sinh y$ which is the hyperbola $\frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin^2\alpha} = 1$ when $\alpha \neq \frac{k\pi}{2}$. When $\alpha = 0$, $w = \frac{e^{-y}+e^y}{2}$ which is the part of real axis with $u \geq 1$, the image of the line $Rez = \pi$ under $\cos z$ is $w = -\frac{e^{-y}+e^y}{2}$ which is the part of real axis $u \leq -1$, and the image of $\alpha = \frac{pi}{2}$ is imaginary axis $Rew = 0$. So the image of the infinite strip $0 \leq Rez \leq \pi$ is the whole complex plane. The map is 1-1 in $\Omega = \{0 < Rez < \pi$, but not 1-1 on its closure, e.g., $T(i) = T(-i)$.

P536, #10, (a) We first note $T(z) = \frac{z-1}{z+1}$ maps $\{-1 < x < 1, y = 0\}$ to $\{-\infty < u \leq 0, v = 0\}$ and maps $\{z = e^{i\theta}, 0 < \theta < \pi\}$ to $\{v = \frac{2\sin\theta}{|1+e^{i\theta}|^2}, u = 0, 0 < \theta < \pi\}$ which is the positive part of imaginary axis. As $T(i) = \frac{-3+4i}{5}$, the image of the upper half disc is $\{u < 0, v > 0\}$. Since $w = (T(z))^2$, the image of the upper half disc is the lower half plane.

(b), the inverse transformation will take the lower half plane to the upper half disc if we choose a branch for log as $0 < arg \leq 2\pi$.

P550, #14, Since bilniear transformation maps "circles" to "circles", as the unit circle passes 1, $T(z) = \frac{z+1}{z-1}$ maps 1 to ∞, the image of the unit circle is a straight line. As $T(-1) = 0, T(i) = -i$, this straight line must be the imaginary axis $\{u = 0\}$.

P550, #20, The image of $Rez = 1$ is $w = \frac{1+iy}{iy} = 1 - \frac{i}{y}$ which is the line$Rew = 1$. And the image of $Rez = 2$ is $w = \frac{2+iy}{1+iy} = 1 + \frac{1}{1+iy}$. Similar argument as in *P527, #10,*, it is the circle centered at $\frac{3}{2}$ of radius $\frac{1}{2}$. Also since $T(\frac{3}{2}) = 3$, the image of under T is the domain which is right of the line $Rew = 1$ and outside of the circle $\{|w - \frac{3}{2}| = \frac{1}{2}\}$.

P550, #24, (a), since $T(-1) = \infty$, T is of the form $T(z) = \frac{az+b}{z+1}$. As $\frac{ai+b}{i+1} = 1 + i$ and $\frac{-ai+b}{-i+1} = 1 - i$, we solve $a = 2, b = 0$, i.e. $T9z) = \frac{2z}{z+1}$.

(b), Since $T(1) = \infty$, the image of the unit circle is a straight line passing through $1 + i$ and $1 - i$, so it must be the line $Rew = 1$. As $T(2) = \frac{4}{3} > 1$, the image of $|z| > 1$ is $Rew > 1$.

P568, #2, (a), Since $Logz = Log|z| + iArg(z)$, Log maps wedge $0 \leq argz \leq \alpha$ to the strip $0 \leq Imw \leq \alpha(< \pi)$

(b), certainly $\phi_1(u, v) = \frac{T_2 - T_1}{\alpha} v + T_1$ is a harmonic function with the given boundary values.

(c), We note $argz = \tan^{-1}(\frac{y}{x})$, therefore transform back to the wedge, we obtain the harmonic function with given boundary values

$$\phi(x,y) = \frac{T_2 - T_1}{\alpha} argz + T_1 = \frac{T_2 - T_1}{\alpha} \tan^{-1}(\frac{y}{x}) + T_1$$

(d), As $-\frac{T_2-T_1}{\alpha} Log|z|$ is a harmonic conjugate of ϕ, we obtain that $\phi(x,y) = -i\frac{T_2-T_1}{\alpha} Logz + T_1$ is the complex temperature.

P568, #3, As in the previous question, we need to find a harmonic function with prescribed boundary values and its conjugate. The map $w = -\frac{1}{z}$ maps the domain to $\Omega_1 = \{|w| < 1, Imw > 0\}$ and the upper half unit circle to the upper half unit circle and the other part of boundary to $-1 \leq Rew \leq 1, Imw = 0$. By Example 2 on page 561, the complex potential of this problem is equal to $\frac{-10iLog(\frac{1-1/z}{1+1/z})^2}{\pi} = \frac{-10iLog(\frac{z-1}{1+z})^2}{\pi}$.