Problem 1 [10], Let D be a region for which Green's Theorem holds. Suppose u is harmonic; that is,

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, \quad \forall (x,y) \in D. \tag{1}$$

Prove that

$$\int_{\partial D} \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy = 0.$$

Problem 2 [10], Suppose $p \in D$ such that $\bar{B}_R(p) \subset D$, and suppose u is continuous in D and u satisfies Laplace's equation (1) on $D \setminus \{p\}$. Assume that $\int_{\partial D} \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy = 0$, prove

$$u(p) = \frac{1}{2\pi R} \int_{\partial B_R(p)} u ds.$$

[Hint: consider $I(\rho) = \frac{1}{\rho} \int_{\partial B_\rho(p)} u ds$, for $0 < \rho \leq R$, using Green's Theorem to deduce that $\frac{d}{d\rho} I(\rho) = 0$, then calculate $\lim_{\rho \to 0} I(\rho) = 2\pi u(p)$.]

Problem 3 [10], Let $B = \{x^2 + y^2 \leq 1\}$, and $\forall \delta > 0$, denote $B_\delta = \{x^2 + y^2 \leq \delta\}$. Suppose f is a continuous function and $\|\nabla f\| \leq 1$ on B, suppose

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{x^2+y^2}, \quad in \quad B \setminus \{0\}.$$

Use the boundedness of ∇f to show

$$\lim_{\delta \to 0} \int_{\partial B_\delta} f_y dx - f_x dy = 0.$$

Use this fact and the Green Theorem to evaluate

$$\int_{\partial B} f_y dx - f_x dy.$$

Problem 4 [10], Evaluate the integral $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the portion of the surface of a sphere defined by $x^2+y^2+z^2 = 1$ and $x+y+z \geq 1$, and where $\mathbf{F} = \mathbf{r} \times (\mathbf{i}+\mathbf{j}+\mathbf{k})$, $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, by observing that $\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial \Sigma} \mathbf{F} \cdot d\mathbf{r}$ for any other surface Σ with the same boundary as S. By picking Σ appropriately, $\int\int_\Sigma (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ may be easy to compute. Show that this is the case if Σ is taken to be the portion of the plane $x + y + z = 1$ inside the circle ∂S.

Problem 5 [10], For a surface S and a fixed vector $\mathbf{v}$, prove that

$$2 \int\int_S \mathbf{v} \cdot \mathbf{n} dS = \int_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{s},$$

where $\mathbf{r}(x,y,z) = (x,y,z)$.

Problem 6 [10],

(a) Show that $\mathbf{F} = \frac{-\mathbf{r}}{\|\mathbf{r}\|^3}$ is the gradient of $f(x,x,z) = \frac{1}{r}$ $(r = \sqrt{x^2+y^2+z^2})$.

(b) What is the work done by the force $\mathbf{F} = \frac{-\mathbf{r}}{\|\mathbf{r}\|^3}$ in moving a particle from a point $\mathbf{r}_0 \in \mathbb{R}^3$ to "∞", again, where $\mathbf{r}(x, y, z) = (x, y, z)$.

Problem 7 [10], Let $\mathbf{F} = \frac{-GmM\mathbf{r}}{\|\mathbf{r}\|^3}$ be the gravitational force field defined in $\mathbb{R}^3 \setminus \{0\}$.

(a) Show that $div\mathbf{F} = 0$.

(b) Show that $\mathbf{F} \neq curl\mathbf{G}$ for any C^1 vector field $\mathbf{G}$ on $\mathbb{R}^3 \setminus \{0\}$.

Problem 8 [10], Evaluate the surface integral $\int\int_S \mathbf{F} \cdot \mathbf{n} dA$, where $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + z(x^2 + y^2)^2\mathbf{k}$ and S is the surface of the cylinder $x^2 + y^2 \leq 1, 0 \leq z \leq 1$.

Problem 9 [10], Suppose $\mathbf{F}$ is tangent to the closed surface $S = \partial W$ of a region W. Prove that

$$\int\int\int_W (div\mathbf{F}) dV = 0.$$

Problem 10. [10] Suppose f is a continuous function in $\mathbb{R}^3 \setminus (0, 0, 0)$ and suppose

$$\|\nabla f(X)\| \leq \frac{1}{\|X\|}, \quad \frac{\partial^2 f}{\partial x^2}(X) + \frac{\partial^2 f}{\partial y^2}(X) + \frac{\partial^2 f}{\partial z^2}(X) = \frac{1}{\|X\|^2}, \quad \forall X \in \mathbb{R}^3 \setminus (0, 0, 0).$$

Denote $B_r = \{\|X\|^2 \leq r^2\}$ for $r > 0$, show that,

$$\lim_{r \to 0} \int\int_{\partial B_r} \nabla f \cdot d\mathbf{S} = 0.$$

Use it and the Divergence Theorem to evaluate

$$\int\int_{\partial B_1} \nabla f \cdot d\mathbf{S} = 4\pi.$$