Problem 1 [10], Let $\mathbf{c}$ be a smooth path.

(a) Suppose $\mathbf{F}$ is perpendicular to $\mathbf{c}'(t)$ at the point $\mathbf{c}(t)$. Show that

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0.$$

(b) If $\mathbf{F}$ is parallel to $\mathbf{c}'(t)$ at the point $\mathbf{c}(t)$, show that

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \|\mathbf{F}\| ds.$$

(By parallel to $\mathbf{c}'(t)$ we mean that $\mathbf{F}(\mathbf{c}(t)) = \lambda(t)\mathbf{c}'(t)$, where $\lambda(t) > 0$.)

Problem 2 [10], Find a parametrization of the surface $x^2 + 3xy + z^2 = 2, z > 0$, and use it to find the tangent plane at the point $x = 1, y = 1/3, z = 0$. Compare your answer with that using level sets.

Problem 3 [10], Let Φ be a regular surface at (u_0, v_0) (i.e., Φ is C^1 and $T_u \times T_v \neq 0$ at (u_0, v_0)).

(a) Use the implicit function theorem to show that the image of Φ near (u_0, v_0) is the graph of a C^1 function of two variables. If the z component of $T_u \times T_v$ is nonzero, we can write it as $z = f(x, y)$.

(b) Show that the tangent plane at $\Phi(u_0, v_0)$ defined by the plane spanned by T_u and T_v coincides with the tangent plane of the graph of $z = f(x, y)$ at this point.

Problem 4 [10], Find the area of the surface defined by $z = xy$ and $x^2 + y^2 \leq 2$.

Problem 5 [10], Show that for the vectors T_u and T_v, we have the formula

$$\|T_u \times T_v\| = \sqrt{[\frac{\partial(x,y)}{\partial(u,v)}]^2 + [\frac{\partial(y,z)}{\partial(u,v)}]^2 + [\frac{\partial(z,x)}{\partial(u,v)}]^2}.$$

Problem 6 [10], Evaluate the integral

$$\int\int_S (1-z)dS,$$

where S is the graph of $z = 1 - x^2 - y^2$, with $x^2 + y^2 \leq 1$.

Problem 7 [10], Let S be a sphere of radius r and $\mathbf{p}$ be a point inside or outside the sphere (but not on it). Show that

$$\int\int_S \frac{1}{\|\mathbf{x} - \mathbf{p}\|} dS = \begin{cases} 4\pi r, & \text{if } \mathbf{p} \text{ is inside S} \\ 4\pi r^2/d, & \text{if } \mathbf{p} \text{ is outside S} \end{cases}$$

where d is the distance from $\mathbf{p}$ to the center of the sphere nd integration is over the sphere. (Hint: assume $\mathbf{p}$ is on the z-axis. Then change variables and evaluate. Why this assumption on $\mathbf{p}$ justified?)

Problem 8 [10], Evaluate the surface integral

$$\int\int_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$ and S is the surface parameterized by $\Phi(u, v) = (2\sin u, 3\cos u, v)$, with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$.

Problem 9 [10], Evaluate $\int\int_S(\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F} = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$ and S is the surface $x^2 + y^2 + z^2 = 16, z \geq 0$. (Let $\mathbf{n}$, the unit normal, be upward pointing.)

Problem 10 [10], Prove the following mean-value theorem for surface integrals: If $\mathbf{F}$ is a continuous vector field, then

$$\int\int_S \mathbf{F} \cdot \mathbf{n} dS = [\mathbf{F}(Q) \cdot \mathbf{n}(Q)A(S),$$

for some $Q \in S$, where $A(S)$ is the area of S. [Hint: Prove it for real functions first, by reducing the problem to one of a double integra: Show that if $g \geq 0$, then

$$\int\int_D fgdA = f(Q)\int\int_D gdA$$

for some $Q \in D$ (do it by considering $(\int\int_D fgdA)/(\int\int_D gdA)$ and using the intermediate value theorem).]