Problem 1 [10], Let $\mathbf{c}$ be a smooth path in $\mathbb{R}^3$.

(a) Suppose $\mathbf{F}$ is perpendicular to $\mathbf{c}'(t)$ at the point $\mathbf{c}(t)$. Show that

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0.$$

(b) If $\mathbf{F}$ is parallel to $\mathbf{c}'(t)$ at the point $\mathbf{c}(t)$, show that

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \|\mathbf{F}\| ds.$$

(By parallel to $\mathbf{c}'(t)$ we mean that $\mathbf{F}(\mathbf{c}(t)) = \lambda(t)\mathbf{c}'(t)$, where $\lambda(t) > 0$.)

Problem 2 [10], Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, where $P(x,y) = \frac{-y}{x^2+y^2}, Q(x,y) = \frac{x}{x^2+y^2}$ are functions defined for $(x,y) \in \mathbb{R}^2 \setminus (0,0)$, show that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \forall (x,y) \in \mathbb{R}^2 \setminus (0,0).$$

Prove that $\mathbf{F}$ is not a gradient vector field in $\mathbb{R}^2 \setminus (0,0)$ by verifying

$$\int_{\gamma} \mathbf{F} \cdot d\vec{\mathbf{s}} \neq 0, \text{ where } \quad \gamma = \{x^2 + y^2 = 1\}.$$

Problem 3 [10], Let Φ be a regular surface at (u_0, v_0) (i.e., Φ is C^1 and $T_u \times T_v \neq 0$ at (u_0, v_0)).

(a) Use the implicit function theorem to show that the image of Φ near (u_0, v_0) is the graph of a C^1 function of two variables. If the z component of $T_u \times T_v$ is nonzero, we can write it as $z = f(x,y)$.

(b) Show that the tangent plane at $\Phi(u_0, v_0)$ defined by the plane spanned by T_u and T_v coincides with the tangent plane of the graph of $z = f(x,y)$ at this point.

Problem 4 [10], Find the area of the surface defined by $z = 2xy$ and $x^2 + y^2 \leq 1$.

Problem 5 [10], Compute the Gauss curvature of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1.$$

Problem 6 [10], Evaluate the integral

$$\int\int_S (2 - z) dS,$$

where S is the graph of $z = 2 - x^2 - y^2$, with $x^2 + y^2 \leq 2$.

Problem 7 [10], Let S be a sphere of radius r and $\mathbf{p}$ be a point inside or outside the sphere (but not on it). Show that

$$\int\int_S \frac{1}{\|\mathbf{x}-\mathbf{p}\|} dS = \begin{cases} 4\pi r, & \text{if } \mathbf{p} \text{ is inside S} \\ 4\pi r^2/d, & \text{if } \mathbf{p} \text{ is outside S} \end{cases}$$

where d is the distance from $\mathbf{p}$ to the center of the sphere nd integration is over the sphere. (Hint: assume $\mathbf{p}$ is on the z-axis. Then change variables and evaluate. Why this assumption on $\mathbf{p}$ justified?)

Problem 8 [10], Evaluate the surface integral

$$\int\int_S \mathbf{F}\cdot d\mathbf{S}$$

where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$ and S is the surface parameterized by $\Phi(u,v) = (2\sin u, 3\cos u, v)$, with $0 \le u \le 2\pi$ and $0 \le v \le 1$.

Problem 9 [10], Evaluate $\int\int_S (\nabla\times\mathbf{F})\cdot d\mathbf{S}$, where $\mathbf{F} = (x^2+y-4)\mathbf{i} + 3xy\mathbf{j} + (2xz+z^2)\mathbf{k}$ and S is the surface $x^2+y^2+z^2 = 16, z \ge 0$. (Let $\mathbf{n}$, the unit normal, be upward pointing.)

Problem 10 [10], Prove the following mean-value theorem for surface integrals: If $\mathbf{F}$ is a continuous vector field, then

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = [\mathbf{F}(Q)\cdot\mathbf{n}(Q)A(S),$$

for some $Q \in S$, where $A(S)$ is the area of S. [Hint: Prove it for real functions first, by reducing the problem to one of a double integra: Show that if $g \ge 0$, then

$$\int\int_D fg dA = f(Q)\int\int_D g dA$$

for some $Q \in D$ (do it by considering $(\int\int_D fgdA)/(\int\int_D gdA)$ and using the intermediate value theorem).]