First Assignment, due on September 21, 2015.

Problem 1, Show that the subset $D = \{(x, y)|x \neq 1 \ \ and \ \ y > 0\}$ is an open set in $\mathbb{R}^2$.

Problem 2, Compute the following limits if they exist:
(a), $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{\sqrt{x^2+y^2}}$, (b), $\lim_{(x,y)\to(0,0)} \frac{1-\cos xy}{xy^2}$, (c), $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

Problem 3, $\forall \mathbf{x} \in \mathbb{R}^n$, define $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n | |\mathbf{y} - \mathbf{x}| < r\}$.
(a) Prove that for $\mathbf{x} \in \mathbb{R}^n$ and $0 < s < t, B_s(\mathbf{x}) \subset B_t(\mathbf{x})$.
(b) Prove that if U and V are neighborhoods of $\mathbf{x} \in \mathbb{R}^n$, then so are $U \bigcap V$ and $U \bigcup V$.
(c) Prove that the boundary points of an open interval $(a, b) \subset \mathbb{R}$ are the points a and b.

Problem 4, Suppose $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{y}$. Show that there is a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(\mathbf{x}) = 1, f(\mathbf{y}) = 0$, and $0 \leq f(\mathbf{z}) \leq 1$ for every $\mathbf{z}$ in $\mathbb{R}^n$.

Problem 5, Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ satisfy $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq K\|\mathbf{x} - \mathbf{y}\|^\alpha$ for all $\mathbf{x}$ and $\mathbf{y}$ in Ω for positive constant K and α. Show that f is continuous. (Such functions are called **Hölder-continuous** or, if $\alpha = 1$, **Lipschitz-continuous**.)

Problem 6, Find the partial derivatives $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ for function $w(x, y) = \sin(xe^{xy}) \cos y$.

Problem 7, Show that the function $f(r, \theta) = r \cos 2\theta, r > 0$ (in polar coordinates) is differentiable at each point in its domain. Decide if it is C^1.

Problem 8, Compute the matrix of partial derivatives of $f(x, y) = (e^x, \sin xy)$.

Problem 9, Evaluate the gradient of $f(x, y, z) = \log(x^2 + y^2 + z^2)$ at $(1, 1, 0)$.

Problem 10, Suppose f is a Hölder-continuous functions with $\alpha > 1$ in $B_1(0)$ (see problem 5) with $f(0) = 0$, prove that $f \equiv 0$ in $B_1(0)$. (Hint: What is the derivative of such function?)