MATH3A3 TEST #1, (2003)

Problem 1 Let $x_n = \sin\frac{n\pi}{2} + \frac{1}{n}$, and let $S = \{x_1, x_2, x_3, ...\}$.
(a) [5], find $\sup(S)$ and $\inf(S)$;
(b) [5], find $\limsup x_n$ and $\liminf x_n$.

Solution. (a), Since $\sin\frac{n\pi}{2}$ takes three values $1, 0, -1$, and $\frac{1}{n}$ is decreasing. So $2 = x_1 \geq x_n$ for all n. Since $\frac{1}{n} > 0$, we have $x_n > -1$ for all n. But, for $n = 4k+3$, $x_n = -1 + \frac{1}{n}$. Clearly, we get $\sup(S) = 2$ and $\inf(S) = -1$.

(b). As $x_n > -1$ and for $n = 4k+3$, $x_n \to -1$ as $k \to \infty$. We get $\liminf x_n = -1$. Also, for any N fixed, $x_n \leq 1 + \frac{1}{N}$ for all $n \geq n$. So $\sup\{x_N, x_{N+1}, \cdots\} \leq 1 + \frac{1}{N}$. From this we get $\limsup x_n \leq 1$ since N is arbitrary. On the other hand, taking $n = 4k+1$, $x_n = 1 + \frac{1}{n} \to 1$ as $k \to \infty$, and taking $n = 4k+3$, $y_n = -1 \to -1$. We conclude that $\limsup x_n = 1$.

Problem 2, Test the convergence of the following series
(a) [5], $\sum_{n=1}^{\infty}(-1)^n\frac{100^n}{n!}$;
(b) [5], $\sum_{n=1}^{\infty}\frac{n^2-101}{n^3+200}$.

Solution. (a). We use Ratio test. $\frac{|a_{n+1}|}{|a_n|} = \frac{100}{n+1} \to 0$ as $n \to \infty$. So the series is convergent.

(b). Let $a_n = \frac{n^2-101}{n^3+200}$ and $b_n = \frac{1}{2n}$. We have for $n \geq 20$, $a_n \geq b_n$. Since $\sum_n b_n$ is divergent by p-test, by Comparison test, $\sum_n a_n$ is divergent.

Problem 3 [5], Suppose A is a subset in a metric space X, let
$A^* = \{$ all the accumulation points of A $\}$. Show that A^* is closed.

Solution. We need to prove that $X \setminus A^*$ is open. For any point $x \in X \setminus A^*$, since x is not an accumulation point of A, there is $\epsilon > 0$ such that $B_\epsilon(x)\setminus\{x\}$ contains no points of A. Now, for any $y \in B_\epsilon(x)$, let $\delta = \min\{d(x,y), \epsilon - d(x,y)\}$. We have $\delta > 0$, and $B_\delta(y) \subset B_\epsilon(x)\setminus\{x\}$. Therefore, $B_\delta(y)$ contains no point of A and y is not an accumulation point of A. This means $B_\epsilon(x) \subset X \setminus A^*$. That is, $X \setminus A^*$ is open.