Solutions of MATH3A3 TEST #2 (Corrected Version)

Problem 1 [10], Is the function $g(x) = \frac{1}{\sqrt{1+x^2}} \sin x$ uniformly continuous in $\mathbb{R}$? Is the function $f(x) = x \sin x$ uniformly continuous in $\mathbb{R}$?

Solution. g is uniformly continuous in $\mathbb{R}$. We compute that $g^{'}(x) = \frac{\cos x}{\sqrt{1+x^2}} - \frac{x \sin x}{(1+x^2)^{\frac{3}{2}}}$. Therefore, $|g^{'}(x)| \leq 2, \forall x \in \mathbb{R}$. By Mean value Theorem, $g(x) - f(y) = g^{'}(z)(x-y)$ for some z between x, y. We get $|g(x) - g(y)| < 2|x-y|$. $\forall \epsilon > 0$, let $\delta = \frac{\epsilon}{2}$, it follows $|g(x) - g(y)| < \epsilon, \forall |x-y| < \delta$. f is not uniformly continuous in $\mathbb{R}$. Let $x_n = 2n\pi$ and $y_n = x_n + \frac{1}{n}$. We have $|x_n - y_n| = \frac{1}{n} \to 0$, but $|f(x_n) - f(y_n)| = 2n\pi \sin \frac{1}{n} \to 2\pi$.

Problem 2 [9], Let $f_n(x) = \frac{x^{2n+1}}{n^2 + x^{2n}}$, $n = 1, 2, \cdots$. Is is convergent uniformly in $[-100, 100]$? Is the sequence of functions $\{f_n\}_{n=1}^{\infty}$ convergent uniformly in $\mathbb{R}$?

Solution. For any $|x| \leq 1$, $|f_n(x)| \leq \frac{1}{n^2} \to 0$, so $f_n(x)$ converges to 0 uniformly in $[-1, 1]$. But for $|x| > 1$, As $\frac{n^2}{x^n} \to 0$, we get $f_n(x) \to x$ for all $|x| > 1$. It follows that $\{f_n(x)\}$ is convergent to $f(x)$ pointwise in $\mathbb{R}$, where $f(x) = 0$ when $|x| \leq 1$ and $f(x) = x$ when $|x| > 1$. The limit function $f(x)$ is not continuous at $|x| = 1$. So the sequence is not uniformly convergent in $[-100, 100]$ and it is also not uniformly convergent in $\mathbb{R}$.

Problem 3 [6] Let $V_n \subset \mathbb{R}$ be open bounded sets, $V_n \neq \emptyset$, and $cl(V_n) \subset V_{n-1}$. Prove that $\cap_{n=1}^{\infty} V_n = \cap_{n=1}^{\infty} cl(V_n)$ and deduce from this that $\cap_{n=1}^{\infty} V_n \neq \emptyset$.

Solution. Set $A_n = cl(V_n)$. Since V_n is bounded, so A_n is bounded for each n. Therefore A_n is compact. By the assumption, $\emptyset \neq V_n \subset A_n \subset V_{n-1} \subset A_{n-1}$, from this we get $\cap_{n=1}^{\infty} V_n = \cap_{n=1}^{\infty} A_n = \cap_{n=1}^{\infty} cl(V_n)$. Since $\emptyset \neq A_n \subset A_{n-1}$, by the Nested Set Property, $\cap_{n=1}^{\infty} A_n \neq \emptyset$.

Problem 4 [6] Let f_n be a sequence of continuous functions defined in the interval $[-1, 1]$. Suppose that $|f_{n+1}(x) - f_n(x)| \leq \frac{100}{n^2}$ for all $-1 \leq x \leq 1$ and for all $n \geq 1$. Show that f_n converges uniformly to some continuous function $f(x)$ defined in $[-1, 1]$.

Solution. Set $g_n(x) = f_{n+1}(x) - f_n(x)$. By the assumption, $|g_n(x)| \leq \frac{100}{n^2}$ for all $|x| \leq 1$. Since $\sum_{n=1}^{\infty} \frac{100}{n^2}$ is convergent by p-test, $\sum_{n=1}^{\infty} g_n(x)$ is uniformly convergent by M-test. Note $f_{n+1}(x) - f_1(x) = \sum_{k=1}^{n} g_k(x)$, we conclude that $\{f_n\}$ is uniformly convergent on $[-1, 1]$. So f is continuous.