Solutions of Assignment #5

253, #2 For $f_n(1) = 0$ for all n and for $0 \le x < 1$, $\lim n^3 x^n \to 0$. We get $\lim_{n\to\infty} f_n(x) = (1-x)\lim_{n\to\infty} n^3 x^n = 0$ for each $0 \le x < 1$. Therefore $f_n(x)$ converges pointwise to 0 on $[0,1]$ On the other hand, $\int_0^1 f_n(x)dx = n^3 \int_0^1 x^n(1-x)dx = n^3(\frac{1}{n+1} - \frac{1}{n+2}) = \frac{n^3}{(n+1)(n+2)} \to \infty$, therefore f_n is not uniformly convergent on $[0,1]$.

P.274, #2. Not necessary. For example, $f_n(x) = (-1)^n(x + \frac{x^2}{n})$ for $0 \le x \le 1$.

P.274, #4. Since $\mathcal{B}$ is bounded, we have $|f(x)| \le M$ for all $f \in \mathcal{B}$ and for all $0 \le x \le 1$. In turn, $|I(f)| \le \int_0^1 |f(x)|dx \le M$ for all $f \in \mathcal{B}$, and the set $A = \{I(f)|f \in \mathcal{B}\}$ is bounded in $\mathbb{R}$. Let $L = \sup A$. We have $f_n \in \mathcal{B}$, $I(f_n) \to L$. Now, $\{[f_n\}$ is bounded and equi-continuous, it has a subsequence $\{f_{n_k}\}$, and $f_{n_k}(x) \to f_0(x)$ uniformly on $[0,1]$. By Theorem 5.3.1, $L = \int_0^1 f_0(x)dx$. Since $\mathcal{B}$ is closed, $f_0 \in \mathcal{B}$.

P.286 #2. Suppose the degrees of $\{p_n\}$ is bounded by N. Let $x_0, x_1, \cdots, x_N$ be any N distinct points in $\mathbb{R}$. For each polynomial p with degree $\le N$, $p(x) = \sum_{i=0}^N \pi_i(x)\frac{p(x_i)}{\pi_i(x_i)}$, where $\pi_i(x) = (x-x_0)(x-x_1)...(x-x_N)/(x-x_i)$ (this follows from the Fundamental Theorem of Algebra). We have $p_n(x) = \sum_{i=0}^N \pi_i(x)\frac{p_n(x_i)}{\pi_i(x_i)}$ for each n and for all $x \in \mathbb{R}$. We now pick $x_0, x_1, \cdots, x_N \in [0,1]$. Since $\{p_n\}$ is uniformly convergent to $f(x)$ on $[0,1]$, $p_n(x_i) \to f(x_i)$ for $i = 0, 1, \cdots, N$. Taking the limit on the above formula for $p_n(x)$, we conclude that $f(x) = \sum_{i=0}^N \pi_i(x)\frac{f(x_i)}{\pi_i(x_i)}$, and f is a polynomial. Contradiction.

P.289, #2. We use Abel test. Let $f_n(x) = (-1)^n\frac{1}{n}$ and $\varphi_n(x) = x^n$. We have $\sum f_n(x) = \sum(-1)^n\frac{1}{n}$ converges uniformly on $[0,1]$, and $|\varphi_n(x)| \le 1$ for all n and for all $0 \le x \le 1$. Therefore, $\sum \frac{(-1)^n x^n}{n}$ converges uniformly on $[0,1]$.

P.294, #2 First, for any $|x| < 1$, $\sum_{n=0}^\infty x^n = \frac{1}{1-x}$. the series is uniformly convergent on $[-1+\delta, 1-\delta]$ for any $1 > \delta > 0$. Therefore, we can differentiate the series term-by-term for any $|x| < 1$ to get $\frac{1}{(1-x)^2} = \sum_{n=1} nx^{n-1} = \sum_{n=0}(n+1)x^n$.

P.316, #8. Not necessary. See question 2 of page 253.

P.316, #28. It is uniformly convergent on $[0,396]$ to 0, since $nay\epsilon > 0$, $|f_n(x) - 0| = \frac{|x|}{n} \le \frac{396}{n} < \epsilon$ for all $x \in [0,396]$ and for all $n > \frac{396}{\epsilon}$. Also, we note that $f_n(x) \to 0$ for all $x \in \mathbb{R}$. But it is not uniformly convergent in $\mathbb{R}$, since $f_n(n) = 1$ for all n.

P.316, #50. We first show that there is N such that degree of p_n is bounded by N for any n. If this is not the case, there exists a subsequence $\{p_{n_k}\}$ such that degree of $p_{n_{k+1}}$ is at least one higher than the degree of p_{n_k}. If m is the degree of p_{n_k} and l the degree of $p_{n_{k+1}}$, then $l > m$. Since x^l is the dominating term in $p_{n_{k+1}}$ as $|x| \to \infty$, there is $x_k \in \mathbb{R}$ such that $|p_{n_{k+1}}(x_k) - p_{n_k}(x_k)| \ge 1$. So we have $\lim_{k\to\infty} |p_{n_{k+1}}(x_k) - p_{n_k}(x_k)| \ge 1$, contradiction to the assumption that $p_n(x)$ uniformly convergent in $\mathbb{R}$. So degrees of p_n is bounded by some N for all n. By problem 2 of page 286, for any N distinct points $x_0, x_1, \cdots, x_N$, $p_n(x) = \sum_{i=0}^N \pi_i(x)\frac{p_n(x_i)}{\pi_i(x_i)}$ for each n and for all $x \in \mathbb{R}$. Since $\{p_n\}$ is uniformly convergent to $f(x)$ on $\mathbb{R}$, $p_n(x_i) \to f(x_i)$ for $i = 0, 1, \cdots, N$. We obtain that $f(x) = \sum_{i=0}^N \pi_i(x)\frac{f(x_i)}{\pi_i(x_i)}$, and f is a polynomial.

P.316, #62(b). Since $\log 1 = 0$, the series makes sense only we sum up from 2. As we know that $\frac{|x|^k}{\log k} \to \infty$ when $|x| > 1$. And since $< 0\frac{1}{\log k} < 1$ for $k > 2$, the series is convergent for $|x| < 1$. The series is convergent at $x = -1$ by alternating test, and it is divergent at $x = 1$ since $\frac{1}{\log k} \geq \frac{1}{k}$ when k large, and $\sum \frac{1}{k}$ is divergent.