Solutions of Assignment #4

191, #3 Since K is compact and f is continuous, the $\max_{x\in K} f(x) = L$ is attained at some $x_0 \in K$. Since the single point $\{L\} \subset \mathbb{R}$ is closed and f is continuous, the pre-image set $M = f^{-1}(L)$ is closed in K. Again, since K is compact, M is a closed subset of K, M is compact.

P.196, #6. (a). Let (X, d) and (Y, ρ) are metric spaces. By the definition, $f : X \to Y$ is not uniformly continuous in X, if and only if there is $\epsilon_0 > 0$ such that $\forall \delta_k > 0$ small, there exist $x_k, y_k \in X$ with $d(x_k, y_k) \le \delta_k$, $\rho(f(x_k), f(y_k)) \ge \epsilon_0$. Since $\frac{1}{n} \to 0$, for any $\delta_k > 0$, there is n, $\frac{1}{n} \le \delta_k$. So $f : X \to Y$ is not uniformly continuous in X, if and only if there exist $\epsilon_0 > 0$ and $x_n, y_n \in X$ with $d(x_n, y_n) \le \frac{1}{n}$, $\rho(f(x_n), f(y_n)) \ge \epsilon_0$. (b). Let $x_n = n$ and $y_n = n + \frac{1}{2n}$. We have $|x_n - y_n| = \frac{1}{2n} < \frac{1}{n}$, $|f(x_n) - f(y_n)| = 1 + \frac{1}{4n^2} \ge 1$.

P.203, #3. Since f is a polynomial and non-constant, it is continuous and non-constant in $[0, 1]$. Since $[0, 1]$ is compact. There exist $x_0, y_0 \in [0, 1]$ such that $f(x_0) = \min_{x\in[0,1]} f(x) = m$ and $f(y_0) = \max_{x\in[0,1]} f(x) = M$. By the assumption, $f(0) = f(1)$. One of x_0, y_0 must not be end points $0, 1$. If $0 < x_0 < 1$, then it is a local minmum of f in $\mathbb{R}$. If $0 < y_0 < 1$, then it is a local maxmum of f in $\mathbb{R}$.

P.231, #2(a) For $x_0 \in B \subset A$ and any $\epsilon > 0$, since g is continuous in A, there is $\delta > 0$ such that if $d(x, x_0) < \delta$, $\|f(x) - f(x_0)\| < \epsilon$. In particular, this is true for $x \in B$ with $d(x, x_0) < \delta$. That is, $g|B$ is continuous.

P.231, #6(b). Let $G = \{(x, f(x))| \quad x \in \mathbb{R}\}$ be the graph of f. Assume f is continuous. If $(x_0, y_0) \in G^c$, we have $y_0 \ne f(x_0)$. That is, there is $\epsilon_0 > 0$, $|y_0 - f(x_0)| \ge \epsilon_0$. Since f is continuous, there is $\delta > 0$, such that $\forall |x - x_0| < \delta$, $|f(x) - f(x_0)| \le \frac{\epsilon_0}{4}$. In turn we have $|y_0 - f(x)| \ge |y_0 - f(x_0)| - |f(x) - f(x_0)| \ge \frac{3\epsilon_0}{4}$. From this, we get $\forall y \in \mathbb{R}$ with $|y - y_0| \le \frac{\epsilon_0}{2}$ and $|x - x_0| < \delta$, $|f(x) - y| \ge |f(x) - y_0| - |y_0 - y| \ge \frac{\epsilon_0}{4}$. That is, G^c is open, so G is closed (note that we have not used the boundedness of f yet). On the other hand, if $f(x) \le M$ for all $x \in \mathbb{R}$ and G is closed. For each $x_0 \in \mathbb{R}$, we want to show if $x \to x_0$, $f(x) \to f(x_0)$. If this is not the case, there exist $\epsilon_0 > 0$, $x_n \to x_0$, $|f(x_n) - f(x_0)| \ge \epsilon_0 > 0$. Since $|f(x_n)| \le M$, there is a subsequence n_k, such that $f(x_{n_k}) \to y_0$ for some $y_0 \in \mathbb{R}$ and $y_0 \ne f(x_0)$. But $x_{n_k} \to x_0$, by the closeness of G, $(x_0, y_0) \in G$. This means $y_0 = f(x_0)$. Contradiction. The result is not true if f is not bounded. Example: $f(x) = 0$ for $x \le 0$ and $f(x) = \frac{1}{x}$ if $x > 0$. The graph of f is closed, but f is not continuous.

P.231, #24. (a). If f is uniformly continuous, by the definition, we have $\rho(f(x_k), f(y_k)) \to 0$ if $d(x_k, y_k) \to 0$. On the other hand, if f is not uniformly continuous, there exist $\epsilon_0 > 0$, x_k, y_k such that $d(x_k, y_k) \le \frac{1}{k}$ but $\rho(f(x_k), f(y_k)) \ge \epsilon_0, \forall k$. (b), Since f is uniformly continuous, for any $\epsilon > 0$, there is $\delta > 0$, such that $\rho(f(x), f(y)) < \epsilon$ if $d(x, y) < \delta$. Now, since $\{x_k\}$ is Cauchy, there is N such that for all $n, m \ge N$, $d(x_n, x_m) < \delta$. In turn, we get (let $x = x_n, y = x_m$) $\rho(f(x - n), f(x_m)) < \epsilon$. that is, $\{f(x_n)\}$ is Cauchy. (c), For any $y \in cl(A)$, there exists $\{x_n\} \subset A$ such that $x_n \to y$. In particular, $\{x_n\}$ is Cauchy. By (b), $\{f(x_n)\}$ is Cauchy. Since N is complete, $\lim f(x_n)$ exists. We want to define $f(y) = \lim f(x_n)$. We need to show that it is independent of the choice of $\{x_n\}$. Suppose there is another sequence $\{y_n\} \subset A$, such that $y_n \to y$. Let $z_n = x_n$ if n even and $z_n = y_n$ if n odd. We have $z_n \to y$. Again by (b), $\{f(z_n)\}$ is Cauchy and $\lim f(z_n)$ exists as N is complete. From this we conclude that $\lim f(x_n) = \lim f(y_n)$. Since f is uniformly continuous in A. For $\epsilon > 0$, there is $\delta > 0$, $\rho(f(x), f(y)) < \frac{\epsilon}{2}$ if $d(x, y) < 2\delta$. Finally, for any $x, y \in cl(A)$ with $d(x, y) < \delta$, we can find $x_n, y_n \in A$ such that $d(x, x_n) < \frac{\delta}{2}, d(y, y_n) < \frac{\delta}{2}$ and $\rho(f(x), f(x_n)) < \frac{\epsilon}{4}$ and $\rho(f(y), f(y_n)) < \frac{\epsilon}{4}$. We note that $d(x_n, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n) < 2\delta$. Now $\rho(f(x), f(y)) \le \rho(f(x), f(x_n)) + \rho(f(x_n), f(y_n)) + \rho(f(y_n), f(y)) < \epsilon$. That is f is uniformly continuous in $cl(A)$. From the definition of f in $cl(A)$, the extension is unique.

P.231, #26. Since $\lim_{x \to a^+} f^{'}(x)$ exists, there is $\delta_0 > 0$ and $M > 0$ such that $|f^{'}(x)| \le M$ for all $x - a \le 2\delta_0$. Since $[a + \delta_0, b]$ is compact and so f is uniformly continuous on this closed interval. $\forall \epsilon > 0$, there is $\delta_1 > 0$, $|f(x) - f(y)| < \epsilon$ if $x, y \in [a + \delta_0, b], |x - y| < \delta_1$. We may pick $\delta_1 < \delta_0$. If $|x - y| < \delta_1 < \delta_0$, and one of x, y is in $(a, a + \delta_0)$, we must have $x, y \in (a, a + 2\delta_0)$. By Mean Value Theorem, $f(x) - f(y) = f^{'}(z)(x - y)$ for some z between x and y. So $z \in (a, a + 2\delta_0)$, and $|f^{'}(z)| \le M$. That is $|f(x) - f(y)| \le M|x - y|$. Now, we pick $\delta = \min(\delta_1, \frac{\epsilon}{2M})$, then for all $x, y \in (a, b]$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

P.231, #30. (a). First f is continuous in $[0, \infty)$. It is uniformly continuous on $[0, 2]$ since $[0, 2]$ is compact. $\forall \epsilon > 0$, there is $\delta_1 > 0$, such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in [0, 2]$ and $|x - y| < \delta_1$. We may pick $\delta_1 < \frac{1}{2}$. For $z \ge 1$, we have $0 < f^{'}(z) = \frac{1}{2\sqrt{z}} < 1$. Therefore, $\forall x, y \in [1, \infty)$, by Mean Value Theorem there is z between x, y, $|f(x) - f(y)| = |f^{'}(z)(x - y)| \le |x - y|$. As in the previous problem pick $\delta = \min(\delta_1, \epsilon)$, it is straightforward to verify that $|f(x) - f(y)| < \epsilon, |x - y| < \delta$.

P.244, #2. No. Define $f(x) = x, 0 \le x < 1$ and $f(1) = 0$. We have $f_n(x) \to f(x)$ pointwise in $[0, 1]$. As f is not continuous and f_n are continuous, f_n can not converge to f uniformly.

P.247, #2. It is uniformly convergent by M-test. Let $g_n(x) = \frac{x^n}{n^2}$, we have $|g_n(x)| \le \frac{1}{n^2}$. By p-test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, so $\sum_{n=1}^{\infty} g_n(x)$ is uniformly convergent.