Solutions of third assignment

P.157 #5. No. Example: $A = \{\frac{1}{n} | n = 1, 2, \cdots\}$. $x = 0$ is the only accumulation point of A, but A is not closed (so it is not compact). If, in addition, assume A is closed, then A is compact.

P.172, #1(c). It is compact since it is closed and bounded. If $n > 1$, it is also path-connected, so it is connected. If $n = 1$, it is not connected.

P.172, #6. By the Nested Set Property, $\cap_{k=1}^{\infty} F_k \neq \emptyset$. We claim it can not have two points. Suppose this is not the case, there are $x, y \in \cap_{k=1}^{\infty} F_k$ with $x \neq y$. Therefore, $d(x, y) = \delta > 0$. As $x, y \in F_k$ for all k, we get $diameter(F_k) \geq \delta$ independent of k. This is a contradiction to the assumption that $diameter(F_k) \to 0$.

P.172, #8. Since $x_k \in A$ and A is compact, x_k has a subsequence $\{x_{k_j}\}$ converges to some point $x \in A$. Since $\{x_k\}$ is Cauchy, then the whole sequence must also be convergent to x.

P.172, #16. By triangle inequality $|\|x_k\| - \|x\|| \leq \|x_k - x\|$, therefore $\|x_k\| \to \|x\|$ if $x_k \to x$. The converse is false, e.g., $x_k = ((-1)^k + \frac{1}{k}, 0)$ in $\mathbb{R}^2$, we have $\|x_k\| \to \|(1, 0)\| = 1$, but x_k is not convergent. For $A = \{x \in \mathbb{R}^n | \|x\| \leq 1\}$, we want to show every accumulation point of A is in A. Suppose y is an accumulation point of A, there exist $x_k \in A$, $x_k \to y$. Since $\|x_k\| \leq 1$, by the previous assertion, $\|x_k\| \to \|y\|$. So $\|y\| \leq 1$, i.e., $y \in A$.

P.172, #20. We need to verify every accumulation point of A is in A. If this is not true, there is $x \in A^c$, such that x is an accumulation point of A. $\forall y \in A$, let $\delta_y = \frac{d(y,x)}{2}$, since $x \in A^c$, $\delta_y > 0$ for all $y \in A$. Let $U_y = B_{\delta_y}(y)$. We have $\cup_{y \in A} U_y$ covers A. Since A compact, there exist $y_, \cdots, y_N$, such that $U_{y_1} \cup \cdots \cup U_{y_N}$ covers A. Let $\delta = \min\{\delta_{y_1}, \cdots, \delta_{y_N}\}$, by our choice of δ_y, $B_\delta(x) \cap U_{y_i} = \emptyset$ for all $i = 1, \cdots, N$. This implies $B_\delta(x) \cap A = \emptyset$, a contradiction to x is an accumulation point of A.

P.172, #25. Let $y_n = \sin n$, $\{y_n\}$ is a bounded sequence in $\mathbb{R}$. There is subsequence y_{n_k} which is convergent to some x. That is, $\sin n_k \to x$ as $k \to \infty$.

P.172, #29. A is obviously closed and bounded in $\mathbb{R}^2$, so A is compact. Yes, A is connected. In fact, it is path-connected.

P. 182, #3. We need to show that any accumulation point w of A is in A. For such w, there exist $(x_n, y_n) \in A$ such that $(x_n, y_n) \to w$. We have $0 \leq f(x_n, y_n) \leq 1$, since f is continuous, $\lim_{n\to\infty} f(x_n, y_n) = f(w)$. We conclude that $0 \leq f(w) \leq 1$, i.e., $w \in A$.

P. 184, #3. Let $B = [1, +\infty)$, $f(x) = \frac{1}{1+x^2}$. We have B is closed, and $f(B) = (0, \frac{1}{2}]$ which is not closed. This is not possible if in addition B is bounded, since that would imply B is compact. Then $f(B)$ would be compact, so it would be closed.