**Solutions of second assignment (P.143)**

*1(a).* The set $A = (0, 1)$ is open, since for any point $0 < x < 0$, the interval $(x - \delta, x + \delta)$ is inside $(0, 1)$ for $\delta = \min(x, 1 - x)$. $(0, 1)$ is not closed since it compliment is $B = (-\infty, 0] \cup [1, \infty)$. For $x = 0$, for any $\epsilon > 0$, interval $(-\epsilon, \epsilon)$ always intersects $A$.

*2(h).* Let $A = \{x \in \mathbb{R}^n | \ \|x\| = 1\}$. The compliment of $A$ is $B = \{x \in \mathbb{R}^n | \ \|x\| < 1$, or $\|x\| > 1\}$. $B$ is open, so $A$ is closed. We have $cl(A) = A$. For any point $x \in A$, any $\epsilon > 0$, the ball $B_\epsilon(x)$ of radius $\epsilon$ centered at $x$ always intersects $B$. Therefore, $A$ has no interior points. The same argument yields that every $x \in A$ is a boundary point, so $bd(A) = A$.

*7.* For any $x \in U$, since $U$ is open, there is $\delta > 0$, $B_\delta(x) \subset U$. That is, $x$ is not an boundary point. So $U \subset cl(U) \backslash bd(U)$. On the other hand, $\forall y \in cl(U))$, $\forall \delta > 0$, $B_\delta(y) \cap U \neq \emptyset$. If $y$ is not in $bd(U)$, there is $\delta > 0$, $B_\delta(y)$ is either in $U$ or in its compliment. Combining the above, $\forall y \in cl(U) \setminus bd(U)$, there is $\delta > 0$, $B_\delta(y) \subset U$. This proves $U = cl(U) \setminus bd(U)$. This conclusion in general not true: $U = [0, 1]$, $cl(U) \setminus bd(U) = (0, 1)$.

*17.* Since $\sum_{n=1}^\infty \|x_n\|$ is convergent, $\forall \epsilon$, there is $N$, such that $\forall n > m \geq N$, $\sum_{k=m}^n \|x_k\| < \epsilon$. Now, we have $\| \sum_{k=m}^n x_k \sin k\| \leq \sum_{k=m}^n \|x_k \sin k\| = \sum_{k=m}^n \|x_k\| < \epsilon$. From this, we conclude that $\sum_{n=1}^\infty x_n \sin n$ is convergent.

*26.* First, it is easy to see that $1 \leq a_n$. We have for all $n \geq 1$,

$$a_{n+1} - a_n = (1 + \frac{1}{1 + a_n}) - (1 + \frac{1}{1 + a_{n-1}} = \frac{a_{n-1} - a_n}{(1 + a_n)(1 + a_{n-1})}.$$

In turn, $|a_{n+1} - a_n| = |\frac{a_{n-1}-a_n}{(1+a_n)(1+a_{n-1})}| \leq \frac{|a_n - a_{n-1}|}{4}$. By induction, $|a_{n+1} - a_n| \leq \frac{|a_1 - a_0|}{4^n}$. Now, for $n > m$, we have $|a_n - a_m| \leq \sum_{k=m}^{n-1} |a_{k+1} - a_k| \leq \sum_{k=m}^{n-1} \frac{|a_1 - a_0|}{4^n} \leq \frac{|a_1 - a_0|}{4^{m-1}}$. From this we conclude that $\{a_n\}$ is Cauchy. So the sequence is convergent to a limit $a$. We take the limit on the equation $a_n = 1 + \frac{1}{1+a_{n-1}}$, we get $a = 1 + \frac{1}{1+a}$. We solve this equation to get either $a = \sqrt{2}$ or $a = -\sqrt{2}$. We conclude $a = \sqrt{2}$ since $a_n \geq 1$ for all $n$.

*29.* Yes. If $x$ is not an accumulation point of $A$ and $B$. Then there is a neighborhood $U$ of $x$, $U$ contains no points of $A$ and $B$ other than $x$. Therefore, $x$ can not be an accumulation point of $A \cup B$.

*34.* By induction, $d(x_{n+1}, x_n) \leq r^{n-1} d(x_2, x_1)$. For any $n > m$, we have $d(x_n, x_m) \leq \sum_{j=m}^{n-1} d(x_{j+1}, x_j) \leq \sum_{j=m}^{n-1} r^{j-1} d(x_2, x_1) \leq r^{m-1} \frac{d(x_2, x_1)}{1-r}$. This shows that $\{x_n\}$ is Cauchy in $\mathbb{R}^k$, so it is convergent.

*43.* By the definition, $x_n \geq \sqrt{3}$. We claim that $x_n < 3$. This can be seen by induction. It is true for $n = 1$, suppose it is true for $n$, $x_{n+1} = \sqrt{3 + x_n} \leq \sqrt{3 + 3} < 3$. We calculate that $x_{n+1} - x_n = \sqrt{3 + x_n} - \sqrt{3 + x_{n-1}} = \frac{x_n - x_{n-1}}{\sqrt{3+x_n}\sqrt{3+x_{n-1}}}$. Since $x_2 - x_1 > 0$, we conclude that $x_{n+1} - x_n > 0$. So $\{x_n\}$ is a bounded increasing sequence, and it is convergent to a limit $x$ (Note, we may also compute $|x_{n+1} - x_n| \leq \frac{|x_n - x_{n-1}|}{\sqrt{3+x_n}\sqrt{3+x_{n-1}}} \leq \frac{|x_n - x_{n-1}|}{3}$ to get $|x_{n+1} - x_n| \leq \frac{|x_2 - x_1|}{3^{n-1}}$. The same argument as in problem #26 yields that $\{x_n\}$ is convergent). $x$ must satisfy the equation $x = \sqrt{3 + x}$. So either $x = \frac{1-\sqrt{13}}{2}$ or $x = \frac{1+\sqrt{13}}{2}$. We must have $x = \frac{1+\sqrt{13}}{2}$ since $x_n \geq \sqrt{3}$ for all $n$ (so $x$ must be positive).

*52(a).* Let $|x_n| \leq e^{-n}$, $\sum_{n=1}^\infty e^{-n}$ is convergent, by comparison Theorem, $\sum_{n=1}^\infty x_n$ is absolutely convergent.

*52(f).* We use ratios test. $|\frac{x_{n+1}}{x_n}| = \frac{(1+\frac{1}{n})^3}{3} \to \frac{1}{3} < 1$. The series is convergent.