Solutions of first assignment

P.48 #4. Yes. Since A and B are bounded from below, $\inf A$ and $\inf B$ exist. $\forall x \in A, y \in B$, it is clear $x+y \geq \inf A + \inf B$, so $\inf A + \inf B$ is a lower bound of $A+B$. On the other hand, $\forall \epsilon > 0$, there are $x \in A, y \in B$ such that $x < \inf A + \epsilon/2$ and $y < \inf B + \epsilon/2$. That is, there exist $x \in A, y \in B$ such that $x + y < \inf A + \inf B + \epsilon$, i.e.,$\inf A + \inf B = \inf(A+B)$.

P.51, #1. For any $N > M$, $\sum_{n=M}^{N} \frac{1}{3^n} = \frac{1}{3^M} \frac{1-\frac{1}{3^{N-M+1}}}{1-\frac{1}{3}} < \frac{1}{3^{M-1}}$. For any $\epsilon > 0$, there is N, such that $\frac{1}{3^{N-1}} < \epsilon$. If $n > m \geq N$, we have

$$|x_n - x_m| \leq \sum_{k=m}^{n-1} |x_{k+1} - x_k| < \sum_{k=m}^{n-1} \frac{1}{3^k} < \frac{1}{3^{m-1}} \leq \frac{1}{3^{N-1}} < \epsilon.$$

That is, $\{x_n\}$ is a Cauchy sequence. So it is convergent.

P.51, #4. We want to prove that, $\forall \epsilon > 0$, there is N, such that if $n \geq N$, $|x_n| < \epsilon$. Since $\{x_n\}$ is Cauchy, there is N_1 such that $|x_n - x_m| < \epsilon/2$ for all $n, m \geq N_1$. Let $\tilde{\epsilon} = \min\{\frac{\epsilon}{2}, \frac{1}{N_1}\}$. By the assumption, there is $n_0 > \frac{1}{\tilde{\epsilon}}$, such that $|x_{n_0}| < \tilde{\epsilon}$. Let $N = \frac{1}{\tilde{\epsilon}}$, we have $N \geq N_1$, and $\tilde{\epsilon} \leq \frac{\epsilon}{2}$. For any $n \geq N$, we get $|x_n| \leq |x_n - x_{n_0}| + |x_{n_0}| < \frac{\epsilon}{2} + \tilde{\epsilon} \leq \epsilon$.

P.98, #7. We divide into two cases. Case 1, one of the set has no upper bound, say A has no upper bound. In this case, $\sup A = +\infty$, and there is sequence $x_n \in A$ such that $x_n \to +\infty$. For any $y \in B$, $y + x_n \to +\infty$. That is $\sup(A+B) = \sup(A) + \sup(B) = +\infty$. Case 2, both A and B are bounded. In this case, $\sup A$ and $\sup B$ exist and finite. For and $x \in A, y \in B$, $x + y \leq \sup A + \sup B$, so $\sup A + \sup B$ is an upper bound of $A + B$. For any $\epsilon > 0$, there exist $x \in A, y \in B$ such that $x > \sup A - \frac{\epsilon}{2}, y > \sup B - \frac{\epsilon}{2}$. That is, there exist $x + y \in A + B$, such that $x + y > \sup A + \sup B - \epsilon$. This mean $\sup A + \sup B$ is the least upper bound of $A + B$.

P.98, #15. To show $\{x_n\}$ is Cauchy, we need to prove that $\forall \epsilon > 0$, there is N, such that for $n > m \geq N$, $|x_n - x_m| < \epsilon$. For any $\epsilon > 0$, there is N such that $\frac{1}{2^{N-2}} d(x_2, x_1) < \epsilon$. By induction, $d(x_{k+1}, x_k) \leq d(x_k, x_{k-1})/2$ implies $d(x_{k+1}, x_k) \leq \frac{1}{2^{k-1}} d(x_2, x_1)$. Now, for $n > m \geq N$,

$$d(x_n, x_m) \leq \sum_{k=m}^{n-1} d(x_{k+1}, x_k) \leq \sum_{k=m}^{n-1} \frac{d(x_2, x_1)}{2^{k-1}} = \frac{d(x_2, x_1)}{2^{m-1}} \sum_{k=0}^{n-m-1} \frac{1}{2^k} \leq \frac{d(x_2, x_1)}{2^{m-2}} < \epsilon.$$

P.98, #17. If a is a lower bound for S, we have $y \geq a, \forall y \in S$. We conclude that $y \geq \sup\{x \in R |\ x \text{ is a lower bound for } S\}$, for all $y \in S$. So it is a lower bound for S. On the other hand, if b is a lower bound of S, $b \in \{x \in R |\ x \text{ is a lower bound for } S\}$. So $b \leq \sup\{x \in R |\ x \text{ is a lower bound for } S\}$. That is $\inf S = \sup\{x \in R |\ x \text{ is a lower bound for } S\}$.

P.98, #22(a). In general, $\limsup(x_n + y_n) \neq \limsup x_n + \limsup y_n$. For example, let $x_n = (-1)^n(1 + \frac{1}{n})$ and $y_n = -x_n$. We have $x_n + y_n = 0$ for all n, but $\limsup x_n = \limsup y_n = 1$.

P.98, #23. Since $x \geq 0$ for all $x \in P$, 0 is a lower bound of P. For any $\epsilon > 0$, there is k, $\frac{1}{k} < \epsilon$. By the assumption, there is $x_k \in P$, $x_k \leq \frac{1}{k} < \epsilon = 0 + \epsilon$. That is, there is $x_k \in P$, $x_k < 0 + \epsilon$. Since $\epsilon > 0$ is arbitrary, 0 is the greatest lower bound of P.