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Regularity of a Class of Quasilinear Degenerate Elliptic Equations

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1. INTRODUCTION

In this paper, we consider the regularity of degenerate elliptic quasilinear equations in the form

$$\sum_{i,j} \partial_j(a_{ij}(x, u)\partial_i u) = f, \tag{1}$$

where $a_{ij}(x, u) \in C^\infty(\mathbf{R}^n \times \mathbf{R})$, semi-positive. Our study of the problem is motivated by the regularity problem for degenerate Monge–Ampère equations

$$\det(u_{ij}) = k, \tag{2}$$

where k is a nonnegative function.

When $n = 2$, if u is a $C^{1,1}$ convex solution of the equation, suppose p is a point such that the hypersurface $(x, u(x))$ is nonplanar near $(p, u(p))$. In a neighborhood of p, by performing a partial Legendre transformation (see Section 5 for details), one may reduce the regularity of u near p to the equation

$$\partial_1^2 w + \partial_2(k(x, w)\,\partial_2 w) = 0, \tag{3}$$

which is in a form of Eq. (1).

In general, convex solution u of (2) is at most in $C^{1,1}$ if k is only assumed to be smooth and nonnegative (e.g., see [CKNS] and [G1]). In [G1], a sharp sufficient condition was introduced to establish $C^{1,1}$ regularity of (2). A basic question left is: When is a solution u of (2) smooth, or better than $C^{1,1}$? We remark that if k is positive and smooth, by elliptic theory u is smooth. One may expect u to be smooth if the decay of k near its null set is under control, say of finite type. Unfortunately, this is not true. The function $u(x) = |x|^{2+2/n}$ provides an example that the solution u of Eq. (2)

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is merely in $C^{2,2/n}$ even when k is analytic, vanishing at only one point of order 2. What is wrong with the example is that the mean curvature of the hypersurface $(x, u(x))$ is vanishing at the point $k=0$. This suggests that we should only expect higher regularity of the solution u of (2) away from the plannar points of the hypersurface $(x, u(x))$. This is what we will prove for the case $n=2$ in the last section of this paper.

The difficulty concerned with Eq. (1) lies on the mixture of degeneracy and nonlinearity. If the equation is elliptic, the regularity follows from De Giorgi–Nash–Moser theory and Schauder theory. On the other hand, if a_{ij} is independent of u, Eq. (1) is linear. There is well-developed hypoelliptic theory. In particular, the theory of linear second order subelliptic operators with smooth real coefficients is quite complete. Deep theorems have been obtained by Hörmander [H] and Kohn [K1, K2] for sums of squares of vectors fields and for operators related to $\bar{\partial}_b$, and by Fefferman and Phong [FP] and Olenik and Radkevitch [OR] for general second order differential operators with smooth nonnegative principal symbols. For each second order degenerate operator L, one can associate a suitable metric in such a way that it is as natural for the operator L as the Euclidean metric is for the Laplace operator (see [FP] and [NSW]). The subellipticity of the operator L can be completely characterized in terms of the geometry of the associated metric, which is the Fefferman–Phong condition.

There have been some works on degenerate nonlinear elliptic equations in connection with Bony's theory of paradifferential operators [B]. Under some initial smoothness assumptions on the solution with some subelliptic estimates, one may prove the solution is C^∞ by paradifferential calculus (e.g., see [X]). (As for our Eq. (1), the $C^{1,1}$ initial assumption on u will suffice). To our knowledge, all the regularity results for degenerate quasilinear elliptic equations are based on the a priori assumption that u is in $C^{1,1}$ or more, which nevertheless is very hard to get. For example, the initial $C^{1,1}$ assumption on w in (3) corresponds to $C^{2,1}$ regularity of u in (2) as they are related by the partial Legendre transformation. So far, there is no better regularity result than $C^{1,1}$ for the degenerate Eq. (2) prior to this paper. Here, we will establish regularity results for (1) with the $C^{0,1}$ initial smoothness assumption on the solution u. As a consequence, we will prove a C^∞ regularity result for the degenerate Eq. (2) in dimension two.

Our results are based on recent developments on De Giorgi–Nash–Moser theory for degenerate equations (see [F] for an up-to-date bibliography). In [F], some sufficient conditions were introduced to obtain Hölder regularity of solutions of degenerate linear equations. Those conditions, we shall call them "subellipticity conditions," are in some way a version of the Fefferman–Phong condition in the nonsmooth case. Under these conditions, Harnack inequalities were proved for weak solutions of the degenerate equations in divergence form. The Harnack inequality together with the

Sobolev–Poincaré inequality gives Hölder regularity of weak solutions of equations in the form

$$\sum_{i,j}^{n} \partial_i(a_{ij}(x)\,\partial_j u) = \nabla_\lambda f + f_0, \tag{4}$$

where $\nabla_\lambda = (\lambda_1 \partial_1, ..., \lambda_n \partial_n)$, $\lambda_j \in C^{0,1}$, and $a_{ij}(x)\,\xi_i \xi_j$ is controlled by $\sum \lambda_i^2 \xi_i^2$ from below and above.

The main contribution of this paper is to obtain C^∞ smoothness of the solution u of Eq. (1) based on the assumptions that u is in $C^{0,1}$, and that the linearized equation satisfies subellipticity conditions. Our key step is the proof of a commutator lemma in Section 4. This lemma singles out the first order terms of the Calderón commutator with a careful control of the coefficients. The difficulty is the limited smooth assumption; otherwise, the lemma is a trivial consequence of the usual Kohn–Nirenberg formula. The proof of the lemma makes use of symbol decomposition of pseudodifferential operators with limited smoothness (see [T]). A similar calculus was also used previously by Guan and Sawyer in [GS] for the oblique derivative problem.

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2. SUBELLIPTICITY CONDITIONS AND MAIN RESULTS

We first recall subellipticity conditions in [F].

Let $\Omega \subset \mathbf{R}^n$ be a domain. Let $(a_{ij}(x))$ be a nonnegatively definite matrix function in Ω.

Definition 1. Let $T = \sum_{j=1}^n \alpha_j \partial_j$ be a vector field in Ω. We say T is a subunit with respect to $(a_{ij}(x))$ if

$$\sum_{j=1}^{n} (\alpha_j \xi_j)^2 \leqslant \sum_{i,j}^{n} a_{ij}(x)\,\xi_i \xi_j \tag{5}$$

for all $\xi \in \mathbf{R}^n$. If γ is an absolutely continuous curve in Ω, we say γ is a subunit curve (with respect to $(a_{ij}(x))$) if γ' is a subunit vector field. If $x, y \in \Omega$, we define a distance function $d(x, y)$ (with respect to $(a_{ij}(x))$) as

$$d(x, y) = \inf \{t > 0 \mid \text{there is a subunit curve } \gamma: [0, t] \to \Omega \text{ such that } \gamma(0) = x, \gamma(t) = y\}. \tag{6}$$

$d(x, y) = \infty$ if the above set is empty. Let $x \in \Omega$, $r > 0$ be fixed. Let

$$C_j(x, r) = \{\gamma_j(t), 0 \leqslant t \leqslant r \mid \gamma(t) = (\gamma_1(t), ..., \gamma_n(t)) \text{ be a subunit curve with } \gamma(0) = x\} \tag{7}$$

for $j = 1, 2, ..., n$.

Subelliptic conditions for degenerate linear Eqs. (4) in [F] can be stated as follows:

(SE1). (i) $\exists \nu > 0$, such that

$$\nu \sum_{j=1}^{n} \lambda_j^2(x)\, \xi_j^2 \leqslant \sum_{i,j}^{n} a_{ij}(x)\, \xi_i \xi_j \leqslant \frac{1}{\nu} \sum_{j=1}^{n} \lambda_j^2(x)\, \xi_j^2 \tag{8}$$

$\forall (x, \xi) \in \Omega \times \mathbf{R}^n$, where

(ii) $\lambda_1, ..., \lambda_n$ are in $C^{0,1}(\bar{\Omega})$, and d associated with ∂_λ is finite in Ω.

(iii) Let $\Lambda_i(x, r) = \max_{s_j \in C_j(x, r)} \lambda_i(s_1, ..., s_n)$. $\forall x_0 \in K \subset\subset \Omega$ there is a neighborhood U of x_0, such that, if $0 < \varepsilon_j \leqslant |\xi_j| \leqslant 1$ for $j = 1, ..., n$ and if we denote by $H(\cdot, x, \xi) = (H_1, ..., H_n)$ the integral curve of the vector field $\xi_1 \lambda_1 \partial_1 + \cdots + \xi_n \lambda_n \partial_n$ starting from x, we assume

$$\int_0^t \lambda_j(H(s, x, \xi))\, dx \geqslant t C_{\varepsilon_1, ..., \varepsilon_n} \Lambda_j(x, t) \tag{9}$$

for $j = 1, ..., n$, where $C_{\varepsilon_1, ..., \varepsilon_n}$ is independent of $t \in (0, t_0)$, $x \in U$ and $\xi \in \prod_{j=1}^{n} [\varepsilon_j, 1]$.

When $n = 2$, we may assume $\lambda_1 = 1$. The condition (SE1) can be replaced by

(SE2). With the condition (iii) in (SE1) replaced by: There is a constant $c > 0$, such that

$$\int_0^t \lambda_2(x + s, y)\, ds \geqslant ct \max_{x < z < x + t} \lambda_2(z, y). \tag{10}$$

Now we have following result in dimension 2.

THEOREM 2. *Suppose $u \in C^{0,1}(\Omega)$, u is a weak solution of the equation*

$$\partial_1^2 u + \partial_2(k(x, u)\, \partial_2 u) = f \tag{11}$$

with $k(x,u)\geqslant 0, k(x,u)\in C^{\infty}(\Omega\times\mathbf{R})$. *Let* $\tilde{k}(x)=k(x,u(x))$. *If* $\tilde{k}^{1/2}$ *satisfies subelliptic condition* (SE2), *then* $u\in C^{\infty}(\Omega)$.

The next is a simple application of our Theorem 2 to the degenerate Monge–Ampére equations.

THEOREM 3. *If u satisfies the degenerate Monge–Ampère equation,*

$$u_{xx}u_{yy}-u_{xy}^2=k(x,y) \tag{12}$$

near the origin. Suppose $k\geqslant 0, k\in C^{\infty}(\mathbf{R}\times\mathbf{R})$, *and* $u\in C^{1,1}$. *Suppose that near the origin* $u_{yy}\geqslant C_0>0$, *and that there exist constants* $A>0$ *and* $B\geqslant 0$, *and positive integers* $l\leqslant m$, *such that*

$$\frac{1}{A}(x^{2l}+By^{2m})\leqslant k(x,y)\leqslant A(x^{2l}+By^{2m}). \tag{13}$$

Then, $u\in C^{\infty}$ *near the origin.*

The proof of Theorem 3 together with discussions of the Monge–Ampére equations will be given in the last section of the paper. As for the regularity of the Eq. (1), we need some definitions.

DEFINITION 4. Let L be a linear operator of the form

$$Lu=\sum_{i,j}^{n}\partial_i(a_{ij}(x)\,\partial_j u)$$

with $a_{ij}\in C^{0,1}(\Omega)$, $(a_{ij})\geqslant 0$, and $\operatorname{tr}(a_{ij}(x))>0$ for all x in Ω. We say L satisfies subunit condition in Ω if there is a constant C such that,

$$|A_{x_k}(x)\cdot\xi|^2\leqslant C\,{}^t\xi A\xi, \tag{14}$$

$\forall\xi\in\mathbb{R}$, $\forall x\in\Omega$ and $\forall k=1,2,...,n$. We say L is α-subelliptic, if there is $\alpha>0$, such that, for any subunit vector fields $T_1,...,T_N$, and for any $f_0,f_1,...,f_N\in L^{\infty}(\Omega)$, if v is a weak solution of the equation

$$Lv=f_0+T_1f_1+\cdots+T_Nf_N, \tag{15}$$

then $v\in C^{\alpha}(\Omega)$, and for each $K\subset\subset\Omega$,

$$\|v\|_{C^{\alpha}(K)}\leqslant C$$

where C depends only on K, $\|f_j\|_{L^{\infty}(\Omega)}$, $\|v\|_{L^2(\Omega)}$, $\|T_j\|_{C^1}$.

DEFINITION 5. Let L be a α-subelliptic linear operator in Ω. We say L is elliptic extendible in Ω, if for any x in Ω there is a neighborhood $U\subset\subset V\subset\subset\Omega$ of x, and a smooth second order differential operator of divergence form $\tilde{L} = \nabla \cdot \tilde{A}^2 \cdot \nabla$ such that the operator $L^* = L + \tilde{L}$ is $\tilde{\alpha}$-subelliptic for some $\tilde{\alpha} > 0$, L^* is elliptic near ∂V, and $L^* = L$ in U.

The following is a general regularity result for degenerate Eq. (1).

THEOREM 6. *Suppose $u \in C^{0,1}(\Omega)$, u is a weak solution of the equation*

$$\sum_{i,j}^{n} \partial_i (a_{ij}(x, u)\, \partial_j u) = f(x) \tag{16}$$

with $a_{ij} \in C^\infty(\Omega \times \mathbf{R})$, $(a_{ij}) \geqslant 0$, $f \in C^\infty(\Omega)$. Let $\tilde{a}_{ij}(x) = a(x, u(x))$. If the operator $L = \sum_{i,j}^{n} \partial_i (\tilde{a}_{ij}(x)\, \partial_j)$ satisfies subelliptic condition (SE1), *and L is subunit and elliptic extendible in Ω, then $u \in C^\infty(\Omega)$.*

The proofs of Theorem 2 and Theorem 6 will rely on a commutator lemma. Before we state the lemma, we introduce some notations.

DEFINITION 7. Let Λ^s, $s \in \mathbf{R}$, be Hölder–Zygmund spaces. We denote $C^s_* = \Lambda^s$ if $s > 0$, $s \notin \mathbf{Z}$, and $C^\ell_* = C^{\ell, 0}$, if $\ell \in \mathbf{Z}^+$. If B is a linear operator, B is bounded from $\Lambda^{s+m}_{\text{comp}}(\mathbf{R}^n) \to \Lambda^s_{\text{loc}}(\mathbf{R}^n)$ for some $m \in \mathbf{R}$, and $\forall 0 < s \leqslant t$, then we say $B \in O^m_t$.

COMMUTATOR LEMMA. *Let $a(x, u) \in C^\infty(\Omega \times \mathbf{R})$, suppose $u \in C^t_*(\Omega)$ for some $t \geqslant 1$. Let*

$$|D|^s = (1 - \Delta)^{s/2} \tag{17}$$

for $x \in \mathbf{R}$, where $\Delta = \partial_1^2 + \cdots + \partial_n^2$. Then, for $s < t$, $\forall \varepsilon > 0$, there are operators $B_j \in O^{s-1+\varepsilon}_{t-s}$, $i = 1, 2, \ldots, n+1$, and $B_0 \in O^{s-2+\varepsilon}_{t-s}$, such that

$$[\,|D|^s, a(x, u(x))] = \sum_{j=1}^{n} a_{x_j}(x, u(x))\, B_j + a_u(x, u(x))\, B_{n+1} + B_0. \tag{18}$$

We will use the Commutator Lemma to prove Theorems 2 and 6 in the next section. The proof of the Commutator Lemma will be postponed to Section 4.

3. PROOF OF THE THEOREMS

In this section, we will prove Proposition 8 below. Theorems 2 and 6 will follow from the proposition and a result due to Franchi (stated as Theorem 9 in this section). The following proposition paves a way from $C^{0,1}$ to C^∞ for the solutions of (1).

PROPOSITION 8. *Suppose $u \in C^{0,1}(\Omega)$, u is a weak solution of the equation*

$$\sum_{i,j}^{n} \partial_i (a_{ij}(x, u)\, \partial_j u) = f(x) \tag{19}$$

with $a_{ij} \in C^\infty(\Omega \times \mathbf{R})$, $(a_{ij}) \geqslant 0$, $f \in C^\infty(\Omega)$. Then $u \in C^\infty(\Omega)$, if the linear operator

$$Lv = \sum_{i,j}^{n} \partial_i (\tilde{a}_{ij}(x)\, \partial_j v), \tag{20}$$

where $\tilde{a}_{ij}(x) = a(x, u(x))$, is subunit and α-subelliptic for some $\alpha > 0$, and if it is elliptic extendible in Ω.

The basic idea involved in proving the proposition is simple. It repeats, following three steps: (i) differentiate the equation; (ii) rewrite the resulting equation in the right form using the Commutator Lemma; and (iii) apply the subellipticity assumption to get higher regularity. Since the real proof is quite lengthy, we will give a formal and hueristic argument first.

Let $A(x, u) = (a_{ij}(x, u))$. Equation (16) can be rewritten as

$$\nabla \cdot A(x, u) \cdot \nabla u = f. \tag{21}$$

If we differentiate (21) and let $v = \partial_k u$, the v satisfies the equation

$$\nabla \cdot A(x, u) \cdot \nabla v = \partial_k f - \nabla (A_{x_k}(x, u) + A_u(x, u)\, \partial_k u) \cdot \nabla u. \tag{22}$$

Since $A(x, u(x))$ is subunit, and u is in $C^{0,1}$, we may write (22) as

$$\nabla \cdot A(x, u) \cdot \nabla v = \tilde{f} + \sum_{i=1}^{n} T_i \tilde{f}_i \tag{23}$$

with T_i subunit, $\tilde{f}, \tilde{f}_1, ..., \tilde{f}_N \in L^\infty$. By the subellipticity assumption, $v \in C^\alpha(\Omega)$ for some $\alpha > 0$.

We hope to differentiate (22) to get higher regularity. But, since v is only in C^α for some $0 < \alpha < 1$, we can not apply the usual differentiation. This is where fractional differentiation comes in. Suppose $u \in C_*{}^{1+\beta}$ for some $0 < \beta \leqslant 1$; we want to show that $u \in \Lambda^{1+\beta+\alpha/2}$. We already have $\beta \geqslant \alpha$. Let $w = |D|^{(\beta - \alpha/2)} v$. Apply $|D|^{(\beta-\alpha/2)}$ on Eq. (22); w then satisfies the equation

$$\begin{aligned} \nabla \cdot A(x, u) \cdot \nabla w = \partial_k \cdot |D|^{(\beta-\alpha/2)} f - \nabla \cdot A_{x_k}(x, u)(\nabla \cdot |D|^{(\beta-\alpha/2)} u) \\ - \nabla \cdot A_u(x, u)(|D|^{(\beta-\alpha/2)} \cdot (\partial_k u \nabla u)) \\ + \nabla \cdot [A(x, u), |D|^{(\beta-\alpha/2)}] \cdot \nabla v \\ + \nabla \cdot [A_{x_k}(x, u), |D|^{(\beta-\alpha/2)}] \cdot \nabla u \\ + \nabla \cdot [A_u(x, u), |D|^{(\beta-\alpha/2)}](\partial_k u \nabla u). \end{aligned} \tag{24}$$

By the Commutator Lemma,

$$[A(x,u), |D|^{(\beta-\alpha/2)}] = \sum_{j=1}^{n} A_{x_j}(x,u)\, B_j + A_u(x,u)\, B_{n+1} + B_0 \tag{25}$$

with $B_j \in O^{\beta-1+\varepsilon+\alpha/2}_{1+\alpha/2}$, $j=1, 2, ..., n+1$, $B_0 \in O^{\beta-2+\varepsilon+\alpha/2}_{1+\alpha/2}$, and

$$[A_{x_k}(x,u), |D|^{(\beta-\alpha/2)}], [A_u(x,u), |D|^{(\beta-\alpha/2)}] \in O^{\beta-1+\varepsilon+\alpha/2}_{1}. \tag{26}$$

If we note that $B_j \nabla v \in \Lambda^{\alpha/2-\varepsilon}$, $j=1, 2, ..., n+1$, $\nabla u, \partial_k u \in C^{\beta}_{*}$, $\beta \geqslant \alpha$, we have

$$\nabla \cdot A(x,u) \cdot \nabla w = f^* + \sum_{i=1}^{N} \tilde{T}_i f_i^* \tag{27}$$

with $f^*, f_i^* \in C^{\alpha/2+\varepsilon}_{*}$, $i=1, 2, ..., N$, and $\tilde{T}_i$ subunit. If we pick $0<\varepsilon<\alpha/2$, again, by the subellipticity assumption, $w \in C^{\alpha}$. That is, $u \in \Lambda^{1+\beta+\alpha/2}$. From this we conclude that $u \in C^{2+\alpha/2}_{*}$.

The whole process would go through if Du is a weak solution of (22). Since u is only assumed in $C^{0,1}$, this is not clear. One would like to try elliptic approximation, but a C^1 a priori estimate is needed for the approximation. This is what is involved in the following proof.

Proof of the Proposition. For any x_0 in Ω, by the assumption there is a neighborhood $U \subset\subset V \subset\subset \Omega$ of x_0, and there exists a C^{∞} matrix function $\tilde{A}(x)$ in Ω, such that $\tilde{A}$ equals 0 in U, and $A^* = A(x,u) + \tilde{A}^2(x)$ is positive definite near ∂V, and $L^* = \nabla(A^* \nabla)$ is subunit and $\tilde{\alpha}$-subelliptic. For any $\phi(x) \in C_0^{\infty}(U)$, then $\tilde{u} = \phi u$ is a weak solution of the equation

$$\nabla(A^*(x,u)\, \nabla \tilde{u}) = \phi f + (\nabla \phi)\, A(x,u)(\nabla u) + \nabla(A(x,u)\, u(\nabla \phi)). \tag{28}$$

For any $\varepsilon > 0$, let $A^{\varepsilon}(x,u) = A^*(x,u) + \varepsilon I$. Let v^{ε} be the solution of the Dirichlet problem of equation

$$\nabla(A^{\varepsilon}(x,u)\, \nabla v) = \phi f + (\nabla \phi)\, A(x,u)(\nabla u) + \nabla(A(x,u)\, u(\nabla \phi)), \tag{29}$$

with $v|_{\partial V} = 0$.

Now we indicate C, which may vary line by line, a constant independent of ε.

Step 1. $\|v^{\varepsilon}\|_{C^{1,\tilde{\alpha}}(\bar{V})} \leqslant C$. By elliptic theory v^{ε} is in $H^2_p(V)$, for any $0<p<\infty$. Furthermore,

$$|v^{\varepsilon}(x)| \leqslant C, \tag{30}$$

$\forall x$ in V, and

$$\|v^\varepsilon\|_{C^{1,\alpha}(K)} \leqslant C, \tag{31}$$

for any $K \subset\subset \bar{V}$ near ∂V and away from $\bar{U}$.

Claim: $\|v^\varepsilon\|_{C^1(\bar{V})} \leqslant C$. Suppose this is not true. There is a sequence ε_j, with $\|v^{\varepsilon_j}\|_{C^1(V)} = c_j > j$. Let $w_j = v^{\varepsilon_j}/c_j$, where w_j satisfies the equation

$$\nabla(A^{\varepsilon_j}(x, u)\, \nabla w) = \frac{1}{c_j}\,(\phi f + (\nabla\phi)\, A(x, u)(\nabla u) + \nabla(A(x, u)\, u(\nabla\phi))), \tag{32}$$

with $w|_{\partial V} = 0$.

If we differentiate (32), let $D = \partial_k$, $g_j = Dw_j$, then g_j is in $H^1_p(V)$ and is a weak solution of the equation

$$\begin{aligned}\nabla(A^{\varepsilon_j}(x, u)\, \nabla g_j) = \frac{1}{c_j}\, 2((\nabla\phi)\, A(x, u)(\nabla Du) + F\\ -\nabla((A^{\varepsilon_j}_{x_k}(x, u) + A^{\varepsilon_j}_u(x, u)\, \partial_k u)\, \nabla w_j),\end{aligned} \tag{33}$$

where $F = F(x, u, \nabla u, \phi, \nabla\phi, f, \nabla f)$. As u is in $C^{0,1}$, F is a bounded function. Since L is subunit, we have

$$|\nabla\phi \cdot A \cdot \xi|^2 \leqslant C^t\xi A\xi,$$

$$|A_{x_k} \cdot \xi|^2 \leqslant C^t\xi A\xi,$$

and

$$|\nabla u|\, |A_u \cdot \xi|^2 \leqslant C^t\xi A\xi, \tag{35}$$

for some constant $C > 0$, $\forall \xi \in \mathbf{R}^n$. Using the fact $u \in C^{0,1}$, we may write (33) as

$$\nabla(A^{\varepsilon_j}(x, u)\, \nabla g_j) = \tilde{f} + \sum_{i=1}^{n} T_i \tilde{f}_i$$

with T_i subunit, $\tilde{f}, \tilde{f}_1, ..., \tilde{f}_N \in L^\infty$, with their L^∞ norms bounded independent of ε_j. By assumption, (30), and (31),

$$\|g_j\|_{C^{\tilde{\alpha}}(\bar{V})} \leqslant C.$$

Therefore, there is a subsequence which we still write as w_j, which is convergent in $C^{1,\tilde{\alpha}/2}(\bar{V})$ to a function w_0. Then w_0 satisfies the equation

$$\nabla(A^*(x, u)\, \nabla w_0) = 0 \tag{36}$$

with $w_0|_{\partial V}=0$. But the C^1 norm of w_0 is 1. This contradicts the uniqueness of the weak solution of (36). Therefore, the claim is true.

Now, if we differentiate (29), $Dv^\varepsilon=\partial_k v^\varepsilon$ satisfies

$$\nabla(A^\varepsilon(x,u)\,\nabla Dv^\varepsilon)=2(\nabla\phi)\,A(x,u)(\nabla Du)+F$$
$$-\nabla((A^\varepsilon_{x_k}(x,u)+A^\varepsilon_u(x,u)\,\partial_k u)\,\nabla v^\varepsilon), \tag{37}$$

where F is as in (33), and it is a bounded function. Using the same argument in the proof of the claim, we conclude that v^ε is uniformly bounded in $C^{1,\tilde\alpha}$ norm. Passing ε to 0, we get $\tilde u$ is in $C^{1,\tilde\alpha}$. We conclude that u is in $C^{1,\tilde\alpha}(U)$.

Step 2. Suppose $u\in C_*^{1+\beta}$ for some $\tilde\alpha\leqslant\beta\leqslant 1$, $\|v^\varepsilon\|_{C_*^{1+\beta}(\bar V)}\leqslant C$, then $\|v^\varepsilon\|_{C_*^{1+\beta+\tilde\alpha/2}(\bar V)}\leqslant C$, and $u\in C_*^{1+\beta+\tilde\alpha/2}$.

We only need to show

$$\|v^\varepsilon\|_{C_*^{1+\beta+\tilde\alpha/2}(\bar V)}\leqslant C. \tag{38}$$

Let $w^\varepsilon=|D|^{(\beta-\tilde\alpha/2)}Dv^\varepsilon$. Applying $|D|^{(\beta-\tilde\alpha/2)}$ to Eq. (37), w^ε satisfies the equation

$$\begin{aligned}\nabla(A^\varepsilon(x,u)\,\nabla w^\varepsilon)=\;&|D|^{(\beta-\tilde\alpha/2)}F+2(\nabla\phi)\,A(x,u)(\nabla\,|D|^{(\beta-\tilde\alpha/2)}\,Du)\\&+2[\,|D|^{(\beta-\tilde\alpha/2)},(\nabla\phi)\,A(x,u)]\,\nabla Du\\&-\nabla A^\varepsilon_{x_k}(x,u)(\nabla|D|^{(\beta-\tilde\alpha/2)}v^\varepsilon)\\&-\nabla A^\varepsilon_u(x,u)(|D|^{(\beta-\tilde\alpha/2)}(\partial_k u\nabla v^\varepsilon))\\&+\nabla[A^\varepsilon(x,u),|D|^{(\beta-\tilde\alpha/2)}]\,\nabla Dv^\varepsilon\\&+\nabla[A^\varepsilon_{x_k}(x,u),|D|^{(\beta-\tilde\alpha/2)}]\,\nabla v^\varepsilon\\&+\nabla[A^\varepsilon_u(x,u),|D|^{(\beta-\tilde\alpha/2)}](\partial_k u\,\nabla v^\varepsilon).\end{aligned} \tag{39}$$

We note that F is in C^β since u and v^ε are in $C^{1,\beta}$. By the Commutator Lemma,

$$[A^\varepsilon_u(x,u),|D|^{(\beta-\tilde\alpha/2)}]=\sum_{j=1}^n A^\varepsilon_{x_j}(x,u)\,B_j+A^\varepsilon_u(x,u)\,B_{n+1}+B_0 \tag{40}$$

with $B_j\in O^{\beta-1+\tilde\varepsilon+\tilde\alpha/2}_{1+\tilde\alpha/2}$, $j=1,2,\ldots,n+1$, $B_0\in O^{\beta-2+\tilde\varepsilon+\tilde\alpha/2}_{1+\tilde\alpha/2}$, and

$$[A^\varepsilon_{x_k}(x,u),|D|^{(\beta-\tilde\alpha/2)}],\;[A^\varepsilon_u(x,u),|D|^{(\beta-\tilde\alpha/2)}]\in O^{\beta-1+\tilde\varepsilon+\tilde\alpha/2}_1. \tag{41}$$

If we note that $B_j \nabla v \in \Lambda^{\tilde{\alpha}/2-\tilde{\varepsilon}}$, $j=1, 2, ..., n+1$, ∇u, $\partial_k u \in C^{\beta}_{*}$, $\beta \geqslant \tilde{\alpha}$, we have

$$\nabla(A^{\varepsilon}(x, u) \nabla w^{\varepsilon}) = f^{*} + \sum_{i=1}^{N} \tilde{T}_i f_i^{*} \tag{42}$$

with f^{*}, $f_i^{*} \in C_{*}^{\tilde{\alpha}/2-\tilde{\varepsilon}}$, $i=1, 2, ..., N$, and $\tilde{T}_i$ a subunit. If we pick $0<\tilde{\varepsilon}<\tilde{\alpha}/2$, by assumption and the ellipticity of Eq. (37) near the boundary,

$$\|w^{\varepsilon}\|_{C^{\tilde{\alpha}}(\bar{V})} \leqslant C.$$

That is, (38) holds for any $\tilde{\alpha} \leqslant \beta \leqslant 1$.

Now, at each stage, when we reach $u \in C^m$ for $m \geqslant 2, m \in \mathbf{Z}$, we apply D^{m-1} on Eq. (37), using the assumption for the linear equation, to get $u \in C_{*}^{m+\tilde{\alpha}}$. Once $u \in C_{*}^{m+\beta}$ for $\beta \geqslant \tilde{\alpha}$, by using $|D|^{\beta-\tilde{\alpha}/2}$ on the resulting equation, repeating the previous argument (using the Commutator Lemma), we can conclude $u \in C_{*}^{m+1+\tilde{\alpha}/2}(U)$. Since x_0 is arbitrary, the proof of the proposition is complete.

Now, Theorem 6 can be easily deduced from Proposition 8 and the next theorem due to Franchi [F].

THEOREM 9. *Suppose $a_{ij}(x)$ are bounded and measurable, and $(a_{ij}(x)) \geqslant 0$ satisfies subelliptic condition* (SE1) (*or* (SE3) *if $n=2$*). *Let $T_1, ..., T_N$ be subunit vector fields in Ω with $C^{0,1}$ coefficients. If u is a weak solution of the equation*

$$\sum_{i,j} \partial_i (a_{ij}(x) \partial_j u) = f_0 + T_1 f_1 + \cdots + T_N f_N \tag{43}$$

with $f_0, f_1, ..., f_N \in L^{\infty}(\Omega)$. Then, there is $\alpha>0$ depending only on Ω and Condition (SE1), *such that $u \in C^{\alpha}(\Omega)$, and for each $K \subset\subset \Omega$,*

$$\|u\|_{C^{\alpha}(K)} \leqslant C$$

where C depends only on K, $\|f_j\|_{L^{\infty}}$, $\|T_j\|_{C^1}$, and (SE1) (*or* (SE2) *when* $n=2$).

Remark 10. In [F], the right-hand side of Eq. (43) is of the form $\nabla_{\lambda} f$. One can easily adapt the proof in [F] for Theorem 9. Elliptic extendible condition Theorem 6 has been removed recently in [G2].

Proof of Theorem 2. Since $k(x, u) \geqslant 0$, $k(x, u) \in C^2(\Omega \times \mathbb{R})$ and $u \in C^{0,1}$, L is subunit. We only need to check the elliptic extendibility of the operator $Lv = \partial_1^2 v + \partial_2(\lambda^2(x)) \partial_2 v)$, if $\lambda(x)$ satisfies subelliptic condition (SE2). For any $x_0=(a, b)$ in Ω, by (SE2) and (10), there is $T>0$, such that

$\lambda(a-T, b)$ and $\lambda(a+T, b)$ are positive. We may assume $b=0$. Since $\lambda(x)$ is continuous, there is $\delta>0$, such that $\lambda(a-T, y)$ and $\lambda(a+T, y)$ are positive for $|y|<\delta$. We pick a smooth nonnegative function $h(y)$ with $h(y)=0$ when $|y|<\delta/4$ and $h(y)=\delta^2$ when $|y|>\delta/2$. Let $\lambda_*=(\lambda^2+h)^{1/2}$. It's easy to check that λ_* satisfies (SE2), and the operator $L^*v=\partial_1^2 v+\partial_2({\lambda_*}^2\partial_2 v)$ is elliptic near the boundary of a neighborhood V of x_0. Now Theorem 2 follows from Proposition 8 and Theorem 10.

4. DECOMPOSITION OF PSEUDODIFFERENTIAL OPERATORS AND PROOF OF THE COMMUTATOR LEMMA

In this section, we shall use the symbol smoothing method for pseudo-differential operators to prove our Commutator Lemma in Section 2.

First, we recall some facts about pseudodifferential operators with limited smoothness.

Definition 11. Let $\sigma(x, \xi)$ be a symbol, suppose $\nabla_\xi^\beta \sigma \in C_*^s$ for each $\beta\in \mathbf{Z}_+^n$. We say $\sigma\in C_*^s S_{1,\delta}^m$ do for some $0\leqslant\delta\leqslant 1$, if

$$\|\nabla_x^\alpha \nabla_\xi^\beta \sigma(x,\xi)\|_{C^\gamma(K)}\leqslant C_{\alpha,\beta,\gamma,K}(1+|\xi|)^{m-|\beta|+\delta(|\alpha|+\gamma)} \tag{44}$$

$\forall x\in K\subset\subset\Omega$, $|\xi|\geqslant 1$, $|\alpha|+\gamma\leqslant s$. We say a pseudodifferential operator $P\in C_*^s \mathscr{S}_{1,\delta}^m$, if P has a symbol $p(x,\xi)\in C_*^s S_{1,\delta}^m$.

The following two propositions will be useful. The proofs of the propositions can be found, for example, in [T].

Proposition 12. *If $p\in C_*^s S_{1,\delta}^m$ for some $0\leqslant\delta<1$, then $\forall\delta<\gamma<1$, there are symbols $p^\#(x,\xi)\in C^\infty S_{1,\gamma}^m$, $p^b(x,\xi)\in C_*^s S_{1,\gamma}^{m-(\gamma-\delta)s}$ such that*

$$p(x,\xi)=p^\#(x,\xi)+p^b(x,\xi). \tag{45}$$

Moreover, if $p\in C_*^s S_{1,0}^m$, $p^\#$ in the decomposition (45) *has the following property*:

$$\nabla_x^\beta p^\#(x,\xi)\in C^\infty S_{1,\gamma}^m \qquad \textit{if} \quad |\beta|\leqslant s; \tag{46}$$

and

$$\nabla_x^\beta p^\#(x,\xi)\in C^\infty S_{1,\gamma}^{m+\gamma(|\beta|-s)} \qquad \textit{if} \quad |\beta|>s. \tag{47}$$

We have the following mapping property for pseudodifferential operators in $C_*^s \mathscr{S}_{1,\delta}^m$.

PROPOSITION 13. *If* $P \in C^s_* \mathscr{S}^m_{1,\delta}$ *for some* $s > 0$, *then*

$$P: \Lambda^{r+m} \to \Lambda^r \qquad \textit{for} \quad 0 \leqslant r < s.$$

PROPOSITION 14. *If* $P \in C^s_* \mathscr{S}^m_{1,1}$, *then*

$$P: \Lambda^{r+m+\varepsilon} \to \Lambda^r \qquad \textit{for} \quad 0 \leqslant r < s.$$

LEMMA 15. *If* $P \in C^s_* \mathscr{S}^m_{1,\delta}$ *for some* $s > 0, 0 \leqslant \delta < 1$, *then* $P \in O^m_s$.

Proof. By Proposition 12 we may write

$$P = P^\# + P^b$$

with $P^\# \in C^\infty \mathscr{S}^m_{1,\gamma}$, $P^b \in C^s_* \mathscr{S}^{m-s(\gamma-\delta)}_{1,\gamma}$. If we pick $\gamma = (1+\delta)/2$, then $P^b \in C^s_* \mathscr{S}^{m-s(1-\delta)/2}$, with $(s/2)(1-\delta) > 0$. By Propositions 13 and 14, $P^\# \in O^m_\infty$, $P^b \in O^m_s$. That is, $P \in O^m_s$.

Now, we are ready to prove the Commutator Lemma.

Proof of the Commutator Lemma. Suppose $u \in C^t_*$, we also have $u \in C^t_* S^0_{1,0}$. That is, we may view u as a symbol in $C^t_* S^0_{1,0}$. By Proposition 13, we may decompose u as

$$u = u^\# + u^b \tag{48}$$

with $u^\# \in C^\infty S^0_{1,\gamma}$, $u^b \in C^t_* S^{0-t\gamma}_{1,\gamma}$, $1 > \gamma > 0$, to be chosen later. Since u is real, we may assume $u^\#(x,\xi)$, $u^b(x,\xi)$ are also real. We write

$$a(x, u(x)) = a(x, u^\#(x,\xi)) + \{a(x, u(x)) - a(x, u^\#(x,\xi))\}. \tag{49}$$

By the Taylor expansion, using the fact $u^b \in C^t_* S^{-t\gamma}_{1,\gamma}$, we have

$$\begin{aligned} a(x, u(x)) - a(x, u^\#(x,\xi)) &= a_u(x, u(x))(u^\#(x,\xi) - u(x)) \\ &\quad + O((u^\#(x,\xi) - u(x))^2) \\ &= -a_u(x, u(x))\, u^b(x,\xi) + O((u^\#(x,\xi) - u(x))^2) \\ &\equiv -a_u(x, u(x))\, u^b(x,\xi) (C^t_* S^{-2t\gamma}_{1,\gamma}). \end{aligned} \tag{50}$$

For the simplicity, in the rest of the proof, we will not distinguish symbols and pseudodifferential operators associated with them. For two symbols, σ_1, σ_2, we denote $\sigma_1 \cdot \sigma_2$ for the operator with symbol $\sigma_1 \cdot \sigma_2$, and we denote $\sigma_1 \circ \sigma_2$ as the composition operator of σ_1 and σ_2.

Now,

$$[|D|^s, a(x,u)] \equiv [|D|^s, a(x,u^\#)] - [|D|^s, a_u(x,u)\, u^b] \bmod (O^{s-2t\gamma}_{t-s}). \tag{51}$$

For $[|D|^s, a(x,u^\#)]$, we use standard pseudodifferential calculus,

$$[|D|^s, a(x,u^\#)] = \sum_{|\beta| \geqslant 1} C_\beta \nabla^\beta_x (a(x,u^\#)) \cdot \nabla^\beta_\xi (1+|\xi|^2)^{s/2}. \tag{52}$$

We denote $a_{\beta,j}(x,z) = (\partial\beta/\partial x^\beta)(\partial/\partial z^j)\, a(x,z)$, where (x,z) are independent variables in $\Omega \times \mathbf{R}$. Using the chain rule, Eq. (52) now can be written as

$$\begin{aligned}
[|D|^s, a(x,u^\#)] = {} & \sum_{|\beta|=1} C_\beta a_{\beta,0}(x,u^\#) \cdot \nabla^\beta_\xi (1+|\xi|^2)^{s/2} \\
& + \sum_{|\beta|\geqslant 1} C_\beta a_{0,1}(x,u^\#)\, \nabla^\beta_x u^\# \cdot \nabla^\beta_\xi (1+|\xi|^2)^{s/2} \\
& + \sum_{|\beta|\geqslant 2} C_\beta a_{\beta,0}(x,u^\#) \cdot \nabla^\beta_\xi (1+|\xi|^2)^{s/2} \\
& + \sum_{|\beta|\geqslant 2} \sum_{\substack{|\beta'|+|j|\leqslant|\beta| \\ j\geqslant 1,\, |\beta'|\geqslant 1}} C_\beta a_{\beta',j}(x,u^\#) \\
& \times \prod_{\substack{i=1 \\ \alpha_1+\cdots+\alpha_j=(\beta-\beta')}}^{j} \nabla^{\alpha_i}_x u^\# \cdot \nabla^\beta_\xi (1+|\xi|^2)^{s/2}.
\end{aligned} \tag{53}$$

By Proposition 12, $u^\#$ satisfies

$$\begin{aligned}
&\partial^\alpha_x u^\# \in C^\infty S^0_{1,\gamma} && \text{if} \quad |\alpha| \leqslant t \\
&\partial^\alpha_x u^\# \in C^\infty S^{\gamma(|\alpha|-t)}_{1,\gamma} && \text{if} \quad |\alpha| > t.
\end{aligned} \tag{54}$$

Since $t \geqslant 1$, $\nabla^\alpha_x u^\# \in C^\infty S^{\gamma(|\alpha|-1)}_{1,\gamma}$, we have

$$\prod_{\substack{i=1 \\ \alpha_1+\cdots+\alpha_j=\beta-\beta'}}^{j} \nabla^{\alpha_i}_x u^\# \in C^\infty S^{\gamma\sum_{i=1}^j(|\alpha_i|-1)}_{1,\gamma} = C^\infty S^{\gamma|\beta|-\gamma|\beta'|-\gamma j}_{1,\gamma}.$$

Therefore, the third term and the fourth term in (53) are in $C^\infty S^{s-2\gamma}_{1,\gamma}$, and we have

$$\begin{aligned}
[|D|^s, a(x,u^\#)] \equiv {} & \sum_{|\beta|=1} C_\beta a_{\beta,0}(x,u^\#) \cdot \nabla^\beta_\xi (1+|\xi|^2)^{s/2} \\
& + \sum_{|\beta|\geqslant 1} C_\beta a_{0,1}(x,u^\#)\, \nabla^\beta_x u^\# \cdot \nabla^\beta_\xi (1+|\xi|^2)^{s/2} \\
& \bmod (O^{s-2\gamma}_\infty),
\end{aligned} \tag{55}$$

while

$$a_{\beta,0}(x,u^{\#})=a_{\beta,0}(x,u)+O(u^b)\equiv a_{\beta,0}(x,u) \qquad \mathrm{mod}(O_t^{-t\gamma}), \tag{56}$$

and

$$a_{0,1}(x,u^{\#})=a_{0,1}(x,u)+O(u^b)\equiv a_{0,1}(x,u) \qquad \mathrm{mod}(O_t^{-t\gamma}). \tag{57}$$

Putting (56) and (57) into (55), we get

$$[|D|^s, a(x,u^{\#})]\equiv \sum_{j=1}^{n} a_{x_j}(x,u)\,\tilde{B}_j+a_u(x,u)\,\tilde{B}_{n+1} \qquad \mathrm{mod}(O_t^{s-2\gamma}) \tag{58}$$

with $\tilde{B}_j\in O_t^{s-1}$, $j=1,2,...,n+1$.

We now deal with $[|D|^s, a_u(x,u)\,u^b]$.

$$\begin{aligned}[|D|^s, a_u(x,u)\,u^b] &= |D|^s\circ a_u(x,u)\,u^b-a_u(x,u)\,u^b\circ|D|^s\\ &=|D|^s\circ a_u(x,u)\circ u^b-a_u(x,u)\circ(u^b\circ|D|^s).\end{aligned} \tag{59}$$

We have

$$\begin{aligned}|D|^s\circ a_u(x,u)\circ u^b &= |D|^s\circ a_u(x,u^{\#})\circ u^b+|D|^s\circ[a_u(x,u)-a_u(x,u^{\#})]\circ u^b\\ &=a_u(x,u^{\#})\circ|D|^s\circ u^b+[|D|^s, a_u(x,u^{\#})]\circ u^b\\ &\quad+|D|^s\circ[a_u(x,u)-a_u(x,u^{\#})]\circ u^b.\end{aligned} \tag{60}$$

By standard pseudodifferential calculus, (54), and (57)

$$[|D|^s, a_u(x,u^{\#})]\in O_\infty^{s-1}, \qquad a_u(x,u)-a_u(x,u^{\#})\in O_t^{-t\gamma}. \tag{61}$$

We get from (60), (61), and (57),

$$\begin{aligned}|D|^2\circ a_u(x,u)\circ u^b &\equiv a_u(x,u^{\#})\circ|D|^s\circ u^b \qquad \mathrm{mod}(O_{t-s}^{s-2\gamma})\\ &\equiv a_u(x,u)|D|^s\circ u^b \qquad \mathrm{mod}(O_{t-s}^{s-2\gamma}).\end{aligned} \tag{62}$$

From (59), (62), (58), and (51)

$$\begin{aligned}[|D|^s, a(x,u) &\equiv \sum_{j=1}^{n} a_{x_j}(x,u)\,\tilde{B}_j+a_u(x,u)\,\tilde{B}_{n+1}\\ &\quad\times a_u(x,u)(|D|^2\circ u^b-u^b\circ|D|^s) \qquad \mathrm{mod}(O_{t-s}^{s-2\gamma})\\ &\equiv \sum_{j=1}^{n} a_{x_j}(x,u)\,B_j+a_u(x,u)\,B_{n+1} \qquad \mathrm{mod}(O_{t-s}^{s-2\gamma})\end{aligned} \tag{63}$$

with $B_j \in O_t^{s-1}$, $j=1, ..., n$, $B_{n+1} \in O_{t-s}^{s-\gamma}$. If we let $\gamma = 1-\varepsilon/2$, the lemma follows.

Remark 16. (i) Though we assume $a \in C^\infty(\Omega \times \mathbf{R})$ in the Commutator Lemma, this is not necessary. In fact, if we use the composition formula in [GS], we can prove that if $a \in C^4(\Omega \times \mathbf{R})$, $u \in C^t_*$, for some $1 \leqslant t < 2$, if $0 < s < 1$ then $\forall \varepsilon > 0$, and there are $B_j \in O_{t-s}^{t-1+\varepsilon}$, $j=1, ..., n+1$, $B_0 \in O_{t-s}^{s-2+\varepsilon}$, such that

$$[|D|^s, a(x, u(s))] = \sum_{j=1}^{n} a_{x_j}(x, u)\, B_j + a_u(x, u)\, B_{n+1} + B_0. \tag{64}$$

(ii) We may also state corresponding results of Theorems 2 and 6, with $a_{ij} \in C^4_*(\Omega \times \mathbf{R})$ and $f \in C^3_*(\Omega)(k \in C^4_*(\Omega \times \mathbf{R}))$ in place of $a_{ij} \in C^\infty(\Omega \times \mathbf{R})$, $f \in C^\infty(\Omega)(k \in C^\infty(\Omega \times \mathbf{R}))$; with the rest of the assumptions there, we may conclude $u \in C^{3+\alpha}_*(\Omega)$. We note that if we merely assume $a_{ij} \in C^2_*(\Omega)$, $f \in C^1_*(\Omega)(k \in C^2(\Omega))$, we already have $u \in C^{1+\alpha}_*(\Omega)$. And if $a_{ij} \in C^{\ell+1}_*(\Omega)$, $f \in C^{\ell}{}_*(\Omega)$ for some $\ell \in \mathbf{R}$, $\ell \geqslant 3$, then we have $u \in C^{\ell+\alpha}_*(\Omega)$.

5. REGULARITY OF SOLUTIONS OF MONGE–AMPÈRE EQUATIONS

We now apply our regularity results for the degenerate quasilinear equations to the degenerate Monge–Ampère equation

$$\det(u_{ij}) = k, \tag{65}$$

where $k \geqslant 0$. When $k > 0$, the regularity of solution u of (65) is well-understood (e.g., [CNS]). In the degenerate case $k \geqslant 0$, one may obtain $C^{1,1}$ regularity for the solution of (65) under some reasonable assumptions (e.g., [G1]). In general, for the case $k \geqslant 0$, $C^{1,1}$ regularity is the best we can expect. The following example is essentially due to Sibony.

Example A. Let $u(x, y) = \max\{(\max\{(x^2 - \frac{1}{2}), 0\})^2, (\max\{(y^2 - \frac{1}{2}), 0\})^2\}$, $\phi(\theta) = (\cos^2\theta - \sin^2\theta)^2$. u satisfies

$$u_{xx} u_{yy} - u_{xy}^2 = 0 \qquad \text{in} \quad x^2 + y^2 < 1.$$

And

$$u|_{x^2+y^2=1} = \phi(\theta).$$

In Example a, $k\equiv 0$. The next example provides a $k\geqslant 0$, vanishing at only one point of order 2.

EXAMPLE B. Let $u(x)=|x|^{2+1/n}-1$, $x\in\mathbf{R}^n$. u satisfies

$$\begin{cases}\det(u_{ij})=c_n|x|^2 & \text{in } \{|x|<1\}\\ u|_{|x|=1}=0\end{cases}$$

for some $c_n>0$.

In Example b, $x=0$ is a planar point of the hypersurface $S=(x, u(x))$. This example indicates that we can only expect higher regularity of u away from the planar points of $S=(x, u(x))$. Our Theorem 5 more or less indicates this is in fact true.

To prove Theorem 3, we use a partial Legendre transformation to translate the regularity problem of Eq. (12) to a degenerate quasilinear equation of the form (11) (e.g., [S]).

At the origin, we may assume $\nabla u(0)=0$. We let

$$\begin{cases}s=x\\ t=u_y.\end{cases} \tag{66}$$

The change of variables $T(x, y)=(s, t)$ is $C^{0,1}$ near the origin by the assumption of $u\in C^{1,1}$. We have

$$J_T=\begin{bmatrix} s_x & s_y\\ t_x & t_y\end{bmatrix}=\begin{bmatrix} 1 & 0\\ u_{xy} & u_{yy}\end{bmatrix} \tag{67}$$

$$J_T^{-1}=\begin{bmatrix} x_s & x_t\\ y_s & y_t\end{bmatrix}=\begin{bmatrix} 1 & 0\\ -\dfrac{u_{xy}}{u_{yy}} & \dfrac{1}{u_{yy}}\end{bmatrix}. \tag{68}$$

By the assumption $u_{yy}\geqslant C_0>0$ near the origin, T and T^{-1} are $C^{0,1}$ diffeomorphisms near the origin.

LEMMA 17. $\exists R>0, \forall (x', y'), (x'', y'')\in B_R=\{|x|^2+|y|^2<R\}$; *there are constants depending only on* $\|u\|_{C^2(B_R)}$, *and* $\min_{B_R} u_{yy}$, *such that, for* $s'=s(x', y')$, $t'=t(x', y')$, $s''=s(x'', y'')$, $t''=t(x'', y'')$,

$$(s'-s'')^2+(t'-t'')^2\leqslant \gamma_1^2((x'-x'')^2+(y'-y'')^2) \tag{69}$$

$$(x'-x'')^2+(y'-y'')^2\leqslant \gamma_2^2((s'-s'')^2+(t'-t'')^2) \tag{70}$$

and $T(B_{r/r_1})\subset B_r$, $T(B_r)\supset B_{r/r_2}$ *for* $r\leqslant R$.

Proof.

$$t' - t'' = u_y(x', y') - u_y(x'', y'') = \int_0^1 \frac{d}{d\tau} u_y((x'', y'') + \tau(x' - x'', y' - y''))\, d\tau$$

$$= \int_0^1 \{u_{xy}(\cdot)(x' - x'') + u_{yy}(\cdot)(y' - y'')\}\, d\tau.$$

And $|t' - t''| \geqslant -\max_{B_R} |u_{xy}(\cdot)|\, |x' - x''| + \min_{B_R} u_{yy} \cdot |y' - y''|$. We get

$$\begin{aligned} |y' - y''| &\leqslant \frac{1}{\min_{B_R} u_{yy}} |t' - t''| + \frac{\max_{B_R} |u_{xy}|}{\min_{B_R} u_{yy}} |x' - x''| \\ &\leqslant \frac{1}{\min_{B_R} u_{yy}} |t' - t''| + \frac{\max_{B_R} |u_{xy}|}{\min_{B_R} u_{yy}} |s' - s''|. \end{aligned}$$

The important feature of the partial Legendre transformation (66) can be seen from next two lemmas. With the help of the transformation (66), the fully nonlinear Eq. (65) will be transformed to a simple quasilinear equation.

LEMMA 18. *$x(s, t)$, $y(s, t) \in C^{0,1}(TB_R)$ is a weak solution of the equation*

$$z_{ss} + \partial_t(k(x(s, t), y(s, t))\, \partial_t z) = 0. \tag{71}$$

Proof. For $z = x(s, t) = s$, this is trivial. Let $\eta \in C_0^\infty(T(B_R))$,

$$\begin{aligned} &\int_{T(B_R)} (y_s \eta_s + k y_t \eta_t)\, ds\, dt \\ &= \int_{B_R} \left\{ \frac{-u_{xy}}{u_{yy}} \left(\eta_x + \left(\frac{-u_{xy}}{u_{yy}} \right) \eta_y \right) + k \left(\frac{1}{u_{yy}} \right)^2 \eta_y \right\} u_{yy}\, dx\, dy \\ &= \int_B (-u_{xy} \eta_x + u_{xx} \eta_y)\, dx\, dy = 0 \end{aligned}$$

by approximation.

LEMMA 19. *$u \in C_*^{2+\beta}(B_r)$ for $0 < r < R$, $\beta \geqslant 0$ if and only if $y(s, t) \in C_*^{1+\beta}(TB_r)$.*

Proof. If $u \in C_*^{2+\beta}(B_r)$, from (66), $T \in C_*^{1+\beta}$, so $y(s, t) \in C_*^{1+\beta}(TB_r)$. On the other hand, if $y(s, t) \in C_*^{1+\beta}$, by (68), $J_t^{-1} \in C_*^\beta$. We have

$$u_{yy}(x(s, t), y(s, t)), u_{xy}(x(s, t), y(s, t)) \in C_*^\beta, \tag{72}$$

therefore $u_{yy}, u_{xy} \in C_*^m(B_R)$, with $m = \min(\beta, 1)$. This gives $T \in C_*^{1+m}$. By (72) again, $u_{yy}, u_{xy} \in C_*^{\tilde{m}}(B_R)$ with $\tilde{m} = \min(\beta, 2m)$. Repeating the argument, we get $u_{yy}, u_{xy} \in C_*^\beta$. By Eq. (12), and the fact $u_{yy} \geqslant C > 0$, we have $u_{xx} \in C_*^\beta$. That is, $u \in C_*^{2+\beta}$.

Using Theorem 2 and the above lemmas, one can easily prove the following proposition.

Proposition 20. *If u satisfies the degenerate Monge–Ampère equation,*

$$u_{xx}u_{yy} - u_{xy}^2 = k(x, y) \tag{73}$$

near the origin. Suppose $k \geqslant 0$, $k \in C^\infty(\mathbf{R} \times \mathbf{R})$, and $u \in C^{1,1}$. If $u_{yy} \geqslant C_0 > 0$ near the origin, and $k^{1/2}(s, y(s, t))$ satisfies (SE2) *after the transformation T in* (66), *then $u \in C^\infty$ near the origin.*

Now, we give a proof of Theorem 3 in Section 2.

Proof of Theorem 3. By Proposition 20, we only need to verify the condition (10) for $k^{1/2}(s, y(s, t))$. It suffices to show that for the function $g(s, t) = |s|^l + B|y(s, t)|^m$ (here (x, y) and (s, t) are related by the partial Legendre transformation T in (66)), the following is true:

$$\int_0^a g(s+\tau, t)\, d\tau \geqslant Cag(z, t), \tag{74}$$

for any $s < z < s + a$, $0 < a < r \leqslant 1$.

For s and t fixed, let Γ be the line segement $\{(z, t) \mid s \leqslant z \leqslant s + a\}$. If $\max_\Gamma |s|^l \geqslant \max_\Gamma |y(s, t)|^m$, (74) is trivial. So, we assume

$$\max_\Gamma |s|^l < \max_\Gamma |y(s, t)|^m. \tag{75}$$

Let γ be the image of Γ under T^{-1}; we then have

$$\int_0^a g(s+\tau, t)\, d\tau = \int_\gamma (|x|^l + B\,|y|^m). \tag{76}$$

Let $M = \max_\gamma y$, $L = \min_\gamma y$. If $M - L \geqslant (1/4H)a$, where $H > 1$ is a constant such that $(a+b)^m \leqslant H(|a|^m + |b|^m)$ for all real numbers a and b. Since γ is $C^{0,1}$ by Lemma 17, we have

$$\begin{aligned}
\int_\gamma (|x|^l + B|y|^m) &= \int_0^a (|s+x|^l + B\,|y(x)|^m)\,dx \\
&\geqslant \int_0^a |s+x|^l\,dx + CB\int_0^a |y|^m \left|\frac{dy}{dx}\right| dx \\
&\geqslant \int_0^a |s+x|^l\,dx + CB\int_L^M |y^m|\,dy \\
&\geqslant Ca \max_\gamma\, g(x,\, y) \\
&= Ca \max_\Gamma\, g(z,\, t).
\end{aligned} \tag{77}$$

If $M - L \leqslant (1/4H)a$, without loss of generality, we may assume that $M = \max_\gamma |\,y|$. By (75), we have

$$\frac{a}{2} \leqslant \frac{(|s| + |s+a|}{2} \leqslant M^{m/l}. \tag{78}$$

Since $l \leqslant m$ and $0 < a < 1$, on γ,

$$\begin{aligned}
|\,y(x)|^m &\geqslant \frac{M^m}{H} - (M - y(x))^m \\
&\geqslant \frac{M^m}{H} - (M-L)^m \\
&\geqslant \frac{M^m}{H} - \left(\frac{a}{4H}\right)^m \\
&\geqslant \frac{M^m}{2H} + \frac{\left(\frac{a}{2}\right)^l}{2H} - \left(\frac{a}{4H}\right)^m \geqslant \frac{M^m}{2H}.
\end{aligned} \tag{79}$$

Therefore,

$$\int_\gamma (|x|^l + B\,|\,y|^m) \geqslant Ca(\max_\gamma |x|^l + \max_\gamma B\,|\,y|^m) \geqslant Ca \max_\Gamma\, g(z,\, t). \tag{80}$$

The proof is now complete.

Remark 21. (i) By Remark 16, we can also obtain the regularity $u \in C_*^{2+\alpha}$ if $k \in C_*^2$, and $u \in C_*^{\ell+1+\alpha}$ if $k \in C^{\ell}{}_*$, $\ell \geqslant 4$, of course, under the assumption that $k^{1/2}(s,\, y(s,\, t))$ satisfies (SE2).

(ii) By modifying the argument in the proofs of Theorems 2 and 3, we may obtain the corresponding regularity results of Theorem 3 and Proposition 20 for the solution u of the equation

$$u_{xx}u_{yy}-u_{xy}^2=k(x,y)\,g(x,u,\nabla u)$$

with $g\in C^\infty$, $g\geqslant C>0$.

(iii) Finally, if $(x(\xi,\eta), y(\xi,\eta), z(\xi,\eta))$ is a $C^{1,1}$ embedded convex surface in $\mathbf{R}^3$; if the Gauss curvature $k(\xi,\eta)$ is smooth near $(\xi,\eta)=(0,0)$ if $(x(\xi,\eta), y(\xi,\eta), z(\xi,\eta))$ is a graph over (x,y) near the origin; $z_{yy}\geqslant C>0$; and $k(\xi(x,y),\eta(x,y))$ satisfies condition (13), we can show that the surface is smooth near $(\xi,\eta)=(0,0)$ by modifying the proofs of Theorems 2 and 3 and updating the regularity at each stage. Existence of $C^{1,1}$ isometric embedding of $(S^2,g)\to\mathbf{R}^3$ with $K_g\geqslant 0$ has been established in [GL1] and [HZ].

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