REMARKS ON THE HOMOGENEOUS COMPLEX MONGE-AMPÈRE EQUATION

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Dedicated to Professor Linda Rothchild on the occasion of her 60th birthday

This short note concerns the homogeneous complex Monge-Ampère equation arising from the Chern-Levine-Nirenberg holomorphic invariant norms in [9]. In [9], Chern-Levine-Nirenberg found close relationship of the intrinsic norms with the variational properties and regularity of the homogeneous complex Monge-Ampère equation. It is known that solutions to the homogeneous complex Monge-Ampère equation fail to be C^2 in general, since the equation is degenerate and the best regularity is $C^{1,1}$ by examples of Bedford-Forneass [2]. In a subsequential study undertaken by Bedford-Taylor [4], to overcome regularity problem for the homogeneous complex Monge-Ampère equation, they developed the theory of weak solutions and they extended the definition of intrinsic norm to a larger class of plurisubharmonic functions. Furthermore, they related it to an extremal function determined by the weak solution of the homogeneous complex Monge-Ampère equation. Among many important properties of the extremal function, they obtained the Lipschitz regularity for the solution of the homogeneous complex Monge-Ampère equation, and proved an estimate for the intrinsic norm in terms of the extremal function and the defining function of the domain. In [12], the optimal $C^{1,1}$ regularity was established for the extremal function. As a consequence, the variational characterization of the intrinsic norm of Bedford-Taylor is validated, along with the explicit formula for the extended norm speculated in [4].

This paper consists of two remarks related to the results in [12]. First is that the approximation of the extremal function constructed in [12] can be used to show that the extended norm of Bedford-Taylor is in fact exact the same as the Chern-Levine-Nirenberg intrinsic norm, thus it provides a proof of the original Chern-Levine-Nirenberg conjecture. The second is that the results in [12] can be generalized to any complex manifold, with the help of the existence of the plurisubharmonic function obtained in [12]. That function was used in a crucial way in [12] to get C^2 boundary estimate following an argument of Bo Guan [10]. We will use this function to establish global C^1 estimate for the homogeneous complex Monge-Ampère equation on general manifolds. We also refer Chen's work [8] on homogeneous complex Monge-Ampère equation arising from a different geometric context.

Let's recall the definitions of the Chern-Levine-Nirenberg norm [9] and the extended norm defined by Bedford-Taylor [4]. Let M be a closed complex manifold with smooth

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boundary $\partial M = \Gamma_1 \cup \Gamma_0$, set

$$\mathcal{F} = \{u \in C^2(M) \mid u \text{ plurisubharmonic and } 0 < u < 1 \text{ on } M\},$$

$$\mathcal{F}'_k = \{u \in \mathcal{F} \mid (dd^c u)^k = 0, \dim\gamma = 2k-1, \text{ or } du \wedge (dd^c u)^k = 0, \dim\gamma = 2k\}.$$

$\forall \gamma \in H_*(M, \mathbb{R})$ be a homology class in M,

$$N_{2k-1}\{\gamma\} = \sup_{u\in\mathcal{F}} \inf_{T\in\gamma} |T(d^c u \wedge (dd^c u)^{k-1})|, \quad \text{if } \dim\gamma = 2k-1; \tag{1}$$

$$N_{2k}\{\gamma\} = \sup_{u\in\mathcal{F}} \inf_{T\in\gamma} |T(du \wedge d^c u \wedge (dd^c u)^{k-1})|, \quad \text{if } \dim\gamma = 2k, \tag{2}$$

where T runs over all currents which represent γ.

It is pointed out in [9] that the intrinsic norm N_j may also be obtained as the supremum over the corresponding subclass of C^2 solutions of homogeneous complex Hessian equations in $\mathcal{F}'_k$. The most interesting case is $k = 2n-1$, elements of $\mathcal{F}'_{2n-1}$ are plurisubharmonic functions satisfying the homogeneous complex Monge-Ampère equation

$$(dd^c u)^n = 0. \tag{3}$$

In this case, associated to N_{2n-1}, there is an extremal function satisfying the Dirichlet boundary condition for the homogeneous complex Monge-Ampère equation:

$$\begin{cases} (dd^c u)^n = 0 & \text{in } M^0 \\ u|_{\Gamma_1=1} \\ u|_{\Gamma_0=0}, \end{cases} \tag{4}$$

where $d^c = i(\bar\partial - \partial)$, M^0 is the interior of M, and Γ_1 and Γ_0 are the corresponding outer and inner boundaries of M respectively.

Due to the lack of C^2 regularity for solutions of equation (4), an extended norm $\tilde N$ was introduced by Bedford-Taylor [4]. Set

$$\tilde{\mathcal{F}} = \{u \in C(M) | u \text{ plurisubharmonic }, 0 < u < 1 \text{ on } M\}.$$

$$\tilde N\{\gamma\} = \sup_{u\in\tilde{\mathcal{F}}} \inf_{T\in\gamma} |T(d^c u \wedge (dd^c u)^{k-1})|, \quad \text{if } \dim\gamma = 2k-1, \tag{5}$$

$$\tilde N\{\gamma\} = \sup_{u\in\tilde{\mathcal{F}}} \inf_{T\in\gamma} |T(du \wedge d^c u \wedge (dd^c u)^{k-1})|, \quad \text{if } \dim\gamma = 2k, \tag{6}$$

where the infimum this time is taken over smooth, compactly supported currents which represent γ.

$\tilde N$ enjoys similar properties of N, and $N \le \tilde N < \infty$. They are invariants of the complex structure, and decrease under holomorphic maps.

Chern-Levine-Nirenberg observed in [9] that equation (3) also arises as the Euler equation for the functional

$$I(u) = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1}. \tag{7}$$

Let

$$\mathcal{B} = \{u \in \mathcal{F} \mid u = 1 \text{ on } \Gamma_1,\ u = 0 \text{ on } \Gamma_0\}. \tag{8}$$

If $v \in \mathcal{B}$, let γ denote the $(2n-1)$-dimensional homology class of the level hypersurface $v =$ constant. Then $\forall\, T \in \gamma$, if v satisfies $(dd^c v)^n = 0$,

$$\int_T dv \wedge (dd^c v)^{n-1} = \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1} = I(v).$$

Chern-Levine-Nirenberg Conjecture [9]**:** $N\{\Gamma\} = \inf_{u \in \mathcal{B}} I(u)$.

The relationship between the intrinsic norms and the extremal function u of (4) was investigated by Bedford-Taylor [4]. They pointed out that: *if the extremal function u in (4) is C^2*, one has the following important representation formula,

$$\tilde{N}(\{\Gamma_1\}) = \int_{\Gamma_1} \left(\frac{\partial u}{dr}\right)^n d^c r \wedge (dd^c r)^{n-1}, \tag{9}$$

where Γ_1 is the outer boundary of M and r is a defining function of Γ_1. They also observed that if the extremal function u of (4) can be approximated by functions in $\mathcal{F}_n^{'}$, then $N = \tilde{N}$ and the Chern-Levine-Nirenberg conjecture would be valid. The problem is that functions in $\mathcal{F}_n^{'}$ are C^2 plurisubharmonic functions satisfying the homogeneous complex Monge-Ampère equation. It is hard to construct such approximation due to the lack of C^2 regularity for such equation. Though in some special cases, for example on Reinhardt domains ([4]) or a perturbation of them ([1] and [15]), the extremal function is smooth.

We note that in order for equation (4) to have a plurisubharmonic solution, it is necessary that there is a plurisubharmonic subsolution v. Now suppose M is of the following form,

$$M = \bar{\Omega}^* \setminus \left(\bigcup_{j=1}^{N} \Omega_j\right), \tag{10}$$

where $\Omega^*, \Omega_1, \ldots, \Omega_N$ are bounded strongly smooth pseudoconvex domains in $\mathbb{C}^n$. $\bar{\Omega}_j \subset \Omega^*$, $\forall\, j = 1, \ldots, N$, $\bar{\Omega}_1, \ldots, \bar{\Omega}_N$ are pairwise disjoint, and $\bigcup_{j=1}^N \Omega_j$ is holomorphic convex in Ω^*, and $\Gamma_1 = \partial\Omega^*$ and $\Gamma_0 = \bigcup_{j=1}^N \partial\Omega_j$. If $\Gamma = \{v = \text{ constant}\}$ for some $v \in \mathcal{B}$, $\Gamma \sim \{v = 1\} \sim \{v = 0\}$ in $H_{2n-1}(M)$, the hypersurface $\{v = 1\}$ is pseudoconvex, and the hypersurface $\{v = 0\}$ is pseudoconcave. If M is embedded in $\mathbb{C}^n$, v is strictly plurisubharmonic, and M must be of the form (10). The reverse is proved in [12]: if M is of the form (10), there is $v \in PSH(M^0) \cap C^\infty(M)$

$$(dd^c V)^n > 0 \qquad \text{in } M, \tag{11}$$

such that $\Gamma_1 = \{V = 1\}$ and $\Gamma_0 = \{V = 0\}$.

The following was proved in [12].

Theorem 1. *If M is of the form (10), for the unique solution u of (4), there is a sequence $\{u_k\} \subset \mathcal{B}$ such that*

$$\|u_k\|_{C^2(M)} \le C, \quad \forall\ k, \qquad \lim_{k\to\infty} \sup (dd^c u_k)^n = 0.$$

In particular, $u \in C^{1,1}(M)$ and $\lim_{k\to\infty} \|u_k - u\|_{C^{1,\alpha}(M)} = 0, \quad \forall\ 0 < \alpha < 1$. And we have

$$\tilde{N}(\{\Gamma_1\}) = \int_{\Gamma_1} \left(\frac{\partial u}{dr}\right)^n d^c r \wedge (dd^c r)^{n-1} \tag{12}$$

where r is any defining function of Ω. Moreover,

$$\tilde{N}(\{\Gamma_1\}) = \inf_{v\in\mathcal{B}} \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}. \tag{13}$$

In this paper, we establish

Theorem 2. *If M is of the form (10), we have $N(\{\Gamma_1\}) = \tilde{N}(\{\Gamma_1\})$, and the Chern-Levine-Nirenberg conjecture is valid, that is*

$$N(\{\Gamma_1\}) = \inf_{v\in\mathcal{B}} \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}. \tag{14}$$

We will work on general complex manifold M, which may not necessary to be restricted as a domain in $\mathbb{C}^n$. We assume that

$$\begin{aligned}&M \text{ is a complex manifold, } \partial M = \Gamma_1 \bigcup \Gamma_0 \text{ with both } \Gamma_1 \text{ and } \Gamma_0 \text{ are compact}\\ &\text{hypersurfaces of } M, \text{ and there is } V \in \mathcal{B} \text{ such that } (dd^c V(z))^n > 0, \forall z \in M.\end{aligned} \tag{15}$$

We will prove the following generalization of Theorem 1.

Theorem 3. *Suppose M is of the form (15), there is a unique solution u of (4), there exist a constant $C > 0$ and a sequence $\{u_k\} \subset \mathcal{B}$ such that*

$$|\Delta u_k(z)| \le C, \quad \forall\ k, \qquad \lim_{k\to\infty} \sup_{z\in M} (dd^c u_k(z))^n = 0. \tag{16}$$

In particular, $0 \le \Delta u(z) \le C$ and $\lim_{k\to\infty} \|u_k - u\|_{C^{1,\alpha}(M)} = 0, \quad \forall\ 0 < \alpha < 1$. Furthermore,

$$N(\{\Gamma_1\}) = \tilde{N}(\{\Gamma_1\}) = \int_{\Gamma_1} \left(\frac{\partial u}{dr}\right)^n d^c r \wedge (dd^c r)^{n-1} \tag{17}$$

where r is any defining function of Ω. Finally,

$$N(\{\Gamma_1\}) = \inf_{v\in\mathcal{B}} \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}. \tag{18}$$

Theorem 3 implies Theorem 2. The proof Theorem 3 relies on the regularity study of equation (4). It is a degenerate elliptic fully nonlinear equation.

If M is a domain in $\mathbb{C}^n$, Caffarelli-Kohn-Nirenberg-Spruck [6] establishes $C^{1,1}$ regularity for solutions in strongly pseudoconvex domains with homogeneous boundary condition. For the Dirichlet problem (4), some pieces of the boundary are concave. In [12], we made use of the subsolution method of [10] for the second derivative estimates on the boundary (in the real case, this method was introduced by Hoffman-Rosenberg-Spruck [14] and Guan-Spruck in [11]). This type of estimates is of local feature, so the second derivative estimates on the boundary can be treated in the same way. What we will work on is the interior estimates for the degenerate complex Monge-Ampère equation on Kähler manifold. Such C^2 estimate has been established by Yau in [16]. The contribution of this paper is an interior C^1 estimate for the degenerate complex Monge-Ampère equation in general Kähler manifolds.

We remark here that the subsolution V in (15) can be guaranteed if we impose certain holomorphic convexity condition on M as one can use the pasting method developed in [12]. The subsolution V played important role in the proof boundary estimates in [12]. In this paper, the subsolution V will be crucial to prove the interior estimate. Since we are dealing the equation (4) on a general complex manifold, there may not exist a global coordinate chart. Instead, we treat equation (4) as a fully nonlinear equation on Kähler manifold (M, g), where $g = (g_{i\bar{j}}) = (V_{i\bar{j}})$ is defined by the function V in (15). We will work on the following equation with parameter $0 \le t < 1$,

$$\left\{\begin{array}{l} \det(g_{i\bar{j}} + \phi_{i\bar{j}}) = (1-t)\det(g_{i\bar{j}})f, \\ (g_{i\bar{j}} + \phi_{i\bar{j}}) > 0, \\ \phi|_{\Gamma_1} = 0, \\ \phi|_{\Gamma_0} = 0, \end{array}\right. \tag{19}$$

where f is a given positive function ($f = 1$ for (4), but we will consider general positive function f). Equation (19) is elliptic for $0 \le t < 1$. We want to prove that equation (19) has a unique smooth solution with a uniform bound on $\Delta\phi$ (independent of t). We emphasis that V is important for the C^1 estimate solutions to equation (19), and it also paves way for us to use Yau's interior C^2 estimate in [16]. We set $u = V + \phi$, where ϕ is the solution of equation (19). Therefore, u satisfies

$$\left\{\begin{array}{l} \det(u_{i\bar{j}}) = (1-t)f\det(V_{i\bar{j}}) \\ u|_{\Gamma_1} = 1 \\ u|_{\Gamma_0} = 0. \end{array}\right. \tag{20}$$

Theorem 4. *If M as in (15), there is a constant C depending only on M (independent of t) such that for each $0 \le t < 1$, there is a unique smooth solution u of (19) with*

$$|\Delta\phi(z)| \le C, \quad \forall z \in M. \tag{21}$$

We first deduce Theorem 3 from Theorem 4, following the same lines of arguments in [12].

Proof of Theorem 3. For each $0 \le t < 1$, let ϕ^t be the solution of equation (19). Set $u^t = V + \phi^t$. From Theorem 4, there is a sequence of strictly smooth plurisubharmonic functions $\{u^t\}$ satisfying (20). By (21), there is a subsequence $\{t_k\}$ that tends to 1, such that $\{u_{t_k}\}$ converges to a plurisubharmonic function u in $C^{1,\alpha}(M)$ for any $0 < \alpha < 1$. By the Convergence Theorem for complex Monge-Ampère measures (see [3]), u satisfies equation (4). Again by (21), $0 \le \Delta u \le C$.

For the sequence $\{u_k\}$, we have

$$\begin{aligned}\int_M du_k \wedge d^c u_k \wedge (dd^c u_k)^{n-1} &= \int_{\Gamma_1} d^c u_k \wedge (dd^c u_k)^{n-1} - \int_M u_k (dd^c u_k)^n \\ &= \int_{\Gamma_1} \left(\frac{\partial u_k}{\partial r}\right)^n d^c r \wedge (dd^c r)^{n-1} - \int_M u_k (dd^c u_k)^n \\ &= \int_{\Gamma_1} \left(\frac{\partial u_k}{\partial r}\right)^n d^c r \wedge (dd^c r)^{n-1} - (1 - t_k) \int_M u_k (dd^c V)^n.\end{aligned}$$

Since $u_k \to u$ in $C^{1,\alpha}(M)$, $\left(\frac{\partial u_k}{\partial r}\right)^n \to \left(\frac{\partial u}{\partial r}\right)^n$ uniformly on Γ_1. Therefore,

$$\int_{\Gamma_1} d^c u \wedge (dd^c u)^{n-1} = \int_{\Gamma_1} \left(\frac{\partial u}{\partial r}\right)^n d^c r \wedge (dd^c r)^{n-1}.$$

The proof of Theorem 3.2 in [4] yields $\tilde{N}(\{\Gamma_1\}) = \int_{\Gamma_1} d^c u \wedge (dd^c u)^{n-1}$. Since $u = 1$ on Γ_1, by the Stokes Theorem,

$$\tilde{N}(\{\Gamma_1\}) = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1}.$$

$\forall\, v \in \mathcal{B}$ if $v \not\equiv u$, one must have $v < u$ in M^{int}. By the Comparison Theorem,

$$\int_M du \wedge d^c u \wedge (dd^c u)^{n-1} \le \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}.$$

That is

$$\int_M du \wedge d^c u \wedge (dd^c u)^{n-1} \le \inf_{v \in \mathcal{B}} \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}.$$

On the other hand, by the Convergent Theorem for complex Monge-Ampère measures

$$\liminf_{k \to \infty} \int_M du_k \wedge d^c u_k \wedge (dd^c u_k)^{n-1} = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1}.$$

That is,

$$\tilde{N}(\Gamma_1) = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1} = \inf_{v \in \mathcal{B}} \int dv \wedge d^c v \wedge (dd^c v)^{n-1}. \tag{22}$$

Finally, if T is homological to Γ_1, there is ω such that $\partial \omega = \Gamma_1 - T$. For any $v \in \mathcal{B}$,

$$T(d^c v \wedge (dd^c v)^{n-1}) = \int_{\Gamma_1} d^c v \wedge (dd^c v)^{n-1} - \int_\omega (dd^c v)^n.$$

Applying this to u_k, we obtain

$$|T(d^c u_k \wedge (dd^c u_k)^{n-1})| \geq \int_{\Gamma_1} d^c u_k \wedge (dd^c u_k)^{n-1} - (1-t_k) \int_M (dd^c V)^n.$$

This implies

$$N(\Gamma_1) \geq \int_{\Gamma_1} d^c u_k \wedge (dd^c u_k)^{n-1} - (1-t_k) \int_M (dd^c V)^n.$$

Taking $k \to \infty$,

$$N(\Gamma_1) \geq \int_{\Gamma_1} d^c u \wedge (dd^c u)^{n-1} = \tilde{N}(\Gamma_1).$$

Since $\tilde{N}(\{\Gamma_1\}) \geq N(\{\Gamma_1\})$ by definition, we must have $N(\Gamma_1) = \tilde{N}(\Gamma_1)$. The Chern-Levine-Nirenberg conjecture now follows from (22). □

The rest of this paper will be devoted to the proof of Theorem 4.

Proof of Theorem 4. We show that $\forall\ 0 \leq t < 1$, $\exists!\ u_t \in C^\infty$, u_t strongly plurisubharmonic, such that u_t solves (20) and $\exists\ C > 0$, $\forall\ \ 0 \leq t < 1$

$$0 \leq \Delta u \leq C. \tag{23}$$

The uniqueness is a consequence of the comparison theorem for complex Monge-Ampère equations. In the rest of the proof, we will drop the subindex t.

We first note that since u is plurisubharmonic in M^0, and $0 \leq u \leq 1$ on ∂M, the maximum principle gives $0 \leq u(z) \leq 1\ \forall\ z \in M$. The estimate for Δu is also easy. We have $\Delta u = \Delta V + \Delta \phi = n + \Delta \phi$. Here we will make use of Yau's estimate [16]. Let $R_{i\bar{j}i\bar{j}}$ be the holomorphic bisectional curvature of the Kähler metric g, let C be a positive constant such that $C + R_{i\bar{j}i\bar{j}} \geq 2$ for all i, j. Let $\varphi = \exp(-C\phi)\Delta u$.

Lemma 1. *[Yau] There is C_1 depending only on $\sup_M -\Delta f, \sup_M |\inf_{i\neq j} R_{i\bar{j}i\bar{j}}|, \sup_M f, n$, if the maximum of φ is achieved at an interior point z_0, then*

$$\Delta u(z_0) \leq C_1. \tag{24}$$

By Yau's interior C^2 estimate, we only need to get the estimates of the second derivatives of u on the boundary of M. The boundary of M consists of pieces of compact strongly pseudoconvex and pseudoconcave hypersurfaces. The second derivative estimates on strongly pseudoconvex hypersurface have been established in [6]. For general boundary under the existence of subsolution, the C^2 boundary estimate was proved by Bo Guan [10]. In [12], following the arguments in [6, 10], boundary C^2 estimates were established for M as in the form of (10), that is, M is a domain in $\mathbb{C}^n$. As these C^2 estimates are of local feature, they can be adapted to general complex manifolds without any change. Therefore, we have a uniform bound on Δu. Once Δu is bounded, the equation is uniformly elliptic and concave (for each $t < 1$). The Evans-Krylov interior and the Krylov boundary estimates can be applied here to get global $C^{2,\alpha}$ regularity (since they can be localized). In fact, with sufficient smooth boundary data, the assumption of $u \in C^{1,\gamma}$ for some $\gamma > 0$ is suffice to get global $C^{2,\alpha}$ regularity (e.g., see Theorem 7.3 in [7]).

What left is the gradient estimate. We will prove C^1 estimate for solution of equation (20) independent of Δu. We believe this type of estimate will be useful.

Lemma 2. *Suppose ϕ satisfies equation*

$$\det(g_{i\bar{j}} + \phi_{i\bar{j}}) = \det(g_{i\bar{j}})f, \tag{25}$$

where $g_{i\bar{j}} = V_{i\bar{j}}$ for some smooth strictly plurisubharmonic function V and f is a positive function. Let $u = V + \phi$ and $W = |\nabla u|^2$. There exist constants A and C_2 depending only on $\sup_M f^{\frac{1}{n}}, \sup_M |\nabla f^{\frac{1}{n}}|, \sup_M |V|$, $\inf_M R_{i\bar{j}i\bar{j}}$, if the maximum of function $H = e^{AV}W$ is achieved at an interior point p, then

$$H(p) \le C_2. \tag{26}$$

Let's first assume Lemma 2 to finish the global C^1 estimate. We only need to estimate ∇u on ∂M. Let h be the solution of

$$\begin{cases} \Delta h = 0 & \text{in } M^0 \\ h|_{\Gamma_1} = 1 \\ h|_{\Gamma_0} = 0. \end{cases} \tag{27}$$

Since $0 < \det(u_{i\bar{j}}) = (1-t)f_0 \le \det(V_{i\bar{j}})$, and

$$\Delta u > 0 = \Delta h,$$

and

$$u|_{\partial M} = V|_{\partial M} = h|_{\partial M},$$

by the Comparison Principle, $V(z) \le u(z) \le h(z)$, $\forall\ z \in M$. Therefore

$$|\nabla u(z)| \le \max\left(|\nabla V(z)|, |\nabla h(z)|\right) \le c \qquad \forall\ z \in \partial M. \tag{28}$$

i.e., $\max_{\partial M} |\nabla u| \le c$. In turn,

$$\max_M |\nabla u| \le c. \tag{29}$$

We now prove Lemma 2. Suppose the maximum of H is attained at some interior point p. We pick a holomorphic orthonormal coordinate system at the point such that $(u_{i\bar{j}}) = (g_{i\bar{j}} + \phi_{i\bar{j}})$ is diagonal at that point. We also have $\nabla g_{i\bar{j}} = \nabla g^{\alpha\bar{\beta}} = 0$. We may also assume that $W(p) \ge 1$.

All the calculations will be performed at p.

$$\frac{W_i}{W} + AV_i = 0, \quad \frac{W_{\bar{i}}}{W} + AV_{\bar{i}} = 0. \tag{30}$$

We have

$$W_i = \sum u_{\alpha i}u_{\bar{\alpha}} + u_\alpha u_{i\bar{\alpha}}, \quad W_{\bar{i}} = \sum u_{\alpha\bar{i}}u_{\bar{\alpha}} + u_\alpha u_{\bar{i}\bar{\alpha}},$$

$$W_{i\bar{i}} = \sum g^{\alpha\bar{\beta}}_{,i\bar{i}} u_\alpha u_{\bar{\beta}} + \sum[|u_{i\alpha}|^2 + u_\alpha u_{i\bar{i}\bar{\alpha}} + u_{\bar{\alpha}}u_{i\bar{i}\alpha}] + u_{i\bar{i}}^2,$$

$$|W_i|^2 = \sum u_{\bar{\alpha}}u_\beta u_{i\alpha}u_{\bar{i}\bar{\beta}} + |u_i|^2 u_{i\bar{i}}^2 + u_{\bar{i}}u_{i\bar{i}} \sum u_{\bar{\alpha}}u_{i\alpha} + u_i u_{i\bar{i}} \sum u_\alpha u_{\bar{i}\bar{\alpha}}.$$

By (30),

$$\sum u_{\bar\alpha} u_{i\alpha} = -AWV_i - u_i u_{i\bar i}, \quad \sum u_\alpha u_{\bar i\bar\alpha} = -AWV_{\bar i} - u_{\bar i} u_{i\bar i},$$

and by equation (25)

$$(\log\det(u_{i\bar j}))_\alpha = \frac{f_\alpha}{f}.$$

We have

$$|W_i|^2 = |\sum u_{\bar\alpha} u_{i\alpha}|^2 - |u_i|^2 u_{i\bar i}^2 - AWu_{i\bar i}(V_i u_{\bar i} + V_{\bar i} u_i).$$

$$\begin{aligned}
0 \ &\geq \ \sum_i u^{i\bar i}\left(\frac{W_{i\bar i}}{W} - \frac{|W_i|^2}{W^2} + AV_{i\bar i}\right) \\
&= \ \sum u^{i\bar i}\left(\frac{g^{\alpha\bar\beta}_{,i\bar i} u_\alpha u_{\bar\beta}}{W} + AV_{i\bar i}\right) + \frac{1}{W}\sum [u_\alpha(\log\det(u_{i\bar j}))_{\bar\alpha} + u_{\bar\alpha}(\log\det(u_{i\bar j}))_\alpha] \\
&\quad + \sum u^{i\bar i}\left[\left(\frac{|u_{i\alpha}|^2}{W} - \frac{|\sum u_\alpha u_{\bar i\bar\alpha}|^2}{W^2}\right) + \frac{Au_{i\bar i}(V_i u_{\bar i} + V_{\bar i} u_i)}{W}\right] + \sum\left(\frac{u_{i\bar i}}{W} + \frac{|u_i|^2 u_{i\bar i}}{W^2}\right) \qquad (31)\\
&\geq \ \sum u^{i\bar i}\left(\frac{g^{\alpha\bar\beta}_{,i\bar i} u_\alpha u_{\bar\beta}}{W} + AV_{i\bar i}\right) + 2\frac{1}{Wf}[Re\sum u_\alpha f_{\bar\alpha}] \\
&\quad + \sum u^{i\bar i}\left[\left(\frac{|u_{i\alpha}|^2}{W} - \frac{|\sum u_\alpha u_{\bar i\bar\alpha}|^2}{W^2}\right) + 2\frac{Au_{i\bar i}}{W} Re(V_i u_{\bar i})\right]. \qquad (32)
\end{aligned}$$

Since $\frac{g^{\alpha\bar\beta}_{,i\bar i} u_\alpha u_{\bar\beta}}{W}$ is controlled by $\inf_M R_{i\bar j i\bar j}$, it follows from the Cauchy-Schwartz inequality,

$$0 \geq \sum u^{i\bar i}(\inf_M R_{k\bar l k\bar l} + AV_{i\bar i}) - 2\frac{\sum |u_\alpha f_\alpha|}{fW} - 2A\frac{\sum |V_i u_{\bar i}|}{W} \qquad (33)$$

Now we may pick A sufficient large, such that

$$\inf_M R_{k\bar l k\bar l} + A \geq 1.$$

This yields

$$0 \geq \sum u^{i\bar i} - 2\frac{A|\nabla V| + |\nabla \log f|}{W^{\frac{1}{2}}} \geq nf^{-\frac{1}{n}}\left(1 - 2\frac{Af^{\frac{1}{n}}|\nabla V| + |\nabla f^{\frac{1}{n}}|}{W^{\frac{1}{2}}}\right). \qquad (34)$$

Lemma 2 follows directly from (34). □

Add-in-Proof. Recently, a general gradient estimate for complex Monge-Ampère equation $\det(g_{i\bar j} + \phi_{i\bar j}) = f\det(g_{i\bar j})$ on Kähler manifolds has been proved by Blocki in [5], also by the author in [13].

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