A CONSTANT RANK THEOREM FOR SOLUTIONS OF FULLY NONLINEAR ELLIPTIC EQUATIONS

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1. Statement of main results

Convexity is one of important geometric properties associated to the study of partial differential equations, in particular for equations related to problems in differential geometry. There is a vast literature on this subject. In an important development 1985, a technique was devised to deal with the convexity issue via homotopy method of deformation in the work of Caffarelli-Friedman [7]. In [7], the existence of convex solutions for semilinear elliptic equations in two dimensions was proved by a form of deformation lemma using the Strong Maximum Principle (see also the work of Singer-Wong-Yau-Yau [18] for a similar approach). The core of this approach is the establishment of the *Constant Rank Theorem*, that is the rank of the Hessian of the corresponding convex solution is constant. The result in [7] was later generalized to higher dimensions in [16]. The *Constant Rank Theorem* is a refined statement of convexity. This has profound implications in geometry of solutions. The idea of the deformation lemma and the establishment of the *Constant Rank Theorem* can be extended to various nonlinear differential equations in differential geometry involving symmetric curvature tensors. Recently, in connection to the Christoffel-Minkowski problem and problem of prescribing Weingarten curvatures in classical differential geometry, this form of deformation lemma was extended to some equations involving the second fundamental forms of embedded hypersurfaces in $\mathbb{R}^n$ [12, 11, 13]. The *Constant Rank Theorem* shares similar geometric flavors in spirit with a classical theorem of Hartman-Nirenberg [14], where they treated hypersurfaces in $\mathbb{R}^n$ with vanishing spherical Jacobian.

A pertinent question is under what structural conditions for partial differential equations so that the positivity of the symmetric curvature tensor is preserved under homotopy deformation? The purpose of this paper is to establish a general principle in this direction. More specifically, we establish *Constant Rank Theorem* for a wide class of elliptic fully nonlinear equations involving symmetric curvature tensors on Riemannian manifolds.

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Let us fix some notations. Let $\Psi \subset \mathbb{R}^n$ be an open symmetric domain, denote $Sym(n) = \{n \times n \quad \text{real symmetric matrices}\}$, set

$$\tilde{\Psi} = \{A \in Sym(n) : \lambda(A) \in \Psi\}.$$

We assume

$$f \in C^2(\Psi) \quad \text{symmetric and } f_{\lambda_i}(\lambda) = \frac{\partial f}{\partial \lambda_i}(\lambda) > 0, \forall i = 1, \cdots, n, \quad \forall \lambda \in \Psi. \tag{1.1}$$

extend it to $F : \tilde{\Psi} \to R$ by $F(A) = f(\lambda(A))$. Condition (1.1) ensures F is elliptic. We define $\tilde{F}(A) = F(A^{-1})$ whenever $A^{-1} \in \tilde{\Psi}$, we will assume

$$\tilde{F} \quad \text{is locally convex.} \tag{1.2}$$

Condition (1.2) was introduced by Alvarez-Lasry-Lions in [1], where they used it to prove the convexity of viscosity solutions in convex domains in $\mathbb{R}^n$ under state constraints boundary conditions.

To illustrate the nature of our results, we first consider fully nonlinear elliptic equations in domains of $\mathbb{R}^n$.

Theorem 1. *Under conditions (1.1)-(1.2), if u is a C^3 convex solution of the following equation in a domain Ω in $\mathbb{R}^n$*

$$F(u_{ij}(x)) = \varphi(x, u(x), \nabla u(x)), \quad \forall x \in \Omega, \tag{1.3}$$

for some $\varphi \in C^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. If $\varphi(x, u, p)$ is concave in $\Omega \times \mathbb{R}$ for any fixed $p \in \mathbb{R}^n$, then the Hessian (u_{ij}) has constant rank in Ω.

We now treat fully nonlinear equations arising from classical differential geometry treated in [12, 11, 13]. Convexity of a hypersurface is equivalent to the positivity of its second fundamental form.

Let M be an oriented immersed connect hypersurface in $\mathbb{R}^{n+1}$ with a nonnegative definite second fundamental form. Let $\kappa(X) = (\kappa_1(X), \cdots, \kappa_n(X))$ be the principal curvature at $X \in M$. We consider the following curvature equation

$$f(\kappa(X)) = \varphi(X, \vec{n}(X)), \quad \forall X \in M, \tag{1.4}$$

where $\vec{n}(X)$ the unit normal of M at X.

Theorem 2. *Suppose f and F as in Theorem 1. Suppose $\Sigma \subset \mathbb{R}^{n+1} \times \mathbb{S}^n$ is a bounded open set and $\varphi \in C^{1,1}(\Gamma)$ and $\varphi(X, y)$ is locally concave in X variable for any $y \in \mathbb{S}^n$. Let M be an oriented immersed connect hypersurface in $\mathbb{R}^{n+1}$ with a nonnegative definite second fundamental form. If $(X, \vec{n}(X)) \in \Sigma$ for each $X \in M$ and the principal curvatures of M satisfies equation*

(1.4), then the second fundamental form of M is of constant rank. If in addition M is compact, then M is the boundary of a strongly convex bounded domain in $\mathbb{R}^{n+1}$.

We next consider the Christoffel-Minkowski type equation,

$$F(u_{ij} + u\delta_{ij}) = \varphi \quad \text{on} \quad \Omega \subset \mathbb{S}^n, \tag{1.5}$$

where u_{ij} are the second covariant derivatives of u with respect to orthonormal frames on $\mathbb{S}^n$.

Theorem 3. *Let f and F as in Theorem 1, and assume f is of homogeneous degree -1 and Ω is an open domain in $\mathbb{S}^n$. If $0 > \varphi \in C^{1,1}(\Omega)$ and $(\varphi_{ij} + \varphi\delta_{ij}) \le 0$ on Ω, if u is a solution of equation (1.5) with $u_{ij} + u\delta_{ij}$ is nonnegative, then $(u_{ij} + u\delta_{ij})$ of constant rank. If $\Omega = \mathbb{S}^n$, then $(u_{ij} + u\delta_{ij})$ is positive definite everywhere on $\mathbb{S}^n$.*

We now turn to the Riemannian geometry. Let (M, g) be a connected Riemannian manifold, for each $x \in M$, let $\tau(x)$ be the minimum of sectional curvatures at x. A symmetric 2-tensor W on M is call a Codazzi tensor if

$$\nabla_X W(Y, Z) = \nabla_Y W(X, Z),$$

for all tangent vectors X, Y, Z, where ∇ is the Levi-Civita connection.

Theorem 4. *Let F as in Theorem 3, and (M, g) is a connected Riemannian manifold. Suppose $\varphi \in C^2(M)$ with $Hess(\varphi)(x) + \tau(x)\varphi(x)g(x) \le 0$ for every $x \in M$. If W is a semi-positive definite Codazzi tensor on M satisfying equation*

$$F(g^{-1}W) = \varphi \quad \text{on } M, \tag{1.6}$$

then W is of constant rank.

Our arguments also apply to nonlinear parabolic equations as well. There are corresponding parabolic versions of Theorems 1-4. For example, we have the following parabolic version of Theorem 1.

Theorem 5. *Under conditions (1.1)-(1.2), if u is a C^3 convex solution of the following parabolic equation in a domain Ω in $\mathbb{R}^n$ for $0 < t \le T$*

$$u_t(t, x) - F(u_{ij}(t, x)) = -\varphi(t, x, u(x), \nabla u(x)), \quad \forall x \in \Omega, \tag{1.7}$$

for some $\varphi \in C^{1,1}([0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n)$. If $\varphi(t, x, u, p)$ is concave in $\Omega \times \mathbb{R}$ for any fixed $(t, p) \in (0, T) \times \mathbb{R}^n$, if for some $t_0 \in (0, T], x_0 \in \Omega$, $rank(u_{ij}(t_0, x_0)) \le rank(u_{ij}(t, x))$ for all $0 < t \le t_0, x \in \Omega$, then $rank(u_{ij}(t, x)) = constant$ for all $0 \le t \le t_0, x \in \Omega$.

Some remarks are in order.

Remark 1. *The condition (1.2) was first appeared in* [1], *where Alvarez-Lasry-Lions obtained a general structure condition for the convexity of viscosity solutions in convex domains under state constraints boundary conditions. The same condition, together with some proper convex cone condition on* Ψ *and concavity condition on* f, *were also used by Andrews in* [2] *on pinching estimates of evolving closed convex hypersurfaces in* $\mathbb{R}^{n+1}$. *We also note that a slight stronger concavity condition on* $\frac{1}{F(A^{-1})}$ *was used by Urbas in* [19] *for the related work on curvature flow of closed convex hypersurfaces in* $\mathbb{R}^{n+1}$.

Remark 2. *We list some well known examples with condition (1.2) satisfied:* $f(\lambda) = \sigma_k^{\frac{1}{k}}(\lambda)$, $f(\lambda) = (\frac{\sigma_k}{\sigma_l})^{\frac{1}{k-l}}(\lambda)$, $f(\lambda) = -\sigma_k^{-\frac{1}{k}}(\lambda)$, $f(\lambda) = -(\frac{\sigma_k}{\sigma_l})^{-\frac{1}{k-l}}(\lambda)$ *with* $\Psi = \Gamma_k$, *where* $0 \le l < k \le n$, σ_j *the* j*th elementary symmetric function and* $\Gamma_k = \{\lambda \in \mathbb{R}^n | \quad \sigma_j(\lambda) > 0, \forall 1 \le j \le k\}$ *and equation for special Lagrangian* $f(A) = -\sqrt{-1}\log(\frac{\det(I+\sqrt{-1}A)}{\det^{\frac{1}{2}}(I+A^2)})$ *for nonnegative definite* A. *The results in* [7, 16, 12, 11, 13] *should be interpreted as* $f(\lambda) = -\sigma_k^{-\frac{1}{k}}(\lambda)$. *We choose this form for the sake of a simple statement of the condition on* φ. *It should be pointed out that for a specific equation, sometimes certain transformation (e.g., taking* $-\frac{1}{\Delta u} = f$ *instead of* $\Delta u = -\frac{1}{f}$*) may strengthen results. We note that homogeneity assumption is not imposed in Theorem 1 and Theorem 2.*

Remark 3. *The constant rank results in Theorems 1-5 are of local nature in the sense that there is no global or boundary condition imposed on the solutions. It is well known that concavity assumption is important for* $C^{1,1}$ *and* $C^{2,\alpha}$ *estimates of solutions of fully nonlinear equations, e.g., the Evans-Krylov Theorem* [9, 17] *and Caffarelli's interior* $C^{1,1}$ *estimates for concave uniformly elliptic equations* [5, 6]. *Condition (1.2) can be viewed as a dual form in this respect, as the estimation of convexity for a solution is equivalent to the estimation of a positive lower bound of the Hessian of the solution.*

The rest of the paper is organized as follows. We prove Theorem 1 and Theorem 5 in Section 2. The proofs of Theorem 2-3 will be presented in Section 3, modifying the main arguments in the proof of Theorem 1. In the last section, we discuss related results for Codazzi tensors on general Riemannian manifolds, in particular, manifolds with signed harmonic curvature.

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2. Proof of Theorem 1

We first present proof of Theorem 1 to illustrate the main idea to establish a local differential inequality (2.8) near the point where the minimum rank of the Hessian (u_{ij}) is attained. One of the key property we will use is the symmetry of u_{ijk} with respect to indices i,j,k. The proof of Theorem 2 and Theorem 3 will be given in the next section. The main arguments also work for equations on Codazzi tensors in Riemannian manifolds, which we will discuss in the last section.

We define $\dot{f}^k = \frac{\partial f}{\partial \lambda_k}$, $\ddot{f}^{kl} = \frac{\partial^2 f}{\partial \lambda_k \partial \lambda_l}$, $F^{\alpha\beta} = \frac{\partial F}{\partial A_{\alpha\beta}}$ and $F^{\alpha\beta,rs} = \frac{\partial^2 F}{\partial A_{\alpha\beta}\partial A_{rs}}$. The following lemma is well known (e.g., see [3, 2, 19], part (a) of Lemma 1 was known to Caffarelli-Nirenberg-Spruck, it was originally stated in a preliminary version of [8] and it was late removed from the published version).

Lemma 1. (a). *At any diagonal $A \in \tilde{\Psi}$ with distinct eigenvalues, let $\ddot{F}(B,B)$ be the second derivative of F in direction $B \in Sym(n)$, then*

$$\ddot{F}(B,B) = \sum_{j,k=1}^{n} \ddot{f}^{jk} B_{jj}B_{kk} + 2\sum_{j<k} \frac{\dot{f}^j - \dot{f}^k}{\lambda_j - \lambda_k} B_{jk}^2. \tag{2.1}$$

(b). *If $\tilde{F}(A) = -F(A^{-1})$ is concave near a positive definite matrix A, then*

$$\sum_{j,k,p,q=1}^{n} (F^{kl,pq}(A) + 2F^{jp}(A)A^{kq})X_{jk}X_{pq} \geq 0 \tag{2.2}$$

for every symmetric matrix X.

Inequality (2.2) is where condition (1.2) is used (this is the only place it is needed). We will use the following form of Lemma 1.

Corollary 1. *Assume F satisfies condition in Lemma 1(b). Suppose $A \in \tilde{\Psi}$, A is semipositive definite and diagonal. Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ and $\lambda_i > 0, \forall i \geq n-l+1$. Then*

$$\sum_{j,k=n-l+1}^{n} \ddot{f}^{jk}(A)X_{jj}X_{kk} + 2\sum_{n-l+1\leq j<k} \frac{\dot{f}^j - \dot{f}^k}{\lambda_j - \lambda_k} X_{jk}^2 + 2\sum_{i,k=n-l+1}^{n} \frac{\dot{f}^i(A)}{\lambda_k} X_{ik}^2 \geq 0 \tag{2.3}$$

for every symmetric matrix $X = (X_{jk})$ with $X_{jk} = 0$ if $j \leq n-l$.

Proof. (2.3) follows directly from (2.1) and (2.2) if A is positive definite. For semi-definite A, it follows by approximating. $\square$

Proof of Theorem 1. We set $\tilde{\varphi}(x) = \varphi(x, u(x), \nabla u(x))$ and $W = (W_{ij})$ with $W_{ij} = u_{ij}$. We rewrite (1.3) in the following form

$$F(W(x)) = \tilde{\varphi}(x), \quad \forall x \in \Omega. \tag{2.4}$$

Suppose $z_0 \in \Omega$ is a point where W is of minimal rank l. We pick an open neighborhood O of z_0, for any $z \in O$, let $\lambda_1 \leq \lambda_2 ... \leq \lambda_n$ be the eigenvalues of W at z. There is a positive constant $C > 0$ depending only on $\|u\|_{C^3}$, $\|\varphi\|_{C^2}$ and n, such that $\lambda_n \geq \lambda_{n-1} ... \geq \lambda_{n-l+1} \geq C$. Let $G = \{n-l+1, n-l+2, ..., n\}$ and $B = \{1, ..., n-l\}$ be the "good" and "bad" sets of indices respectively. Let $\Lambda_G = (\lambda_{n-l+1}, ..., \lambda_n)$ be the "good" eigenvalues of W at z, for the simplicity of the notations, we also write $G = \Lambda_G$ if there is no confusion.

Since F is elliptic and W is continuous, if O is sufficiently small, we may pick a positive constant A such that

$$\min_{\alpha} F^{\alpha\alpha}(W(x)) \geq \frac{100}{A} \sum_{\alpha,\beta,r,s} |F^{\alpha\beta,rs}(W(x))|, \quad \forall x \in O. \tag{2.5}$$

Set (with the convention that $\sigma_j(W) = 0$ if $j < 0$ or $j > n$)

$$\phi(x) = \sigma_{l+1}(W) + A\sigma_{l+2}(W). \tag{2.6}$$

Following the notations in [7], for two functions defined in an open set $O \subset \Omega$, $y \in O$, we say that $h(y) \lesssim k(y)$ provided there exist positive constants c_1 and c_2 such that

$$(h - k)(y) \leq (c_1|\nabla\phi| + c_2\phi)(y). \tag{2.7}$$

We also write $h(y) \sim k(y)$ if $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$. Next, we write $h \lesssim k$ if the above inequality holds in O, with the constant c_1, and c_2 depending only on $||u||_{C^3}$, $||\tilde{\varphi}||_{C^2}$, n and C_0 (independent of y and O). Finally, $h \sim k$ if $h \lesssim k$ and $k \lesssim h$. In the following, all calculations are at the point z using the relation "$\lesssim$", with the understanding that the constants in (2.7) are under control.

We shall show that

$$\frac{1}{\sigma_l(G)} \sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim \sum_{i\in B} \tilde{\varphi}_{ii}. \tag{2.8}$$

To prove (2.8), we may assume $u \in C^4$ by approximation. For each $z \in O$ fixed, we can rotate coordinate so that W is diagonal at z, and $W_{ii} = \lambda_i, \forall i = 1, ..., n$. We note that since W is diagonal at z, $(F^{\alpha\beta})$ is also diagonal at z and $F^{\alpha\beta,rs} = 0$ unless $\alpha = \beta, r = s$ or $\alpha = r, \beta = s$.

Now we compute ϕ and its first and second derivatives in the direction x_α. The following computations follow mainly from [12]. As W is diagonal at z, $\sigma_{l+2}(W) \leq C\sigma_{l+1}^{\frac{l+2}{l+1}}(W)$, we obtain

$$0 \sim \phi(z) \sim \sigma_{l+1}(W) \sim (\sum_{i\in B} W_{ii})\sigma_l(G) \sim \sum_{i\in B} W_{ii}, \quad (\text{so} \quad W_{ii} \sim 0, \quad i \in B), \tag{2.9}$$

Let W be a $n \times n$ diagonal matrix, we denote $(W|i)$ to be the $(n-1) \times (n-1)$ matrix with ith row and ith column deleted, and denote $(W|ij)$ to be the $(n-2) \times (n-2)$ matrix with i, jth rows and i, jth columns deleted. We also denote $(G|i)$ be the subset of G with λ_i deleted. Since $\sigma_{l+1}(W|i) \lesssim 0$, we have

$$0 \sim \phi_\alpha \sim \sigma_l(G) \sum_{i\in B} W_{ii\alpha} \sim \sum_{i\in B} W_{ii\alpha}. \tag{2.10}$$

(2.9) yields that, for $1 \le m \le l$,

$$\sigma_m(W) \sim \sigma_m(G), \quad \sigma_m(W|j) \sim \begin{cases} \sigma_m(G|j), & \text{if } j \in G; \\ \sigma_m(G), & \text{if } j \in B. \end{cases} \tag{2.11}$$

$$\sigma_m(W|ij) \sim \begin{cases} \sigma_m(G|ij), & \text{if } i,j \in G, i \neq j; \\ \sigma_m(G|j), & \text{if } i \in B, j \in G; \\ \sigma_m(G), & \text{if } i,j \in B, i \neq j. \end{cases}$$

Since W is diagonal, it follows from (2.9) and Proposition 2.2 in [12],

$$\frac{\partial \sigma_{l+1}(W)}{\partial W_{ij}} \sim \begin{cases} \sigma_l(G), & \text{if } i = j \in B, \\ 0, & \text{otherwise,} \end{cases} \tag{2.12}$$

and for $1 \le m \le n$,

$$\frac{\partial^2 \sigma_m(W)}{\partial W_{ij} \partial W_{rs}} = \begin{cases} \sigma_{m-2}(W|ir), & \text{if } i = j, r = s, i \neq r; \\ -\sigma_{m-2}(W|ij), & \text{if } i \neq j, r = j, s = i; \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

From (2.10)-(2.13), we have

$$\sum_{\substack{i \in B \\ j \in G}} \sigma_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} \sim (\sum_{j \in G} \sigma_{l-1}(G|j) W_{jj\alpha}) \sum_{i \in B} W_{ii\alpha} \sim 0, \tag{2.14}$$

$$\sum_{\substack{i,j \in B \\ i \neq j}} \sigma_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} \sim -\sigma_{l-1}(G) \sum_{i \in B} W_{ii\alpha}^2, \tag{2.15}$$

$$\sum_{j \in G, i \in B} \sigma_{l-1}(W|ij) W_{ij\alpha}^2 \sim \sum_{i \in B, j \in G} \sigma_{l-1}(G|j) W_{ij\alpha}^2, \tag{2.16}$$

and if $l \le n-2$ (that is $|B| \ge 2$)

$$\begin{aligned} \sum_{i,j=1}^{n} \frac{\partial^2 \sigma_{l+2}(W)}{\partial W_{ij} \partial W_{rs}} W_{ij\alpha} W_{rs\alpha} &\sim \sum_{i \neq j \in B} \sigma_l(G) W_{ii\alpha} W_{jj\alpha} - \sum_{i \neq j \in B} \sigma_l(G) W_{ij\alpha}^2 \\ &\sim -\sum_{i \in B} \sigma_l(G) W_{ii\alpha}^2 - \sum_{i \neq j \in B} \sigma_l(G) W_{ij\alpha}^2 \\ &\sim -\sigma_l(G) \sum_{i,j \in B} W_{ij\alpha}^2. \end{aligned} \tag{2.17}$$

We note that if $l = n-1$, we have $|B| = 1$, (2.17) still holds since $w_{ii\alpha} \sim 0$ by (2.10).

By (2.11)-(2.16), $\forall \alpha \in \{1, 2, ..., n\}$

$$\begin{aligned}
\phi_{\alpha\alpha} &= A\sigma_{l+2}(W)_{\alpha\alpha} + (\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}})\sigma_{l-1}(W|ij)W_{ii\alpha}W_{jj\alpha} \\
&- (\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}})\sigma_{l-1}(W|ij)W_{ij\alpha}^2 + \sum_i \frac{\partial\sigma_{l+1}(W)}{\partial W_{ii}} W_{ii\alpha\alpha} \\
&\sim \sigma_l(G)\sum_{i\in B} W_{ii\alpha\alpha} + A\sum_{i=1}^n \sigma_{l+1}(W|i)W_{ii\alpha\alpha} - 2\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)W_{ij\alpha}^2 \\
&\quad -(\sigma_{l-1}(G) + A\sigma_l(G))\sum_{i,j\in B} W_{ij\alpha}^2. \qquad (2.18)
\end{aligned}$$

Since $F^{\alpha\beta}$ is diagonal at z, we have

$$\begin{aligned}
\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha} &\sim A\sum_{\alpha=1}^n\sum_{i=1}^n F^{\alpha\alpha}\sigma_{l+1}(W|i)W_{ii\alpha\alpha} + \sum_{\alpha=1}^n F^{\alpha\alpha}[\sigma_l(G)(\sum_{i\in B} W_{ii\alpha\alpha} - A\sum_{i,j\in B} W_{ij\alpha}^2) \\
&\quad -\sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2 - 2\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)W_{ij\alpha}^2]. \qquad (2.19)
\end{aligned}$$

By equation (2.4),

$$\tilde{\varphi}_i = \sum_{\alpha,\beta=1}^n F^{\alpha\beta}W_{\alpha\beta i}, \quad \tilde{\varphi}_{ii} = \sum_{\alpha,\beta,r,s=1}^n F^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi} + \sum_{\alpha,\beta=1}^n F^{\alpha\beta}W_{\alpha\beta ii}.$$

So for any $i \in B$, we have

$$\sum_{\alpha=1}^n F^{\alpha\alpha}W_{\alpha\alpha ii} \sim \tilde{\varphi}_{ii} - \sum_{\alpha,\beta,r,s=1}^n F^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi} \qquad (2.20)$$

As $W_{\alpha\alpha ii} = W_{ii\alpha\alpha}$ and $\sigma_{l+1}(W|i) \sim 0$, from (2.19) and (2.20)

$$\begin{aligned}
\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha} &\sim \sigma_l(G)[\sum_{i\in B}\tilde{\varphi}_{ii} - \sum_{i\in B}\sum_{\alpha,\beta,r,s=1}^n F^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi} - A\sum_{\alpha=1}^n\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2] \\
&\quad -\sigma_{l-1}(G)\sum_{\alpha=1}^n\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2 - 2\sum_{\alpha=1}^n\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)F^{\alpha\alpha}W_{ij\alpha}^2. \qquad (2.21)
\end{aligned}$$

In order to study terms in (2.21), we may assume the eigenvalues of W are distinct at z (if necessary, we perturb W then take limit). In the following we let $\lambda_i = W_{ii}$.

Using (2.1), (2.2) and (2.21), we obtain

$$\begin{aligned}\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha} \sim\ & \sigma_l(G)\sum_{i\in B}\tilde{\varphi}_{ii} - \sigma_l(G)\sum_{i\in B}[\sum_{\alpha,\beta=1}^n \ddot{f}^{\alpha\beta}W_{\alpha\alpha i}W_{\beta\beta i} + 2\sum_{\alpha<\beta}\frac{\dot{f}^\alpha-\dot{f}^\beta}{\lambda_\alpha-\lambda_\beta}W^2_{\alpha\beta i}] \\ & -(\sigma_{l-1}(G)+A\sigma_l(G))\sum_{\alpha=1}^n\sum_{i,j\in B}\dot{f}^\alpha W^2_{ij\alpha} - 2\sum_{\alpha=1}^n\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)\dot{f}^\alpha W^2_{ij\alpha}.\end{aligned} \tag{2.22}$$

As W_{ijk} is symmetric with respect to i,j,k,

$$\begin{aligned}\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha} \sim\ & \sum_{i\in B}\tilde{\varphi}_{ii} - \sum_{i\in B}[\sum_{\alpha,\beta\in B} + \sum_{\alpha,\beta\in G} + 2\sum_{\substack{\alpha\in G\\ \beta\in B}}]\ddot{f}^{\alpha\beta}W_{\alpha\alpha i}W_{\beta\beta i} \\ & -2\sum_{i\in B}[\sum_{\substack{\alpha,\beta\in G\\ \alpha<\beta}} + \sum_{\substack{\alpha\in G\\ \beta\in B}} + \sum_{\substack{\alpha,\beta\in B\\ \alpha<\beta}}]\frac{\dot{f}^\alpha-\dot{f}^\beta}{\lambda_\alpha-\lambda_\beta}W^2_{\alpha\beta i} - A\sum_{\alpha=1}^n\sum_{i,j\in B}\dot{f}^\alpha W^2_{ij\alpha} \\ & -2\sum_{i\in B}[\sum_{\alpha,\beta\in G} + \sum_{\substack{\alpha\in B\\ \beta\in G}}]\frac{\dot{f}^\alpha}{\lambda_\beta}W^2_{\alpha\beta i} - \sum_{i\in B}[\sum_{\alpha,\beta\in B} + \sum_{\substack{\alpha\in G\\ \beta\in B}}](\sum_{k=n-l+1}^n\frac{1}{\lambda_k})\dot{f}^\alpha W^2_{\alpha\beta i}.\end{aligned} \tag{2.23}$$

We note for $\beta,\gamma\in B,\alpha\in G$, $\ddot{f}^{\alpha\beta}\sim\ddot{f}^{\alpha\gamma}$. Thus from (2.10)

$$2\sum_{\substack{\alpha\in G\\ \beta\in B}}\ddot{f}^{\alpha\beta}W_{\alpha\alpha i}W_{\beta\beta i} \sim \sum_{\alpha\in G}\ddot{f}^{\alpha\beta}W_{\alpha\alpha i}(\sum_{\beta\in B}W_{\beta\beta i}) \sim 0.$$

In turn, we may rewrite (2.23) as

$$\begin{aligned}\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha} \sim\ & \sum_{i\in B}\tilde{\varphi}_{ii} - \sum_{i\in B}[\sum_{\alpha,\beta\in B} + \sum_{\alpha,\beta\in G}]\ddot{f}^{\alpha\beta}W_{\alpha\alpha i}W_{\beta\beta i} \\ & -2\sum_{i\in B}[\sum_{\substack{\alpha,\beta\in G\\ \alpha<\beta}} + \sum_{\substack{\alpha\in G\\ \beta\in B}} + \sum_{\substack{\alpha,\beta\in B\\ \alpha<\beta}}]\frac{\dot{f}^\alpha-\dot{f}^\beta}{\lambda_\alpha-\lambda_\beta}W^2_{\alpha\beta i} - A\sum_{\alpha=1}^n\sum_{i,j\in B}\dot{f}^\alpha W^2_{ij\alpha} \\ & -2\sum_{i\in B}[\sum_{\alpha,\beta\in G} + \sum_{\substack{\alpha\in B\\ \beta\in G}}]\frac{\dot{f}^\alpha}{\lambda_\beta}W^2_{\alpha\beta i} - \sum_{i\in B}[\sum_{\alpha,\beta\in B} + \sum_{\substack{\alpha\in G\\ \beta\in B}}](\sum_{k=n-l+1}^n\frac{1}{\lambda_k})\dot{f}^\alpha W^2_{\alpha\beta i}.\end{aligned} \tag{2.24}$$

By the symmetry of W_{ijk} and with the choice of A in (2.5), the term $A\sum_{\alpha=1}^n\sum_{i,j\in B}\dot{f}^\alpha W^2_{ij\alpha}$ dominates all the terms involving W^2_{ijk} if at least two of indices i,j,k are in B. With this observation, we deduce from (2.24) that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha} \sim \sum_{i\in B}\tilde{\varphi}_{ii} - \sum_{i\in B}I_i - \frac{A}{2}\sum_{\alpha=1}^n\sum_{i,j\in B}\dot{f}^\alpha W^2_{ij\alpha}, \tag{2.25}$$

where

$$I_i = \sum_{\alpha,\beta\in G} \ddot{f}^{\alpha\beta} W_{\alpha\alpha i} W_{\beta\beta i} + 2 \sum_{\substack{\alpha,\beta\in G\\ \alpha<\beta}} \frac{\dot{f}^{\alpha} - \dot{f}^{\beta}}{\lambda_\alpha - \lambda_\beta} W^2_{\alpha\beta i} + 2 \sum_{\alpha,\beta\in G} \frac{\dot{f}^{\beta}}{\lambda_\alpha} W^2_{\alpha\beta i}.$$

By (2.3) in Corollary 1, we have

$$I_i \gtrsim 0. \tag{2.26}$$

(2.25) becomes

$$\frac{1}{\sigma_l(G)} \sum_{\alpha=1}^{n} F^{\alpha\alpha} \phi_{\alpha\alpha} \lesssim \sum_{i\in B} \tilde{\varphi}_{ii} - \frac{A}{2} \sum_{\alpha=1}^{n} \sum_{i,j\in B} \dot{f}^{\alpha} W^2_{ij\alpha}, \tag{2.27}$$

We now finish the proof of Theorem 1. Since (u_{ij}) is diagonal at the point,

$$\sum_{i\in B} \tilde{\varphi}_{ii} = \sum_{i\in B} (\varphi_{x_ix_i} + 2\varphi_{x_iu} u_i + \varphi_{uu} u_i^2) + \sum_{i\in B} u_{ii} (2\varphi_{x_ip_i} + \varphi_{p_ip_i} u_{ii} + \varphi_u + 2\varphi_{up_i} u_i) + \sum_{j} \varphi_{p_j} \sum_{i\in B} u_{iij}.$$

By our assumption on φ, (2.9) and (2.10),

$$\sum_{i\in B} \tilde{\varphi}_{ii} \lesssim 0. \tag{2.28}$$

By (2.27),

$$\frac{1}{\sigma_l(G)} \sum_{\alpha=1}^{n} F^{\alpha\alpha} \phi_{\alpha\alpha} \lesssim \sum_{i\in B} \tilde{\varphi}_{ii} \lesssim 0.$$

Theorem 1 then follows from the strong minimum principle. □

Remark 4. *We remark that the major difference of the above proof and the proofs in* [7, 16, 12, 11, 13] *is the choice of the test function ϕ in (2.6). While letting $\phi = \sigma_{l+1}$ as in* [7, 16, 12, 11, 13]*, the calculations there rely heavily on the algebraic properties of the elementary symmetric functions σ_k. The extra term $A\sigma_{l+2}$ in (2.6) paves the way for us to deal with general nonlinear functional F.*

Remark 5. *In the proof of Theorem 1, the condition $\varphi(x, u, p)$ is concave in $\Omega \times \mathbb{R}$ for any fixed $p \in \mathbb{R}^n$ was only used in (2.28). By inspection, assumption that $(\frac{\partial^2 \varphi}{\partial y_i \partial y_j}(x, u(x), \nabla u(x)))$ is semi-negative definite $\forall x \in \Omega$ is suffice to ensure (2.28). In turn, Theorem 1 is valid under this weakened assumption.*

Proof of Theorem 5. The parabolic version of Theorem 1 follows directly from our proof above. We adopt the same notation as in above. We write equation (1.7) in the following form

$$F(u_{ij}) = \varphi^*, \tag{2.29}$$

where $\varphi^* = u_t + \tilde{\varphi}$. As in the proof (2.27) of Theorem 1, in a neighborhood of t_0, x_0,

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \sim \sum_{i\in B}\varphi^*_{ii} - \frac{A}{2}\sum_{\alpha=1}^{n}\sum_{i,j\in B}\dot{f}^{\alpha}W^2_{ij\alpha}.$$

As $\varphi^*_{ii} = u_{tii} + \tilde{\varphi}_{ii}$ and it is easy to check that

$$\sum_{i\in B} u_{tii} \sim \frac{1}{\sigma_l(G)}\phi_t.$$

It follows that

$$-\phi_t + \sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim 0.$$

We can deduce Theorem 5 from the Strong Maximum Principle for parabolic equations. □

3. Curvature equations of hypersurfaces in $\mathbb{R}^{n+1}$

In this section, we convexity problem of fully nonlinear curvature equations of hypersurfaces in $\mathbb{R}^{n+1}$. We refer [10, 12, 11, 13] for geometric background of these type of equations. We prove Theorem 3 first.

Proof of Theorem 3. We work on spherical Hessian $W = (u_{ij} + u\delta_{ij})$ in place of standard Hessian (u_{ij}) in the proof of Theorem 1.

As in the proof of Theorem 1, let $z_0 \in \Omega$ be a point where W is of minimum rank and O is a small open neighborhood of z_0. For any $z \in O \subset \Omega$, we divide eigenvalues of W at z into G and B, the "good" and "bad" sets of indices respectively. Define ϕ as in (2.6). We may assume at the point, W is diagonal under some local orthonormal frames. We want to show that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim \sum_{i\in B}[\varphi_{ii} + \varphi]. \tag{3.1}$$

The same arguments in the proof of Theorem 1 yield (2.9)-(2.10) for $W = (u_{ij} + u\delta_{ij})$, and

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \sim & \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B} W_{ii\alpha\alpha} - \sigma_{l-1}(G)\sum_{i,j\in B} W^2_{ij\alpha} \\ & -2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W^2_{ij\alpha} - A\sigma_l(G)\sum_{i,j\in B} W^2_{ij\alpha}].\end{aligned} \tag{3.2}$$

Since f is of homogeneous degree of -1, $\sum_{\alpha=1}^{n} F^{\alpha\alpha}W_{\alpha\alpha} = -\varphi$, we get

$$
\begin{aligned}
\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \quad \sim \quad & \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B}(W_{\alpha\alpha ii}+W_{ii}-W_{\alpha\alpha}) - \sigma_{l-1}(G)\sum_{\substack{i\in B\\ j\in G}} W_{ij\alpha}^2 \\
& -2\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)W_{ij\alpha}^2 - A\sigma_l(G)\sum_{i,j\in B} W_{ij\alpha}^2] \\
\sim \quad & \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B} W_{\alpha\alpha ii} + (n-l)\sigma_l(G)\varphi - \sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2 \\
& -2\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)W_{ij\alpha}^2 - A\sigma_l(G)\sum_{i,j\in B} W_{ij\alpha}^2].
\end{aligned}
\tag{3.3}
$$

Since W_{ijk} is symmetric respect to indices $\{ijk\}$ (which is used in the derivation from (2.22) to (2.23) in the proof of Theorem 1), as in (2.25), we reduce that

$$
\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \sim \sum_{i\in B}[\varphi_{ii}+\varphi] - \sum_{i\in B} I_i - \frac{A}{2}\sum_{\alpha=1}^{n}\sum_{i,j\in B} f^{\alpha} W_{ij\alpha}^2, \tag{3.4}
$$

where I_i defined similarly as in (2.25). Therefore, (3.1) follows from (2.26). The condition $(\varphi_{ij}+\varphi\delta_{ij}) \leq 0$ yields

$$
\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim 0. \tag{3.5}
$$

It follows from strong minimum principle that W is of constant rank in Ω. If $\Omega = \mathbb{S}^n$, the Minkowski integral formula implies W is of full rank (e.g., see an argument in [12]). $\square$

We now precede to treat curvature equation (1.4). Let W be the second fundamental form of M, equation (1.4) can be rewritten as

$$
F(W(X)) = \varphi(X,\vec{n}), \quad \forall X \in M. \tag{3.6}
$$

Proof of Theorem 2. We let $\tilde{\varphi}(X) = \varphi(X,\vec{n}(X))$. We work on second fundamental form $W = (h_{ij})$ in place of standard Hessian (u_{ij}) in the proof of Theorem 1.

As in the proof of Theorem 1, let $O \subset M$ be an open neighborhood of some point z_0 where the minimum rank of W is attained. For any $z \in O$, we choose a local orthonormal frame $\{e_A\}$ in the neighborhood of z in M with $\{e_1, e_2, ..., e_n\}$ tangent to M and $e_{n+1}(=\vec{n})$ is the normal so that the second fundamental form (W_{ij}) is diagonal at z, we divide eigenvalues of W at z into G and B, the "good" and "bad" sets of indices respectively. Set $\phi = \sigma_{l+1}(W) + A\sigma_{l+2}(W)$ as in (2.6). We want to show

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim \sum_{i\in B}\tilde{\varphi}_{ii}. \tag{3.7}$$

The same arguments in the proof of Theorem 1 yield (2.9)-(2.10) for $W=(h_{ij})$, and

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \quad \sim \quad & \sigma_l(G)\sum_{\alpha=1}^{n}\sum_{i\in B} F^{\alpha\alpha}W_{ii\alpha\alpha} - \sigma_{l-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2 \\ & -2\sum_{\alpha=1}^{n}\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)F^{\alpha\alpha}W_{ij\alpha}^2 - A\sigma_l(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2.\end{aligned} \tag{3.8}$$

It follows from the Gauss equation and (2.9) that

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \quad \sim \quad & \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sum_{i\in B}\sigma_l(G)(W_{\alpha\alpha ii} + W_{ii}W_{\alpha\alpha}^2 - W_{ii}^2 W_{\alpha\alpha}) \\ & -\sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2 - 2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W_{ij\alpha}^2 - A\sigma_l(G)\sum_{i,j\in B} W_{ij\alpha}^2] \\ \sim \quad & \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sum_{i\in B}\sigma_l(G)W_{\alpha\alpha ii} - \sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2 \\ & -2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W_{ij\alpha}^2 - A\sigma_l(G)\sum_{i,j\in B} W_{ij\alpha}^2].\end{aligned}$$

Since by Codazzi formula W_{ijk} is symmetric respect to indices $\{ijk\}$, as in (2.25), we reduce that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\varphi_{\alpha\alpha} \sim \sum_{i\in B}\tilde{\varphi}_{ii} - \sum_{i\in B} I_i - \frac{A}{s}\sum_{\alpha=1}^{n}\sum_{i,j\in B} f^{\alpha}W_{ij\alpha}^2, \tag{3.9}$$

where I_i defined similarly as in (2.25). Now (3.7) follows from (2.26) in the proof of Theorem 1.

We now compute $\tilde{\varphi}_{ii}$. $\forall \quad i\in\{1,2,...,n\}$,

$$\begin{aligned}\tilde{\varphi}(X)_i \quad &= \sum_{A=1}^{n+1}\varphi_{X_A}e_i^A + \varphi_{e_{n+1}}(e_{n+1})_i, \\ \tilde{\varphi}(X)_{ii} \quad &= \sum_{A,C=1}^{n+1}\varphi_{X_A X_C}e_i^A e_i^C + \sum_{A=1}^{n+1}\varphi_{X_A}X_{ii}^A + 2\sum_{A=1}^{n+1}\varphi_{X_A e_{n+1}}e_i^A(e_{n+1})_i \\ & +\varphi_{e_{n+1},e_{n+1}}(e_{n+1})_i(e_{n+1})_i + \varphi_{e_{n+1}}(e_{n+1})_{ii}.\end{aligned}$$

By the Gauss formula and the Weingarten formula for hypersurfaces, it follows that,

$$\sum_{i\in B}\tilde{\varphi}(X)_{ii} \backsim \sum_{i\in B}\sum_{A,C=1}^{n+1}\varphi_{X_A X_C} e_i^A e_i^C. \tag{3.10}$$

By our assumption on φ, we conclude that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim 0. \tag{3.11}$$

The strong minimum principle implies W is of constant rank l. If M is compact, there is at least one point that its second fundamental form is positive definite. Therefore it is positive definite everywhere and M is the boundary of some strongly convex bounded domain in $\mathbb{R}^{n+1}$. □

We note the proof of Theorem 3 is of local nature, there is a corresponding local statement of constant rank result for $W = (u_{ij} + u\delta_{ij})$ as in Theorem 2. If $\Omega = \mathbb{S}^n$, the condition on φ in Theorem 3 is equivalent to say that $\varphi(x)$ is concave in $\mathbb{R}^{n+1}$ after being extended as a homogeneous function of degree 1. Theorem 3 can deduce a positive upper bound on principal curvatures of M if it satisfies (1.5).

Corollary 2. *In addition to the conditions on F in Theorem 3, we assume that F is concave and*

$$\lim_{\lambda\to\partial\Psi} f(\lambda) = -\infty.$$

For any constant $\beta \geq 1$, there exist positive constants $\gamma > 0, \vartheta > 0$ such that if $0 > \varphi(x) \in C^{1,1}(\mathbb{S}^n)$ is a negative function with $\inf_{\mathbb{S}^n}(-\varphi) = 1, \|\varphi\|_{C^{1,1}(\mathbb{S}^n)} \leq \beta$, and $(\varphi_{ij} + (\varphi - \gamma)\delta_{ij}) \leq 0$ on $\mathbb{S}^n$, if u satisfies (1.5) on $\mathbb{S}^n$ with $(u_{ij} + u\delta_{ij}) \geq 0$, then $(u_{ij} + u\delta_{ij}) \geq \frac{1}{\vartheta} I$ on $\mathbb{S}^n$. That is, the principal curvature of convex hypersurface M with u as its support function is bounded from above by ϑ.

Proof of Corollary 2. We argue by contradiction. If the result is not true, for some $\beta \geq 1$, there are sequences functions $0 \geq \varphi^l \in C^{1,1}(\mathbb{S}^n)$ and $u^l \in C^2(\mathbb{S}^n)$, with $\sup_{\mathbb{S}^n}\varphi^l = -1$, $\|\varphi^l\|_{C^{1,1}(\mathbb{S}^n)} \leq \beta$, $(\varphi_{ij} + (\varphi - \frac{1}{l})\delta_{ij}) \leq 0$, $W^l = (u^l_{ij} + u^l\delta_{ij}) \geq 0$ on $\mathbb{S}^n$ and its minimum eigenvalue $\lambda^l_m(x_l) \leq \frac{1}{l}$ at some point $x_l \in \mathbb{S}^n$. Since equation (1.5) is invariant if we transfer $u(x)$ to $u(x) + \sum_{i=1}^{n+1} a_i x_i$, we may assume that

$$\int_{\mathbb{S}^n} u(x)x_j = 0, \quad \forall j = 1, \cdots, n+1.$$

It follows [10, 12, 13] that

$$\|u^l\|_{C^{1,1}(\mathbb{S}^n)} \leq C,$$

independent of l. By the assumption that

$$\lim_{\lambda\to\partial\Psi} f(\lambda) = -\infty,$$

W^l stay in a fixed compact subset of Ψ for all l, and F is uniformly elliptic. By the Evans-Krylov Theorem and Schauder theory,

$$\|u^l\|_{C^{2,\alpha}(\mathbb{S}^n)} \leq C,$$

independent of l. Therefore, there exist subsequences, we still denote φ_l and u^l,

$$\varphi_l \to \varphi \quad \text{in} \quad C^{1,\alpha}(\mathbb{S}^n), \quad u^l \to u \quad \text{in} \quad C^{3,\alpha}(\mathbb{S}^n),$$

for $0 > \varphi \in C^{1,1}(\mathbb{S}^n)$ with $\sup_{\mathbb{S}^n} \varphi = -1$, $(\varphi_{ij} + \varphi\delta_{ij}) \leq 0$ on $\mathbb{S}^n$, u satisfies equation (1.5) and the smallest eigenvalue of $(u_{ij}(x) + u(x)\delta_{ij})$ vanishes at some point x. On the other hand, Theorem 3 ensures $(u_{ij} + u\delta_{ij}) > 0$. This is a contradiction. □

We also have the corresponding consequence of Theorem 2

Corollary 3. *In addition to the conditions on f and F in Theorem 2, we assume that F is concave and*

$$\lim_{\lambda\to\partial\Psi} |f(\lambda)| = \infty.$$

For any constant $\beta \geq 1$, there exist positive constants $\gamma > 0, \vartheta > 0$ such that if $\|\varphi(x)\|_{C^{1,1}(\Gamma)} \leq \beta$, and $\varphi(X,p) - \gamma|X|^2$ is locally concave in X for any $p \in \mathbb{S}^n$ fixed, if M is a compact convex hypersurface satisfying (1.4) with $\|M\|_{C^2} \leq \beta$, then $\kappa_i(X) \geq \vartheta$ for all $X \in M$ and $i = 1, \cdots, n$.

The proof of Corollary 3 is similar to the proof of Corollary 2, we won't repeat it here.

4. Codazzi tensors on Riemannian manifolds

Let (M,g) be a Riemannian manifold, a symmetric 2-tensor W is call a Codazzi tensor if W is closed (viewed as a TM-valued 1-form). W is Codazzi if and only if

$$\nabla_X W(Y,Z) = \nabla_Y W(X,Z),$$

for all tangent vectors X, Y, Z, where ∇ is the Levi-Civita connection. In local orthonormal frame, the condition is equivalent to w_{ijk} is symmetric with respect to indices i, j, k. Codazzi tensors arise naturally from differential geometry. We refer Chapter 16 in [4] for general discussions on Codazzi tensors in Riemannian geometry. The followings are some important examples.

(1) The second fundamental form of a hypersurface is a Codazzi tensor, implied by the Codazzi equation.
(2) If (M,g) is a space form of constant curvature c, then for any $u \in C^\infty(M)$, $W_u = Hess(u) + cug$ is a Codazzi tensor.

(3) If (M, g) has harmonic Riemannian curvature, then the Ricci tensor Ric_g is a Coddazi tensor and its scalar curvature R_g is constant.
(4) If (M, g) has harmonic Weyl tensor, the Schouten tensor S_g is a Codazzi tensor.

The convexity principle we established in the previous sections can be generalized to Codazzi tensors on Riemannian manifolds. We first prove Theorem 4.

Proof of Theorem 4. The proof goes the similar way as in the proof of Proposition 3. We sketch here some necessary modifications.

We work on a small neighborhood of $z_0 \in M$ be a point where $W(z_0)$ is of minimum rank l. Set $\phi(x) = \sigma_{l+1}(W(x)) + A\sigma_{l+2}(W)$ as in (2.6) for $x \in O$. For any $z \in O \subset M$, we choose a local orthonormal frame so that at the point W is diagonal. As in the proof of Theorem 3, we may divide eigenvalues of W at z into G and B, the "good" and "bad" sets of indices respectively with $|G| = l, |B| = n - l$. As before, (2.9)-(2.10) hold for our Codazzi tensor W. We want to show that

$$\frac{1}{\sigma_l(G)} \sum_{\alpha=1}^{n} F^{\alpha\alpha} \phi_{\alpha\alpha} \lesssim \sum_{i \in B} [\varphi_{ii} + \tau\varphi]. \tag{4.1}$$

Our condition on φ implies $\frac{1}{\sigma_l(G)} \sum_{\alpha=1}^{n} F^{\alpha\alpha} \phi_{\alpha\alpha} \lesssim 0$. Theorem 4 would follow from the strong minimum principle.

The Codazzi condition implies W_{ijk} is symmetric. The same computation for ϕ in the proof of Theorem 1 deduces the same formula (3.2) for our Codazzi tensor W. It follows from Ricci identity, (2.9), (3.2) and homogeneity of F,

$$\begin{aligned}
\sum_{\alpha=1}^{n} F^{\alpha\alpha} \phi_{\alpha\alpha} \sim\ & \sum_{\alpha=1}^{n} F^{\alpha\alpha} [\sigma_l(G) \sum_{i \in B} (W_{\alpha\alpha ii} + R_{i\alpha i\alpha}(W_{ii} - W_{\alpha\alpha})) - \sigma_{l-1}(G) \sum_{i,j \in B} W_{ij\alpha}^2 \\
& -2 \sum_{\substack{i \in B \\ j \in G}} \sigma_{l-1}(G|j) W_{ij\alpha}^2 - A\sigma_l(G) \sum_{i,j \in B} W_{ij\alpha}^2] \\
\lesssim\ & \sum_{\alpha=1}^{n} F^{\alpha\alpha} [\sigma_l(G) \sum_{i \in B} (W_{\alpha\alpha ii} - \tau W_{\alpha\alpha}) - \sigma_{l-1}(G) \sum_{i,j \in B} W_{ij\alpha}^2 \\
& -2 \sum_{\substack{i \in B \\ j \in G}} \sigma_{l-1}(G|j) W_{ij\alpha}^2 - A\sigma_l(G) \sum_{i,j \in B} W_{ij\alpha}^2] \\
=\ & \sum_{\alpha=1}^{n} F^{\alpha\alpha} [\sigma_l(G) \sum_{i \in B} W_{\alpha\alpha ii} + (n-l)\tau\sigma_l(G)\varphi - \sigma_{l-1}(G) \sum_{i,j \in B} W_{ij\alpha}^2 \\
& -2 \sum_{\substack{i \in B \\ j \in G}} \sigma_{l-1}(G|j) W_{ij\alpha}^2 - A\sigma_l(G) \sum_{i,j \in B} W_{ij\alpha}^2].
\end{aligned} \tag{4.2}$$

As in (2.25), we reduce that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim \sum_{i\in B}[\varphi_{ii}+\tau\varphi] - \sum_{i\in B} I_i - \frac{A}{2}\sum_{\alpha=1}^{n}\sum_{i,j\in B} f^{\alpha}W_{ij\alpha}^2, \tag{4.3}$$

where I_i defined similarly as in (2.25). (4.1) now follows directly from (2.26). □

The homogeneity assumption in Theorem 4 was used in (4.2) in the above proof. If the sectional curvature is nonnegative, the homogeneity condition can be removed.

Proposition 1. *Let F as in Theorem 1, and (M,g) is a connected Riemannian manifold with nonnegative sectional curvature. Suppose $\varphi \in C^2(M)$ with $Hess(\varphi)(x) \le 0$ for every $x \in M$. If W is a semi-positive definite Codazzi tensor on M satisfying equation*

$$F(g^{-1}W) = \varphi \quad \text{on } M, \tag{4.4}$$

then W is of constant rank.

Proof. The proposition follows the same lines of arguments in the proof of Theorem 4. We deduce immediately from the first line in (4.2) that

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \ \lesssim\ & \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B} W_{\alpha\alpha ii} - \sum_{i\in B,\alpha\in G} R_{i\alpha i\alpha}W_{\alpha\alpha} - \sigma_{l-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2 \\ & -2\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)W_{ij\alpha}^2 - A\sigma_l(G)\sum_{i,j\in B} W_{ij\alpha}^2].\end{aligned} \tag{4.5}$$

We note that

$$\sum_{i\in B,\alpha\in G} R_{i\alpha i\alpha}W_{\alpha\alpha} \ge 0.$$

Then the same argument as in the proof of Theorem 4 yields $\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim 0$. □

Corollary 4. *Suppose (M,g) is a connected Riemannian manifold with nonnegative harmonic Riemannian curvature, then the Ricci tensor is of constant rank. If the* inf *of the smallest eigenvalue or the* sup *of the largest eigenvalue of Ric_g is attained in M, then it must be a constant and its eigenspace is of constant rank. Moreover, if in addition (M,g) has positive harmonic curvature at some point, then (M,g) is Einstein.*

Proof. Since (M,g) has nonnegative harmonic Riemannian curvature, Ric_g is a Codazzi tensor and it is semi-positive definite and the scalar curvature R_g is constant. Let $W = Ric_g$ and $F(W) = \sigma_1(W)$. W satisfies

$$F(g^{-1}W) = c. \tag{4.6}$$

The Corollary 4 now follows from Proposition 1.

Let $\lambda_s(x)$ be the smallest eigenvalue of Ric_g at x, if $\inf_{x\in M}\lambda_s(x)=\lambda_s(x_0)=a$ is attained at some point x_0. Define $W=Ric_g-ag$, then W is a semi-positive definite Codazzi tensor satisfies equation

$$\sigma_1(g^{-1}W)=c-na. \tag{4.7}$$

Proposition 1 implies $\lambda_s(x)=a$ for every $x\in M$, and null space of W is of constant rank. Similarly, if the sup of largest eigenvalue $\lambda_l(x)$ of $Ric_g(x)$ is attained at some point y_0, the similar conclusion follows by considering $W=\lambda_l(y_0)g-Ric_g$.

Suppose (M,g) has positive harmonic curvature at some point x_0. If $\inf_{x\in M}\lambda_s(x)$ is attained, by previous statement we know $\lambda_s(x)$ is constant in M. Let $W=Ric_g-ag$. If W does not vanish identically in a small neighborhood O of x_0 (i.e., $G\neq\emptyset$), then $\sigma_1(g^{-1}W)$ is a positive constant. W satisfies

$$F(W)=\varphi,$$

where $F(W)=\sigma_1(W)$ and $\varphi=c$. The proof of Proposition 1 yields (note that $\nu(x)=\min_{i\neq\alpha}R_{i\alpha i\alpha}(x)>0$ by the positive harmonic curvature assumption),

$$\frac{1}{\sigma_l(W)}\sum_{\alpha=1}^{n}F^{\alpha\alpha}\phi_{\alpha\alpha}\lesssim-\nu\sum_{\alpha\in G}W_{\alpha\alpha}<0. \tag{4.8}$$

This is a contradiction to the strong minimum principle. The similar argument applies if the sup of the largest eigenvalue $\lambda_l(x)$ of Ric_g is attained at some point y_0, by considering $W=\lambda_l(y_0)g-Ric_g$. □

In a special case $n=3$, a metric has a harmonic curvature is equivalent to the vanishing of Cotton tensor, in turn it is equivalent to the locally conformally flatness. The condition of nonnegative harmonic Riemannian curvature in Corollary 4 can be weakened and the result can be strengthened.

Corollary 5. *Suppose (M,g) is a connected 3-dimensional Riemannian manifold with harmonic Riemannian curvature, if the Ricci tensor is nonnegative, then it is of constant rank. If in addition if the smallest or the largest eigenvalue of Ricci tensor is attained in M, then the Ricci tensor is parallel and (M,g) is locally isometric to either S^3_r, $\mathbb{R}^3$ or $S^2_r\times\mathbb{R}$ for some $r>0$.*

Proof. Let $W=Ric_g$, and l is the minimal rank of W in M. Then (4.5) holds. When $n=3$ and Ric_g is nonnegative, if we make R_{ij} diagonal at the point and arrange that $0\le R_{11}\le R_{22}\le R_{33}$, then we have $R_{1212}\le R_{1313}\le R_{2323}$ and $R_{1313}\ge 0$. With these relations, it is straightforward to check that

$$\sum_{i\in B,\alpha\in G}R_{i\alpha i\alpha}W_{\alpha\alpha}\ge 0. \tag{4.9}$$

The same argument in the proof of Proposition 1 yields $\sigma_{l+1}(W)\equiv 0$.

If $\inf_{x\in M}\lambda_s(x)=\lambda_s(x_0)=a$ is attained at some point x_0. Let $W=Ric_g-ag$, (4.9) and the proof of Corollary 4 yields that the rank l of W is constant and $0\le l\le 2$ since B is no empty. In view of (4.5) and (4.9), we must have $R_{i\alpha i\alpha}=0$ for all $i\in B,\alpha\in G$. Since $n=3$, we must have either $l=0$, $l=2$. If $l=2$, we have $R_{1212}=R_{1313}=0$, so $R_{22}=R_{33}=R_{2323}=R=constant$ where R is the scalar curvature. We deduce from this fact together with (4.5) that Ric_g is parallel and M is locally isometric to $S^2_r\times\mathbb{R}$ for some $r>0$. For the case $l=0$, then M is Einstein, since $n=3$ it has constant sectional curvature so (M,g) is locally isometric to either S^3_r or $\mathbb{R}^3$.

Finally, the case the sup of the largest eigenvalue $\lambda_l(x)$ of Ric_g is attained can be treated similar way by considering $W=\lambda_l(y_0)g-Ric_g$. □

The same argument also works for manifolds with non-positive harmonic curvature.

Proposition 2. *Suppose (M,g) is a connected Riemannian manifold with non-positive harmonic Riemannian curvature, then the Ricci tensor is of constant rank.*

Proof. We work on $W=-Ric_g$. Since (M,g) has non-positive harmonic Riemannian curvature, Ric_g is a Codazzi tensor and it is semi-negative definite and the scalar curvature R_g is constant. So W is semi-positive definite and $\sigma_1(g^{-1}W)=c$ is a nonnegative constant. Let $F(W)=\sigma_1(W)$. W satisfies

$$F(g^{-1}W)=c. \tag{4.10}$$

Suppose $z_0\in M$ is the point where W attains the minimal rank l. We choose a small neighborhood O of z_0, set $\phi(x)=\sigma_{l+1}(W(x))+A\sigma_{l+2}(W(x))$ for $x\in O$ as in (2.6). For any $z\in O$, we choose a local orthonormal frame so that at the point W is diagonal. As in the proof of Theorem 3, we may divide eigenvalues of W at z into G and B, the "good" and "bad" sets of indices respectively with $|G|=l,|B|=n-l$. As before, the proposition will follow, if we can show

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n}F^{\alpha\alpha}\phi_{\alpha\alpha}\lesssim 0. \tag{4.11}$$

Following the same computation in the proof of Theorem 1, since W is diagonal at the point, it follows from Ricci identity, (2.9) and (3.2),

$$\begin{aligned}\sum_{\alpha=1}^{n}F^{\alpha\alpha}\phi_{\alpha\alpha}\ \sim\ & \sum_{\alpha=1}^{n}F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B}(W_{\alpha\alpha ii}+R_{i\alpha i\alpha}(W_{ii}-W_{\alpha\alpha}))-\sigma_{l-1}(G)\sum_{i,j\in B}W^2_{ij\alpha}\\ & -2\sum_{\substack{i\in B\\ j\in G}}\sigma_{l-1}(G|j)W^2_{ij\alpha}-A\sigma_l(G)\sum_{i,j\in B}W^2_{ij\alpha}].\end{aligned} \tag{4.12}$$

Since $R_{i\alpha i\alpha} \leq 0$, we have $|R_{i\alpha i\alpha}| \leq W_{ii}$. Again by (2.9), (4.12) becomes

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} &\lesssim \sum_{\alpha=1}^{n} F^{\alpha\alpha}[\sigma_l(G)\sum_{i\in B} W_{ii\alpha\alpha} - \sigma_{l-1}(G)\sum_{i,j\in B} W^2_{ij\alpha} \\ &\quad -2\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j)W^2_{ij\alpha} - A\sigma_l(G)\sum_{i,j\in B} W^2_{ij\alpha}]. \end{aligned} \tag{4.13}$$

As in (2.25), we reduce that

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim \sum_{i\in B}\varphi_{ii} - \sum_{i\in B} I_i - \frac{A}{2}\sum_{\alpha\in G}\sum_{i,j\in B} f^{\alpha} W^2_{ij\alpha}, \tag{4.14}$$

where I_i defined similarly as in (2.25) and $\varphi = c$. (4.11) now follows directly from (2.26). □

Remark 6. *Though we only consider the Codazzi tensors here, all the results in this section remain valid (under the same assumptions on the Riemannian sectional curvature) for any symmetric 2-tensor W satisfying the Ricci identity*

$$W_{ii\alpha\alpha} = W_{\alpha\alpha ii} + R_{i\alpha i\alpha}(W_{ii} - W_{\alpha\alpha}).$$

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