GLOBAL C^2 ESTIMATES FOR CONVEX SOLUTIONS OF CURVATURE EQUATIONS

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Abstract. We establish C^2 a priori estimate for convex hypersurfaces whose principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_n)$ satisfying Weingarten curvature equation $\sigma_k(\kappa(X)) = f(X, \nu(X))$. We also obtain such estimate for admissible 2-convex hypersurfaces in the case $k = 2$. Our estimates resolve a longstanding problem in geometric fully nonlinear elliptic equations considered in [3, 19, 20, 14]

1. introduction

This paper concerns a longstanding problem of the global C^2 estimates for curvature equation in general form

$$\sigma_k(\kappa(X)) = f(X, \nu(X)), \quad \forall X \in M, \tag{1.1}$$

where σ_k is the kth elementary symmetric function, $\nu(X), \kappa(X)$ are the outer-normal and principal curvatures of hypersurface $M \subset \mathbb{R}^{n+1}$ at X respectively. $\sigma_k(\kappa)$, $k = 1, \cdots, n$, are the Weingarten curvatures of the hypersurface M. In the cases $k = 1, 2$ and n, they are the mean curvature, scalar curvature and Gauss curvature respectively.

Equation (1.1) is associated with many important geometric problems. The Minkowski problem ([21, 22, 23, 9]), the problem of prescribing general Weingarten curvature on outer normals by Alexandrov [3, 13], the problem of prescribing curvature measures in convex geometry [2, 22, 15, 14]), the prescribing curvature problem considered [4, 24, 8], all these geometric problems fall into equation (1.1) with special form of f respectively. Equation (1.1) has been studied extensively, it is a special type of general equations systemically studied by Alexandrov in [3]. C^2 estimates are known in many special cases. When $k = 1$, equation (1.1) is quasilinear, C^2 estimate follows from the classical theory of quasilinear PDE. The equation is of Monge-Ampère type if $k = n$, C^2 estimate in this case for general $f(X, \nu)$ is due to Caffarelli-Nirenberg-Spruck [6]. When f is independent of normal vector ν, C^2 estimate has been proved by Caffralli-Nirenberg-Spruck [8] for a general class of fully nonlinear operators F, including $F = \sigma_k, F = \frac{\sigma_k}{\sigma_l}$. If f in (1.1) depends only on ν, C^2 estimate was proved in [13]. Ivochkina [19, 20] considered the Dirichlet problem of equation (1.1) on domains in $\mathbb{R}^n$, C^2 estimate was proved there under some extra conditions on the dependence of f on ν. C^2 estimate was also proved for equation of prescribing curvature measures problem in [15, 14], where $f(X, \nu) = \langle X, \nu \rangle \tilde{f}(X)$. It is of great interest, both in

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geometry and in PDE, to establish C^2 estimate for equation (1.1) for $1 < k < n$ and for general $f(X,\nu)$.

C^2 estimates for equation (1.1) is equivalent to the curvature estimates from above for $\kappa_1, \cdots, \kappa_n$. We state the main results of this paper.

Theorem 1. *Suppose $M \subset \mathbb{R}^{n+1}$ is a closed convex hypersurface satisfying curvature equation (1.1) for some positive function $f(X,\nu) \in C^2(\Gamma)$, where Γ is an open neighborhood of unit normal bundle of M in $\mathbb{R}^{n+1} \times \mathbb{S}^n$, then there is a constant C depending only on n, k, $\|M\|_{C^1}$, $\inf f$ and $\|f\|_{C^2}$, such that*

$$\max_{X\in M, i=1,\cdots,n} \kappa_i(X) \le C. \tag{1.2}$$

Estimate (1.2) is special to equation (1.1). One may ask if estimate (1.2) can be generalized to this type of curvature equations when f depends on (X,ν) as in (1.1). The answer is *no* in general.

Theorem 2. *For each $1 \le l < k \le n$, there exist $C > 0$ and a sequence of smooth positive functions $f_t(X,\nu)$ with*

$$\|f_t\|_{C^3(\mathbb{R}^{n+1}\times\mathbb{S}^n)} + \|\frac{1}{f_t}\|_{C^3(\mathbb{R}^{n+1}\times\mathbb{S}^n)} \le C,$$

and a sequence of strictly convex hypersurface $M_t \subset \mathbb{R}^{n+1}$ with $\|M_t\|_{C^1} \le C$ satisfying quotient of curvatures equation

$$\frac{\sigma_k}{\sigma_l}(\kappa) = f_t(X,\nu), \tag{1.3}$$

such that estimate (1.2) fails.

It is desirable to drop the convexity assumption in Theorem 1. In the case of scalar curvature equation ($k = 2$), we establish estimate (1.2) for starshaped admissible solutions of equation (1.1). The general case $2 < k < n$ is still open.

Following [7], we define

Definition 3. *For a domain $\Omega \subset \mathbb{R}^n$, a function $v \in C^2(\Omega)$ is called k-convex if the eigenvalues $\lambda(x) = (\lambda_1(x), \cdots, \lambda_n(x))$ of the hessian $\nabla^2 v(x)$ is in Γ_k for all $x \in \Omega$, where Γ_k is the Garding's cone*

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \quad \sigma_m(\lambda) > 0, \quad m = 1, \cdots, k\}.$$

A C^2 regular hypersurface $M \subset \mathbb{R}^{n+1}$ is k-convex if $\kappa(X) \in \Gamma_k$ for all $X \in M$.

Theorem 4. *Suppose $k = 2$ and suppose $M \subset \mathbb{R}^{n+1}$ is a closed strictly starshaped 2-convex hypersurface satisfying curvature equation (1.1) for some positive function $f(X,\nu) \in C^2(\Gamma)$, where Γ is an open neighborhood of unit normal bundle of M in $\mathbb{R}^{n+1} \times \mathbb{S}^n$, then there is a constant C depending only on n, k, $\|M\|_{C^1}$, $\inf f$ and $\|f\|_{C^2}$, such that*

$$\max_{X\in M, i=1,\cdots,n} \kappa_i(X) \le C. \tag{1.4}$$

Theorem 1 and Theorem 4 are stated for compact hypersurfaces, the corresponding estimates hold for solutions of equation (1.1) with boundary conditions, with C in the right hand side of (1.2) and (1.4) depending C^2 norm on the boundary in addition.

The proof of above two theorems relies on maximum principles for appropriate curvature functions. The novelty of this paper is the discovery of some new test curvature functions. They are nonlinear in terms of the principal curvatures with some good convexity properties.

With appropriate barrier conditions on function f, one may establish existence results of the prescribing curvature problem (1.1) in general.

Theorem 5. *Suppose $f \in C^2(\mathbb{R}^{n+1} \times \mathbb{S}^n)$ is a positive function and suppose there is a constant $r > 1$ such that,*

$$f(X, \frac{X}{|X|}) \leqslant \frac{\sigma_k(1, \cdots, 1)}{r^k} \quad \textit{for} \quad |X| = r, \tag{1.5}$$

and $f^{-1/k}(X, \nu)$ is a locally convex in $X \in B_r(0)$ for any fixed $\nu \in \mathbb{S}^n$, then equation (1.1) has a strictly convex $C^{3,\alpha}$ solution inside $\bar{B}_r$.

To state a corresponding existence result for 2-convex solutions of the prescribed scalar curvature equation (1.1), we need further barrier conditions on the prescribed function f as considered in [4, 24, 8]. We denote $\rho(X) = |X|$.

We assume that

Condition (1). There are two positive constant $r_1 < 1 < r_2$ such that

$$\begin{cases} f(X, \frac{X}{|X|}) & \geqslant & \frac{\sigma_k(1, \cdots, 1)}{r_1^k}, & \text{for } |X| = r_1, \\ f(X, \frac{X}{|X|}) & \leqslant & \frac{\sigma_k(1, \cdots, 1)}{r_2^k}, & \text{for } |X| = r_2. \end{cases} \tag{1.6}$$

Condition (2). For any fixed unit vector ν,

$$\frac{\partial}{\partial \rho}(\rho^k f(X, \nu)) \leqslant 0, \quad \text{where } |X| = \rho. \tag{1.7}$$

Theorem 6. *Suppose $k = 2$ and suppose positive function $f \in C^2(\bar{B}_{r_2} \setminus B_{r_1} \times \mathbb{S}^n)$ satisfies conditions (1.6) and (1.7), then equation (1.1) has a unique $C^{3,\alpha}$ starshaped solution M in $\{r_1 \leq |X| \leq r_2\}$.*

The organization of the paper is as follow. As an illustration, we give a short proof of C^2 estimate for σ_2-Hessian equation on $\mathbb{R}^2$ in Section 2. Theorem 4 and Theorem 1 are proved in Section 3 and Section 4 respectively. Section 5 is devoted to various existence theorems. Construction of examples of convex hypersurfaces stated in Theorem 2 appears in Section 6.

2. The Hessian equation for $k = 2$

To begin this section, we list one lemma which is well known (e.g., Theorem 5.5 in [5], it was also originally stated in a preliminary version of [7] and was lately removed from the published version).

Lemma 7. *Denote $Sym(n)$ the set of all $n \times n$ symmetric matrices. Let F be a C^2 symmetric function defined in some open subset $\Psi \subset Sym(n)$. At any diagonal matrix $A \in \Psi$ with distinct eigenvalues, let $\ddot{F}(B,B)$ be the second derivative of C^2 symmetric function F in direction $B \in Sym(n)$, then*

$$\ddot{F}(B,B) = \sum_{j,k=1}^{n} \ddot{f}^{jk} B_{jj} B_{kk} + 2 \sum_{j<k} \frac{\dot{f}^j - \dot{f}^k}{\lambda_j - \lambda_k} B_{jk}^2. \tag{2.1}$$

We use standard notation. We let $\kappa(A)$ be eigenvalues of the matrix $A = (a_{ij})$. For equation

$$F(A) = F(\kappa(A)),$$

we define

$$F^{pq} = \frac{\partial F}{\partial a_{pq}}, \text{ and } F^{pq,rs} = \frac{\partial^2 F}{\partial a_{pq} \partial a_{rs}}.$$

For a local orthonormal frame, if A is diagonal at a point, then at this point,

$$F^{pp} = \frac{\partial f}{\partial \kappa_p} = f_p, \text{ and } F^{pp,qq} = \frac{\partial^2 f}{\partial \kappa_p \partial \kappa_q} = f_{pq}.$$

The following facts regarding σ_k will be used throughout this paper, their proof can be found in [17].
(i) $\sigma_k^{pp,pp} = 0$, and $\sigma_k^{pp,qq}(\kappa) = \sigma_{k-2}(\kappa|pq)$;
(ii) $\sigma_k^{pq,rs} h_{pql} h_{rsl} = \sigma_k^{pp,qq} h_{pql}^2 - \sigma_k^{pp,qq} h_{ppl} h_{qql}$.

In what follows, we consider σ_2-Hessian equations in a domain $\Omega \subset \mathbb{R}^{n+1}$:

$$\begin{cases} \sigma_2[D^2 u] &= f(x,u,Du), \\ u|_{\partial\Omega} &= \phi. \end{cases} \tag{2.2}$$

We believe C^2 estimates for equation (2.2) is known. Since we are not able to find any reference in the literature, a proof is produced here to serve as an illustration.

For a symmetric 2-tensor W on a Riemannian manifold (M,g) is call a Codazzi tensor if W is closed (viewed as a TM-valued 1-form). W is Codazzi if and only if

$$\nabla_X W(Y,Z) = \nabla_Y W(X,Z),$$

for all tangent vectors X, Y, Z, where ∇ is the Levi-Civita connection. In local orthonormal frame, the condition is equivalent to w_{ijk} is symmetric with respect to indices i, j, k. Hessian $\nabla^2 u$ of a function $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, is Codazzi. It is well known that the second fundamental form of a hypersurface in $\mathbb{R}^{n+1}$ is a Codazzi tensor by the Codazzi equation.

We need following lemma which is a slightly improvement of Lemma 1 in [14].

Lemma 8. *Assume that $k>l$, $W=(w_{ij})$ is a Codazzi tensor which is in Γ_k. Denote $\alpha=\dfrac{1}{k-l}$. Then, for $h=1,\cdots,n$, we have the following inequality,*

$$\begin{aligned}
(2.3)\qquad & -\frac{\sigma_k^{pp,qq}}{\sigma_k}(W)w_{pph}w_{qqh}+\frac{\sigma_l^{pp,qq}}{\sigma_l}(W)w_{pph}w_{qqh}\\
\geqslant\ & \left(\frac{(\sigma_k(W))_h}{\sigma_k(W)}-\frac{(\sigma_l(W))_h}{\sigma_l(W)}\right)\left((\alpha-1)\frac{(\sigma_k(W))_h}{\sigma_k(W)}-(\alpha+1)\frac{(\sigma_l(W))_h}{\sigma_l(W)}\right).
\end{aligned}$$

Furthermore, for any $\delta>0$,

$$\begin{aligned}
(2.4)\qquad & -\sigma_k^{pp,qq}(W)w_{pph}w_{qqh}+(1-\alpha+\frac{\alpha}{\delta})\frac{(\sigma_k(W))_h^2}{\sigma_k(W)}\\
\geqslant\ & \sigma_k(W)(\alpha+1-\delta\alpha)\left[\frac{(\sigma_l(W))_h}{\sigma_l(W)}\right]^2-\frac{\sigma_k}{\sigma_l}(W)\sigma_l^{pp,qq}(W)w_{pph}w_{qqh}.
\end{aligned}$$

Proof. Define a function

$$\ln F=\ln(\frac{\sigma_k}{\sigma_l})^{1/(k-l)}=\frac{1}{k-l}\ln\sigma_k-\frac{1}{k-l}\ln\sigma_l.$$

Differentiate it twice,

$$\frac{F^{pp}}{F}=\frac{1}{k-l}\frac{\sigma_k^{pp}}{\sigma_k}-\frac{1}{k-l}\frac{\sigma_l^{pp}}{\sigma_l},$$

$$\frac{F^{pp,qq}}{F}-\frac{F^{pp}F^{qq}}{F^2}=\frac{1}{k-l}\frac{\sigma_k^{pp,qq}}{\sigma_k}-\frac{1}{k-l}\frac{\sigma_k^{pp}\sigma_k^{qq}}{\sigma_k^2}-\frac{1}{k-l}\frac{\sigma_l^{pp,qq}}{\sigma_l}+\frac{1}{k-l}\frac{\sigma_l^{pp}\sigma_l^{qq}}{\sigma_l^2}.$$

Using previous two equalities,

$$\frac{1}{\alpha}\frac{F^{pp,qq}}{F}=\alpha\left(\frac{\sigma_k^{pp}}{\sigma_k}-\frac{\sigma_l^{pp}}{\sigma_l}\right)\left(\frac{\sigma_k^{qq}}{\sigma_k}-\frac{\sigma_l^{qq}}{\sigma_l}\right)+\left(\frac{\sigma_k^{pp,qq}}{\sigma_k}-\frac{\sigma_k^{pp}\sigma_k^{qq}}{\sigma_k^2}-\frac{\sigma_l^{pp,qq}}{\sigma_l}+\frac{\sigma_l^{pp}\sigma_l^{qq}}{\sigma_l^2}\right).$$

By the concavity of F, $(F^{pp,qq})\leq 0$. Together with the above identity,

$$-\frac{\sigma_k^{pp,qq}}{\sigma_k}+\frac{\sigma_l^{pp,qq}}{\sigma_l}\geqslant\left(\frac{\sigma_k^{pp}}{\sigma_k}-\frac{\sigma_l^{pp}}{\sigma_l}\right)\left((\alpha-1)\frac{\sigma_k^{qq}}{\sigma_k}-(\alpha+1)\frac{\sigma_l^{qq}}{\sigma_l}\right).$$

Here the meaning of " $\geqslant$ " is for comparison of symmetric matrices. Hence, for each h with $(w_{11h},\cdots,w_{nnh})$, we obtain (2.3). (2.4) follows from (2.3) and the Schwarz inequality. □

Proposition 9. *Suppose $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary. Suppose $f(p,u,x)\in C^2(\mathbb{R}^n\times\mathbb{R}\times\bar{\Omega})$ is a positive function. The Dirichlet problem (2.2) has a global C^2 bound depending on the C^1 bound of u, the domain Ω and the C^1 bound of f.*

Proof. Consider

$$\phi=\max_{|\xi|=1,x\in\Omega}\exp\{\frac{\varepsilon}{2}|Du|^2+\frac{a}{2}|x|^2\}u_{\xi\xi},$$

where ε and a are to be determined later. Suppose that the maximum of ϕ is achieved at some point x_0 in Ω along some direction ξ. We may assume that $\xi=(1,0,\cdots,0)$. Rotating the coordinates if necessary, we may assume the matrix (u_{ij}) is diagonal, and $u_{11}\geqslant u_{22}\cdots\geqslant u_{nn}$ at the point.

We differentiate the function $\log\phi$ twice at x_0,

$$\frac{u_{11i}}{u_{11}}+\varepsilon u_i u_{ii}+ax_i=0, \tag{2.5}$$

and

$$\frac{u_{11ii}}{u_{11}}-\frac{u_{11i}^2}{u_{11}^2}+\sum_k \varepsilon u_k u_{kii}+\varepsilon u_{ii}^2+a\leqslant 0. \tag{2.6}$$

Contract with the matrix $\sigma_2^{ii}u_{11}$,

$$\sigma_2^{ii}u_{11ii}-\sigma_2^{ii}\frac{u_{11i}^2}{u_{11}}+u_{11}\sum_k \varepsilon u_k\sigma_2^{ii}u_{kii}+u_{11}\varepsilon\sigma_2^{ii}u_{ii}^2+a\sum_i\sigma_2^{ii}u_{11}\leqslant 0. \tag{2.7}$$

At x_0, differentiate equation (1.1) twice,

$$\sigma_2^{ii}u_{iij} \quad = \quad f_j+f_u u_j+f_{p_j}u_{jj}, \tag{2.8}$$

and

$$\begin{aligned} &\sigma_2^{ii}u_{iijj}+\sigma_2^{pq,rs}u_{pqj}u_{rsj} \\ =\ & f_{jj}+2f_{ju}u_j+2f_{jp_j}u_{jj}+f_{uu}u_j^2+2f_{up_j}u_ju_{jj}+f_uu_{jj}+f_{p_jp_j}u_{jj}^2+\sum_k f_{p_k}u_{kjj}. \end{aligned} \tag{2.9}$$

Choose $j=1$ in the above equation, and insert (2.9) into (2.7),

$$\begin{aligned} 0\geqslant &-C-Cu_{11}+f_{p_1p_1}u_{11}^2+\sum_k f_{p_k}u_{k11}-\sigma_2^{pq,rs}u_{pq1}u_{rs1} \\ &-\sigma_2^{ii}\frac{u_{11i}^2}{u_{11}}+u_{11}\sum_k\varepsilon u_k\sigma_2^{ii}u_{kii}+u_{11}\varepsilon\sigma_2^{ii}u_{ii}^2+a\sum_i\sigma_2^{ii}u_{11}. \end{aligned}$$

Use (2.5) and (2.8),

$$\sum_k f_{p_k}u_{k11}+u_{11}\sum_k\varepsilon u_k\sigma_2^{ii}u_{kii}=u_{11}\sum_k(\varepsilon u_kf_k+\varepsilon f_uu_k^2-ax_kf_{p_k}).$$

Then

$$\begin{aligned} 0 \quad \geqslant \quad & -C-Cu_{11}+f_{p_1p_1}u_{11}^2-\sum_{p\neq r}u_{pp1}u_{rr1}+\sum_{p\neq q}u_{pq1}^2-\sigma_2^{ii}\frac{u_{11i}^2}{u_{11}} \\ & +u_{11}\varepsilon\sigma_2^{ii}u_{ii}^2+(n-1)au_{11}\sum_k u_{kk}. \end{aligned} \tag{2.10}$$

Choose $k=2,l=1$ and $h=1$ in Lemma 8, we have,

$$-\sum_{p\neq r}u_{pp1}u_{rr1}+(1-\alpha+\frac{\alpha}{\delta})\frac{(\sigma_2)_1^2}{\sigma_2} \quad\geqslant\quad (\alpha+1-\delta\alpha)\sigma_2[\frac{(\sigma_1)_1}{\sigma_1}]^2 \quad\geqslant\quad 0.$$

Inequality (2.10) becomes,

$$
\begin{aligned}
(2.11) \qquad 0 &\geqslant -C - Cu_{11} + f_{p_1p_1}u_{11}^2 + (n-1)au_{11}^2 - C(\sigma_2)_1^2 \\
&\quad + u_{11}\varepsilon\sigma_2^{ii}u_{ii}^2 + 2\sum_{k\neq 1}u_{11k}^2 - \sigma_2^{ii}\frac{u_{11i}^2}{u_{11}} \\
&\geqslant ((n-1)a - C_0)u_{11}^2 + u_{11}\varepsilon\sigma_2^{ii}u_{ii}^2 + 2\sum_{k\neq 1}u_{11k}^2 - \sigma_2^{ii}\frac{u_{11i}^2}{u_{11}},
\end{aligned}
$$

where we have used (2.5) and the Schwarz inequality. We claim that if a is chosen large enough that

$$(n-1)a - C_0 \geqslant 1,$$

then

$$u_{11}\varepsilon\sigma_2^{ii}u_{ii}^2 + 2\sum_{k\neq 1}u_{11k}^2 - \sigma_2^{ii}\frac{u_{11i}^2}{u_{11}} \geqslant 0. \tag{2.12}$$

Inequality (2.11) then yield an upper bound of u_{11}.

We prove the claim (2.12). We may assume that u_{11} is sufficient large. By (2.5) and the Schwarz inequality,

$$\sigma_2^{11}u_{11}\varepsilon u_{11}^2 - \sigma_2^{11}\frac{u_{111}^2}{u_{11}} \geqslant \sigma_2^{11}u_{11}(\varepsilon u_{11}^2 - 2\varepsilon^2u_1^2u_{11}^2 - 2a^2x_1^2). \tag{2.13}$$

If we require

$$\varepsilon \geqslant 3\varepsilon^2\max_{\Omega}|\nabla u|^2, \tag{2.14}$$

and if u_{11} sufficient large, (2.13) is nonnegative. As in [10], we divide it into two different cases. Denote $\lambda_i = u_{ii}$.

(A) $\sum_{i=2}^{n-1}\lambda_i \leqslant \lambda_1$. In this case, for $i \neq 1$, since $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, and $\sigma_2^{ii} = \sigma_1 - \lambda_i$,

$$2u_{11} \geqslant \sigma_2^{nn} \geqslant \sigma_2^{ii}.$$

Hence,

$$2\sum_{k\neq 1}u_{11k}^2 - \sum_{i\neq 1}\sigma_2^{ii}\frac{u_{11i}^2}{u_{11}} \geqslant 0.$$

Combined with (2.13), we obtain (2.12).

(B) $\sum_{i=2}^{n-1}\lambda_i \geqslant \lambda_1$, then $\frac{\lambda_1}{n-2} \leqslant \lambda_2 \leqslant \lambda_1$ and $\sigma_2^{nn} \geqslant 2\lambda_1$. We further divide this case into two subcases.

(B1) Suppose $\sigma_2^{22} \geqslant 1$. Using (2.14), (2.5) and the Schwarz inequality,

$$\begin{aligned}
&u_{11}\varepsilon\sum_{i\neq 1}\sigma_2^{ii}u_{ii}^2 - \sum_{i\neq 1}\sigma_2^{ii}\frac{u_{11i}^2}{u_{11}}\\
=&\sigma_2^{22}u_{11}\varepsilon u_{22}^2 - \sigma_2^{22}\frac{u_{112}^2}{u_{11}} + \sum_{i>2}(\sigma_2^{ii}u_{11}\varepsilon u_{ii}^2 - \sigma_2^{ii}\frac{u_{11i}^2}{u_{11}})\\
\geqslant&\sigma_2^{22}u_{11}(\varepsilon u_{22}^2 - 2\varepsilon^2u_2^2u_{22}^2 - 2a^2x_2^2) + \sum_{i>2}\sigma_2^{ii}u_{11}(\varepsilon u_{ii}^2 - 2\varepsilon^2u_i^2u_{ii}^2 - 2a^2x_i^2)\\
\geqslant&\frac{1}{3}\sigma_2^{22}u_{11}(\varepsilon u_{22}^2 - C) - Cu_{11}\sum_{i>2}\sigma_2^{ii}\\
\geqslant&\frac{\varepsilon}{3}\lambda_1\lambda_2^2 - C\lambda_1^2\\
\geqslant&\frac{\varepsilon}{3(n-2)^2}\lambda_1^3 - C\lambda_1^2,
\end{aligned}$$

it is nonnegative if λ_1 is sufficient large. In view of (2.13), in this subcase, (2.12) holds.

(B2) Suppose $\sigma_2^{22} = \lambda_1 - \lambda_2 + \lambda_n + \sum_{i=2}^{n-1}\lambda_i < 1$. Again, we may assume that λ_1 is sufficient large, then,

$$-\lambda_n = u_{nn} \geqslant \lambda_1 - 1 \geqslant \frac{\lambda_1}{2}.$$

Hence, for ε sufficient small, we have,

$$\begin{aligned}
&u_{11}\varepsilon\sum_{i\neq 1}\sigma_2^{ii}u_{ii}^2 - \sum_{i\neq 1}\sigma_2^{ii}\frac{u_{11i}^2}{u_{11}}\\
=&\sigma_2^{nn}u_{11}\varepsilon u_{nn}^2 - \sigma_2^{nn}\frac{u_{11n}^2}{u_{nn}} + \sum_{1<i<n}(\sigma_2^{ii}u_{11}\varepsilon u_{ii}^2 - \sigma_2^{ii}\frac{u_{11i}^2}{u_{11}})\\
\geqslant&\frac{1}{3}\sigma_2^{nn}u_{11}(\varepsilon u_{nn}^2 - C) - Cu_{11}\sum_{1<i<n}\sigma_2^{ii}\\
\geqslant&\frac{\varepsilon}{6}\lambda_1^2(\lambda_1 - 1)^2 - C\lambda_1^2.
\end{aligned}$$

Here, the first inequality comes from (2.5) and the Schwarz inequality. The process is similar to the first and second inequalities in subcase (B1). The above quantity is nonnegative, if λ_1 is sufficient large. (2.12) follows from (2.13). $\square$

With the C^2 interior estimate, one may obtain a global C^2 estimate if the corresponding boundary estimate is in hand. This type of C^2 boundary estimates have been proved by Bo Guan in [12] under the assumption that Dirichlet problem (2.2) has a subsolution. Namely, there is a function $\underline{u}$, satisfying

$$\begin{cases}\sigma_2[D^2\underline{u}] & \geqslant & f(x,\underline{u},D\underline{u}),\\ \underline{u}|_{\partial\Omega} & = & \phi.\end{cases} \tag{2.15}$$

Theorem 10. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Suppose $f(p, u, x) \in C^2(\mathbb{R}^n \times \mathbb{R} \times \bar{\Omega})$ is a positive function with $f_u \geq 0$. Suppose there is a subsolution $\underline{u} \in C^3(\bar{\Omega})$ satisfying (2.15), then the Dirichlet problem (2.2) has a unique $C^{3,\alpha}, \forall 0 < \alpha < 1$ solution u.*

3. THE SCALAR CURVATURE EQUATION

We consider the global curvature estimates for solution to curvature equation (1.1) with $k = 2$, i.e. the prescribing scalar curvature equation in $\mathbb{R}^{n+1}$. In [11], a global curvature estimate was obtained for prescribing scalar curvature equation in Lorentzian manifolds, where some special properties of the spacelike hypersurfaces were used. It seems for equation (1.1) in $\mathbb{R}^{n1+}$, the situation is different. A new feature here is to consider a nonlinear test function $\log \sum_l e^{\kappa_l}$. We explore certain convexity property of this function, which will be used in a crucial way in our proof.

Set $u(X) = \langle X, \nu(X) \rangle$. By the assumption that M is starshaped with a C^1 bound, u is bounded from below and above by two positive constants. At every point in the hypersurface M, choose a local coordinate frame $\{\partial/(\partial x_1), \cdots, \partial/(\partial x_{n+1})\}$ in $\mathbb{R}^n$ such that the first n vectors are the local coordinates of the hypersurface and the last one is the unit outer normal vector. Denote ν to be the outer normal vector. We let h_{ij} and u be the second fundamental form and the support function of the hypersurface M respectively. The following geometric formulas are well known (e.g., [14]).

$$h_{ij} = \langle \partial_i X, \partial_j \nu \rangle, \tag{3.1}$$

and

$$\begin{array}{rl} X_{ij} = & -h_{ij}\nu \quad \text{(Gauss formula)} \\ (\nu)_i = & h_{ij}\partial_j \quad \text{(Weigarten equation)} \\ h_{ijk} = & h_{ikj} \quad \text{(Codazzi formula)} \\ R_{ijkl} = & h_{ik}h_{jl} - h_{il}h_{jk} \quad \text{(Gauss equation)}, \end{array} \tag{3.2}$$

where R_{ijkl} is the $(4,0)$-Riemannian curvature tensor. We also have

$$\begin{array}{rl} h_{ijkl} = & h_{ijlk} + h_{mj}R_{imlk} + h_{im}R_{jmlk} \\ = & h_{klij} + (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} + (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}. \end{array} \tag{3.3}$$

We need a more explicit version of Lemma 8 for $k = 2$ case.

Lemma 11. *Suppose $W = (w_{ij})$ is a Codazzi tensor which is in Γ_2. For $h = 1, \cdots, n$ and K large so that $\sigma_2 \geqslant \dfrac{1}{K}$, there exist universal constants α large and δ small, such that the following inequality holds,*

$$K(\sigma_2)_h^2 - \sum_{p \neq r} w_{pph}w_{rrh} - \delta w_{hh}\sigma_2^{hh}\frac{w_{hhh}^2}{\sigma_1^2} + \alpha \sum_{i \neq h} w_{iih}^2 \geqslant 0. \tag{3.4}$$

Proof. Consider function

$$Q = \frac{\sigma_2(W)}{\sigma_1(W)}.$$

We have,

$$\sigma_1 Q^{pp,qq} w_{pph} w_{qqh} = \sum_{p\neq q} w_{pph} w_{qqh} - \frac{2(\sigma_2)_h \sum_j w_{jjh}}{\sigma_1} + 2\frac{\sigma_2(\sum_j w_{jjh})^2}{\sigma_1^2}.$$

On the other hand, one may write (e.g. [18])

$$-Q^{pp,qq} w_{pph} w_{qqh} = \frac{\sum_i (w_{iih}\sigma_1 - w_{ii}\sum_k w_{kkh})^2}{\sigma_1^3}.$$

From the above two identities and the Schwartz inequality, with K, α large enough,

$$\begin{aligned}
&-\sum_{p\neq r} w_{pph} w_{rrh} \\
= \ & \frac{\sum_i (w_{iih}\sigma_1 - w_{ii}\sum_k w_{kkh})^2}{\sigma_1^2} - \frac{2(\sigma_2)_h \sum_j w_{jjh}}{\sigma_1} + 2\frac{\sigma_2(\sum_j w_{jjh})^2}{\sigma_1^2} \\
\geqslant \ & \frac{\sigma_2(\sum_j w_{jjh})^2}{\sigma_1^2} - K(\sigma_2)_h^2 + \frac{(w_{hhh}\sigma_1 - w_{hhh}w_{hh} - w_{hh}\sum_{k\neq h} w_{kkh})^2}{\sigma_1^2} \\
&+ \frac{\sum_{i\neq h}(w_{iih}\sigma_1 - w_{ii}w_{hhh} - w_{ii}\sum_{k\neq h} w_{kkh})^2}{\sigma_1^2} \\
\geqslant \ & \frac{\sigma_2(w_{hhh})^2}{\sigma_1^2} - K(\sigma_2)_h^2 + \frac{(w_{hhh}\sigma_2^{hh})^2}{2\sigma_1^2} + \frac{w_{hhh}^2 \sum_{i\neq h} w_{ii}^2}{2\sigma_1^2} - \alpha \sum_{i\neq h} w_{iih}^2.
\end{aligned} \tag{3.5}$$

By (3.5),

$$\begin{aligned}
& K(\sigma_2)_h^2 - \sum_{p\neq r} w_{pph} w_{rrh} - \delta \sigma_2^{hh} w_{hh} \frac{w_{hhh}^2}{\sigma_1^2} + \alpha \sum_{i\neq h} w_{iih}^2 \\
\geqslant \ & \frac{\sigma_2(w_{hhh})^2}{\sigma_1^2} + \frac{w_{hhh}^2 \sum_{i\neq h} w_{ii}^2}{2\sigma_1^2} - \delta \frac{\sigma_2^{hh} w_{hh} w_{hhh}^2}{\sigma_1^2}.
\end{aligned}$$

Since,

$$w_{hh}\sigma_2^{hh} = \sigma_2 - \frac{1}{2}\sum_{a\neq b; a,b\neq h} w_{aa} w_{bb},$$

if δ is sufficient small, we obtain (3.4). $\square$

Theorem 4 is a consequence of the following theorem. A hypersurface M is called strictly starshaped if $u \geq c_0$ for some $c_0 > 0$.

Theorem 12. *Suppose $k = 2$ and suppose $M \subset \mathbb{R}^{n+1}$ is a strictly starshaped 2-convex hypersurface satisfying curvature equation (1.1) for some positive function $f(X,\nu) \in C^2(\Gamma)$, where Γ is an open neighborhood of unit normal bundle of M in $\mathbb{R}^{n+1} \times \mathbb{S}^n$, then there is a constant C depending only on n, k, $\|M\|_{C^1}$, $\inf f$ and $\|f\|_{C^2}$, such that*

$$\max_{X\in M, i=1,\cdots,n} \kappa_i(X) \le C(1 + \max_{X\in\partial M, i=1,\cdots,n} \kappa_i(X)). \tag{3.6}$$

The proof of Theorem 12 is quite technical. The main step is to create a Maximum Principle for an appropriate auxiliary curvature function. For that purpose, we set

$$P = \sum_l e^{\kappa_l}, \quad \phi = \log\log P - (1+\varepsilon)\log u + \frac{a}{2}|X|^2, \tag{3.7}$$

where ε and a are constants which will be determined later. Here, $P = G(h_{ij}) = g(\kappa)$ with g symmetric is smooth. We may assume that the maximum of ϕ is achieved at some point $X_0 \in M$. After rotating the coordinates, we may assume the matrix (h_{ij}) is diagonal at the point, and we can further assume that $h_{11} \geqslant h_{22} \cdots \geqslant h_{nn}$. Denote $\kappa_i = h_{ii}$.

We covariantly differentiate the function ϕ twice at X_0 uisng Lemma 7,

$$\phi_i = \frac{P_i}{P\log P} - (1+\varepsilon)\frac{h_{ii}\langle X, \partial_i\rangle}{u} + a\langle \partial_i, X\rangle = 0, \tag{3.8}$$

and by (2.1),

$$\begin{aligned}
0 \;\geqslant\; & \phi_{ii} \\
= \;& \frac{P_{ii}}{P\log P} - \frac{P_i^2}{P^2\log P} - \frac{P_i^2}{(P\log P)^2} - \frac{1+\varepsilon}{u}\sum_l h_{il,i}\langle \partial_l, X\rangle - \frac{(1+\varepsilon)h_{ii}}{u} \\
& + (1+\varepsilon)h_{ii}^2 + (1+\varepsilon)\frac{h_{ii}^2\langle X,\partial_i\rangle^2}{u^2} + a - auh_{ii} \\
= \;& \frac{1}{P\log P}[\sum_l e^{\kappa_l} h_{llii} + \sum_l e^{\kappa_l} h_{lli}^2 + \sum_{\alpha\neq\beta} \frac{e^{\kappa_\alpha} - e^{\kappa_\beta}}{\kappa_\alpha - \kappa_\beta} h_{\alpha\beta i}^2 - (\frac{1}{P} + \frac{1}{P\log P})P_i^2] \\
& - \frac{(1+\varepsilon)\sum_l h_{iil}\langle \partial_l, X\rangle}{u} - \frac{(1+\varepsilon)h_{ii}}{u} + (1+\varepsilon)h_{ii}^2 + (1+\varepsilon)\frac{h_{ii}^2\langle X,\partial_i\rangle^2}{u^2} \\
& + a - auh_{ii} \\
= \;& \frac{1}{P\log P}[\sum_l e^{\kappa_l} h_{ii,ll} + \sum_l e^{\kappa_l}(h_{il}^2 - h_{ii}h_{ll})h_{ii} + \sum_l e^{\kappa_l}(h_{ii}h_{ll} - h_{il}^2)h_{ll} \\
& + \sum_l e^{\kappa_l} h_{lli}^2 + \sum_{\alpha\neq\beta} \frac{e^{\kappa_\alpha} - e^{\kappa_\beta}}{\kappa_\alpha - \kappa_\beta} h_{\alpha\beta i}^2 - (\frac{1}{P} + \frac{1}{P\log P})P_i^2] \\
& - \frac{(1+\varepsilon)\sum_l h_{iil}\langle \partial_l, X\rangle}{u} - \frac{(1+\varepsilon)h_{ii}}{u} + (1+\varepsilon)h_{ii}^2 + (1+\varepsilon)\frac{h_{ii}^2\langle X,\partial_i\rangle^2}{u^2} \\
& + a - auh_{ii}
\end{aligned} \tag{3.9}$$

Contract with σ_2^{ii},

(3.10)
$$\begin{aligned}
0 &\geqslant \sigma_2^{ii}\phi_{ii}\\
&= \frac{1}{P\log P}[\sum_l e^{\kappa_l}\sigma_2^{ii}h_{ii,ll} + 2f\sum_l e^{\kappa_l}h_{ll}^2 - \sigma_2^{ii}h_{ii}^2\sum_l e^{\kappa_l}h_{ll} + \sum_l \sigma_2^{ii}e^{\kappa_l}h_{lli}^2\\
&\quad + \sum_{\alpha\neq\beta}\sigma_2^{ii}\frac{e^{\kappa_\alpha}-e^{\kappa_\beta}}{\kappa_\alpha-\kappa_\beta}h_{\alpha\beta i}^2 - (\frac{1}{P}+\frac{1}{P\log P})\sigma_2^{ii}P_i^2] + (n-1)a\sigma_1 - 2afu\\
&\quad - \frac{(1+\varepsilon)\sum_l \sigma_2^{ii}h_{iil}\langle\partial_l, X\rangle}{u} - \frac{(1+\varepsilon)2f}{u} + (1+\varepsilon)\sigma_2^{ii}h_{ii}^2 + (1+\varepsilon)\frac{\sigma_2^{ii}h_{ii}^2\langle X,\partial_i\rangle^2}{u^2}.
\end{aligned}$$

At x_0, differentiate equation (1.1) twice,

$$(\sigma_2)_k = \sigma_2^{ii}h_{iik} = d_X f(\partial_k) + h_{kk}d_\nu f(\partial_k), \tag{3.11}$$

and

$$\sigma_2^{ii}h_{iikk} + \sigma_2^{pq,rs}h_{pqk}h_{rsk} \geqslant -C - Ch_{11}^2 + \sum_l h_{lkk}d_\nu f(\partial_l), \tag{3.12}$$

where C is a constant under control.

Insert (3.12) into (3.10),

(3.13)
$$\begin{aligned}
&\sigma_2^{ii}\phi_{ii}\\
\geqslant\ & \frac{1}{P\log P}[\sum_l e^{\kappa_l}(-C - Ch_{11}^2 - \sigma_2^{pq,rs}h_{pql}h_{rsl}) + \sum_l e^{\kappa_k}h_{kll}d_\nu f(\partial_k) + 2f\sum_l e^{\kappa_l}h_{ll}^2\\
&-\sigma_2^{ii}h_{ii}^2\sum_l e^{\kappa_l}h_{ll} + \sum_l \sigma_2^{ii}e^{\kappa_l}h_{lli}^2 + \sum_{\alpha\neq\beta}\sigma_2^{ii}\frac{e^{\kappa_\alpha}-e^{\kappa_\beta}}{\kappa_\alpha-\kappa_\beta}h_{\alpha\beta i}^2 - (\frac{1}{P}+\frac{1}{P\log P})\sigma_2^{ii}P_i^2]\\
&-\frac{(1+\varepsilon)\sum_l \sigma_2^{ii}h_{iil}\langle\partial_l, X\rangle}{u} + (1+\varepsilon)\sigma_2^{ii}h_{ii}^2 + (1+\varepsilon)\frac{\sigma_2^{ii}h_{ii}^2\langle X,\partial_i\rangle^2}{u^2}.\\
&+a\kappa_1 - Ca
\end{aligned}$$

By (3.8) and (3.11),

(3.14)
$$\begin{aligned}
&\sum_k d_\nu f(\partial_k)\frac{\sum_l e^{\kappa_l}h_{llk}}{P\log P} - \frac{1+\varepsilon}{u}\sum_k \sigma_2^{ii}h_{iik}\langle\partial_k, X\rangle\\
=\ & -a\sum_k d_\nu f(\partial_k)\langle X,\partial_k\rangle - \frac{1+\varepsilon}{u}\sum_k d_X f(\partial_k)\langle X,\partial_k\rangle.
\end{aligned}$$

Denote

$$A_i = e^{\kappa_i}(K(\sigma_2)_i^2 - \sum_{p\neq q}h_{ppi}h_{qqi}),\quad B_i = 2\sum_{l\neq i}e^{\kappa_l}h_{lli}^2,\quad C_i = \sigma_2^{ii}\sum_l e^{\kappa_l}h_{lli}^2;$$
$$D_i = 2\sum_{l\neq i}\sigma_2^{ll}\frac{e^{\kappa_l}-e^{\kappa_i}}{\kappa_l-\kappa_i}h_{lli}^2,\quad E_i = (\frac{1}{P}+\frac{1}{P\log P})\sigma_2^{ii}P_i^2.$$

Note that $\log P \geqslant \kappa_1$ and $\dfrac{(\sigma_2)_l^2}{P\log P} \leqslant C\kappa_1 e^{-\kappa_1}$ by (3.11). Since $\kappa_1 e^{-\kappa_1} \leq e$, combining (3.13), (3.14) and using

$$-\sum_l \sigma_2^{pq,rs} h_{pql} h_{rsl} = \sum_{p\neq q} h_{pql}^2 - \sum_l h_{ppl} h_{qql},$$

we find for any $K > 0$,

$$
\begin{aligned}
(3.15)\qquad & \sigma_2^{ii}\phi_{ii} \\
\geqslant\ & -C(a+K) + (a-C)h_{11} + \frac{1}{P\log P}\sum_l e^{\kappa_l}(K(\sigma_2)_l^2 - \sum_{p\neq q} h_{ppl}h_{qql} + \sum_{p\neq q} h_{pql}^2) \\
& + \sum_l \sigma_2^{ii} e^{\kappa_l} h_{lli}^2 + \sum_{\alpha\neq\beta} \sigma_2^{ii} \frac{e^{\kappa_\alpha} - e^{\kappa_\beta}}{\kappa_\alpha - \kappa_\beta} h_{\alpha\beta i}^2 - (\frac{1}{P} + \frac{1}{P\log P})\sigma_2^{ii} P_i^2] \\
& + \varepsilon\sigma_2^{ii} h_{ii}^2 + (1+\varepsilon)\frac{\sigma_2^{ii} h_{ii}^2 \langle X, \partial_i\rangle^2}{u^2} \\
=\ & -C(a+K) + (a-C)h_{11} + \frac{1}{P\log P}\sum_i (A_i + B_i + C_i + D_i - E_i) \\
& + \varepsilon\sigma_2^{ii} h_{ii}^2 + (1+\varepsilon)\frac{\sigma_2^{ii} h_{ii}^2 \langle X, \partial_i\rangle^2}{u^2}.
\end{aligned}
$$

Choose $k = 2, l = 1$, $\delta = 1$(so $\alpha = 1$) and $h = i$ in Lemma 8. Then,

$$-\sum_{p\neq r} h_{ppi} h_{rri} + \frac{(\sigma_2)_i^2}{\sigma_2} \quad\geqslant\quad \sigma_2[\frac{(\sigma_1)_i}{\sigma_1}]^2 \quad\geqslant\quad 0.$$

Hence,

$$K(\sigma_2)_i^2 - \sum_{p\neq r} h_{ppi} h_{rri} \geqslant (\sigma_2)_i^2 (K - \frac{1}{f}) \geqslant 0$$

for K large enough. Therefore, $A_i \geqslant 0$ for K sufficiently large.

Lemma 13. *Suppose*

$$n\kappa_i \leqslant \kappa_1, \quad \forall i \geq 2,$$

then

$$B_i + C_i + D_i - E_i \geqslant 0,$$

if κ_1 *sufficient large.*

Proof. We have,

$$P_i^2 = (e^{\kappa_i} h_{iii} + \sum_{l\neq i} e^{\kappa_l} h_{lli})^2 = e^{2\kappa_i} h_{iii}^2 + 2\sum_{l\neq i} e^{\kappa_i + \kappa_l} h_{lli} h_{iii} + (\sum_{l\neq i} e^{\kappa_l} h_{lli})^2.$$

By the Schwartz inequality,

$$(\sum_{l\neq i} e^{\kappa_l} h_{lli})^2 \leqslant \sum_{l\neq i} e^{\kappa_l} \sum_{l\neq i} e^{\kappa_l} h_{lli}^2.$$

Hence,

$$P_i^2 \leqslant e^{2\kappa_i}h_{iii}^2 + 2\sum_{l\neq i} e^{\kappa_l+\kappa_i}h_{lli}h_{iii} + (P - e^{\kappa_i})\sum_{l\neq i} e^{\kappa_l}h_{lli}^2.$$

In turn,

$$\begin{aligned}
(3.16)\qquad & B_i + C_i + D_i - E_i \\
\geqslant\ & \sum_{l\neq i}(2e^{\kappa_l} + \sigma_2^{ii}e^{\kappa_l} + 2\sigma_2^{ll}\frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i})h_{lli}^2 + \sigma_2^{ii}e^{\kappa_i}h_{iii}^2 - (\frac{1}{P} + \frac{1}{P\log P})\sigma_2^{ii}e^{2\kappa_i}h_{iii}^2 \\
& -(\frac{1}{P} + \frac{1}{P\log P})(P - e^{\kappa_i})\sigma_2^{ii}\sum_{l\neq i} e^{\kappa_l}h_{lli}^2 - 2(\frac{1}{P} + \frac{1}{P\log P})\sigma_2^{ii}\sum_{l\neq i} e^{\kappa_i+\kappa_l}h_{iii}h_{lli} \\
=\ & \sum_{l\neq i}[(2 - \frac{\sigma_2^{ii}}{\log P})e^{\kappa_l} + (\frac{1}{P} + \frac{1}{P\log P})\sigma_2^{ii}e^{\kappa_l+\kappa_i} + 2\sigma_2^{ll}\frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}]h_{lli}^2 \\
& +[1 - (\frac{1}{P} + \frac{1}{P\log P})e^{\kappa_i}]\sigma_2^{ii}e^{\kappa_i}h_{iii}^2 - 2(\frac{1}{P} + \frac{1}{P\log P})\sigma_2^{ii}\sum_{l\neq i} e^{\kappa_i+\kappa_l}h_{iii}h_{lli}.
\end{aligned}$$

As

$$h_{lli}^2 + h_{iii}^2 \geqslant 2h_{lli}h_{iii},$$

$$\begin{aligned}
(3.17)\qquad & \sum_{l\neq i,1}(\frac{1}{P} + \frac{1}{P\log P})\sigma_2^{ii}e^{\kappa_l+\kappa_i}h_{lli}^2 + \sum_{l\neq i,1}(\frac{1}{P} + \frac{1}{P\log P})\sigma_2^{ii}e^{\kappa_l+\kappa_i}h_{iii}^2 \\
\geqslant\ & 2(\frac{1}{P} + \frac{1}{P\log P})\sum_{l\neq i,1}\sigma_2^{ii}e^{\kappa_l+\kappa_i}h_{iii}h_{lli}.
\end{aligned}$$

Combine (3.16) and (3.17),

$$\begin{aligned}
(3.18)\qquad & B_i + C_i + D_i - E_i \\
\geqslant\ & \sum_{l\neq i}[(2 - \frac{\sigma_2^{ii}}{\log P})e^{\kappa_l} + 2\sigma_2^{ll}\frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}]h_{lli}^2 + (\frac{1}{P} + \frac{1}{P\log P})\sigma_2^{ii}e^{\kappa_1+\kappa_i}h_{11i}^2 \\
& +[(\frac{1}{P} + \frac{1}{P\log P})e^{\kappa_1} - \frac{1}{\log P}]\sigma_2^{ii}e^{\kappa_i}h_{iii}^2 - 2(\frac{1}{P} + \frac{1}{P\log P})\sigma_2^{ii}e^{\kappa_i+\kappa_1}h_{iii}h_{11i}. \\
\geqslant\ & \sum_{l\neq i}(2 - \frac{\sigma_2^{ii}}{\log P})e^{\kappa_l}h_{lli}^2 + 2\sigma_2^{11}\frac{e^{\kappa_1} - e^{\kappa_i}}{\kappa_1 - \kappa_i}h_{11i}^2 + \frac{1}{P}\sigma_2^{ii}e^{\kappa_1+\kappa_i}h_{11i}^2 \\
& +[\frac{e^{\kappa_1}}{P} - \frac{1}{\log P}]\sigma_2^{ii}e^{\kappa_i}h_{iii}^2 - 2\frac{1}{P}\sigma_2^{ii}e^{\kappa_i+\kappa_1}h_{iii}h_{11i}.
\end{aligned}$$

By the assumptions in the lemma, for each $i \geqslant 2$,

$$n\kappa_i \leqslant \kappa_1 \text{ and } \sigma_2^{ii} = \kappa_1 + \sum_{j\neq 1,i}\kappa_j.$$

We have, for $i \geqslant 2$,

$$2\log P \geqslant 2\kappa_1 \geqslant \sigma_2^{ii}.$$

Taking κ_1 sufficient large, we have,

$$\frac{e^{\kappa_1}}{2P} \geqslant \frac{1}{2n} \geqslant \frac{1}{\log P}.$$

Expanding e^x and as $n\kappa_i \leqslant \kappa_1$,

$$\begin{aligned}\sigma_2^{11}\frac{e^{\kappa_1}-e^{\kappa_i}}{\kappa_1-\kappa_i} &= \sigma_2^{11}e^{\kappa_i}\frac{e^{\kappa_1-\kappa_i}-1}{\kappa_1-\kappa_i} = \sigma_2^{11}e^{\kappa_i}\sum_{l=1}^{\infty}\frac{(\kappa_1-\kappa_i)^{l-1}}{l!}\\ &\geqslant \sigma_2^{11}e^{\kappa_i}\frac{(\kappa_1-\kappa_i)^3}{4!} \geqslant c_0\kappa_1^3\sigma_2^{11}e^{\kappa_i} \geqslant c_0\kappa_1\sigma_2^{ii}\frac{e^{\kappa_i+\kappa_1}}{P},\end{aligned}$$

for some positive constant c_0. Here, we have used the fact $\kappa_1\sigma_2^{11} \geqslant 2\sigma_2/n$. The lemma follows from (3.18), previous three inequalities, provided κ_1 is sufficiently large. $\square$

Lemma 14. *If*

$$n\kappa_i \leqslant \kappa_1,$$

for some index $i \geq 2$, then if κ_1 sufficient large,

$$B_j + C_j + D_j - (\frac{1}{P} + \frac{2}{n-1}\frac{1}{P\log P})\sigma_2^{jj}P_j^2 \geqslant 0,$$

for any $j \geqslant i$.

Proof. Replace the term $\frac{1}{P\log P}$ by $\frac{2}{n-1}\frac{1}{P\log P}$ in the proof of previous lemma, note that

$$2 - \frac{2}{n-1}\frac{\sigma_2^{jj}}{\log P} \geqslant \frac{1}{\kappa_1}(2\kappa_1 - \frac{2}{n-1}\sigma_2^{jj}) \geqslant 0.$$

Hence, the arguments in the previous proof can be carried out without further changes. $\square$

Lemma 15. *For any fixed index j, if*

$$n\kappa_j > \kappa_1,$$

we have, for sufficient large κ_1, K and sufficient small ε,

$$\frac{1}{P\log P}(A_j + B_j + C_j + D_j - E_j) + (1+\varepsilon)\frac{\sigma_2^{jj}h_{jj}^2\langle X, \partial_j\rangle^2}{u^2} \geqslant 0.$$

Proof. By the Schwarz inequality,

$$\sigma_2^{jj}P_j^2 = \sigma_2^{jj}(\sum_l e^{\kappa_l}h_{llj})^2 \leqslant \sigma_2^{jj}\sum_l e^{\kappa_l}\sum_l e^{\kappa_l}h_{llj}^2.$$

Hence,

$$C_j - \frac{\sigma_2^{jj}P_j^2}{P} \geqslant 0. \tag{3.19}$$

By Lemma 11, for some sufficient large constant C,

$$\sigma_2^{jj}\frac{\kappa_j h_{jjj}^2}{\sigma_1^2} \leqslant C[K(\sigma_2)_j^2 - \sum_{p\neq q}h_{ppj}h_{qqj} + \sum_{l\neq j}h_{llj}^2].$$

Thus,

$$\begin{aligned}
(3.20) \qquad \frac{\sigma_2^{jj}P_j^2}{P\log P} &= \frac{\sigma_2^{jj}}{P\log P}(e^{\kappa_j}h_{jjj}+\sum_{l\neq j}e^{\kappa_l}h_{llj})^2 \\
&\leqslant \frac{C\sigma_2^{jj}}{P\sigma_1}(e^{2\kappa_j}h_{jjj}^2+\sum_{l\neq j}e^{2\kappa_l}h_{llj}^2) \\
&\leqslant C[\sum_{l\neq j}e^{\kappa_l}h_{llj}^2+\frac{\kappa_j\sigma_2^{jj}}{\sigma_1^2}e^{\kappa_j}h_{jjj}^2] \\
&\leqslant C(A_j+B_j+e^{\kappa_j}\sum_{l\neq j}h_{llj}^2).
\end{aligned}$$

We **claim** that

$$\sum_{l\neq j}e^{\kappa_l}h_{llj}^2+\sum_{l\neq j}\sigma_2^{ll}\frac{e^{\kappa_l}-e^{\kappa_j}}{\kappa_l-\kappa_j}h_{llj}^2\geqslant e^{\kappa_j}\sum_{l\neq j}h_{llj}^2.$$

To prove the claim, we divide it two cases.
Case (A): $\kappa_l>\kappa_j$, obviously,

$$e^{\kappa_l}h_{llj}^2+\sigma_2^{ll}\frac{e^{\kappa_l}-e^{\kappa_j}}{\kappa_l-\kappa_j}h_{llj}^2\geqslant e^{\kappa_j}h_{llj}^2.$$

Case (B): $\kappa_l<\kappa_j$, we have

$$\frac{\sigma_2^{ll}}{\kappa_j-\kappa_l}=\frac{\kappa_j-\kappa_l+\sigma_2^{jj}}{\kappa_j-\kappa_l}\geqslant 1.$$

Therefore,

$$e^{\kappa_l}h_{llj}^2+\sigma_2^{ll}\frac{e^{\kappa_l}-e^{\kappa_j}}{\kappa_l-\kappa_j}h_{llj}^2\geqslant e^{\kappa_l}h_{llj}^2+(e^{\kappa_j}-e^{\kappa_l})h_{llj}^2=e^{\kappa_j}h_{llj}^2.$$

The claim is verified. Hence, by (3.20) and the claim,

$$\frac{\sigma_2^{jj}P_j^2}{P\log P}\leqslant c_j(A_j+B_j+D_j).$$

Denote $\delta_j=1/c_j$. It follows from (3.19) and (3.8) that,

$$\begin{aligned}
&\frac{1}{P\log P}(A_j+B_j+C_j+D_j-E_j)+(1+\varepsilon)\frac{\sigma_2^{jj}h_{jj}^2\langle X,\partial_j\rangle^2}{u^2} \\
\geqslant\ & (1+\varepsilon)\frac{\sigma_2^{jj}h_{jj}^2\langle X,\partial_j\rangle^2}{u^2}-\frac{1-\delta_j}{(P\log P)^2}\sigma_2^{jj}P_j^2 \\
=\ & (1+\varepsilon)[(1-(1-\delta_j)(1+\varepsilon))\frac{\sigma_2^{jj}h_{jj}^2\langle X,\partial_j\rangle^2}{u^2}+2(1-\delta_j)\frac{a\sigma_2^{jj}h_{jj}\langle X,\partial_j\rangle^2}{u}] \\
&-(1-\delta_j)a^2\sigma_2^{jj}\langle X,\partial_j\rangle^2.
\end{aligned}$$

The above is nonnegative, if κ_1 sufficiently large, and ε is small enough. $\square$

Proof of Theorem 12. We are in the position to give C^2 estimate. We use a similar argument in the previous section. We need to deal with every index in (3.15). First, we note that $n\kappa_1 > \kappa_1$. By Lemma 15,

$$\frac{1}{P\log P}(A_1+B_1+C_1+D_1-E_1)+(1+\varepsilon)\frac{\sigma_2^{11}h_{11}^2\langle X,\partial_1\rangle^2}{u^2}\geqslant 0. \tag{3.21}$$

We divide into two different cases.
Case (A): Suppose $n\kappa_2 \leqslant \kappa_1$. In this case, we use Lemma 13. For $i \geqslant 2$, note that $A_j \geqslant 0$,

$$\frac{1}{P\log P}(A_i+B_i+C_i+D_i-E_i)\geqslant 0. \tag{3.22}$$

Combine (3.21), (3.22) and (3.15),

$$\sigma_2^{ii}\phi_{ii}\geqslant -C+(a-C)\kappa_1.$$

We obtain C^2 estimate if a is sufficiently large.
Case (B): Suppose $n\kappa_2 > \kappa_1$. We assume that index i_0 satisfies $n\kappa_{i_0} > \kappa_1$ and $n\kappa_{i_0+1} \leqslant \kappa_1$. Hence, for index $j \leqslant i_0$, $n\kappa_j > \kappa_1$. Lemma 15 implies,

$$\frac{1}{P\log P}(A_j+B_j+C_j+D_j-E_j)+(1+\varepsilon)\frac{\sigma_2^{jj}h_{jj}^2\langle X,\partial_j\rangle^2}{u^2}\geqslant 0. \tag{3.23}$$

For index $j \geqslant i_0+1$, by Lemma 14,

$$\begin{aligned}
&\frac{1}{P\log P}(A_j+B_j+C_j+D_j-E_j)+(1+\varepsilon)\frac{\sigma_2^{jj}h_{jj}^2\langle X,\partial_j\rangle^2}{u^2} \\
\geqslant\ & -(1-\frac{2}{n-1})\frac{\sigma_2^{jj}P_j^2}{(P\log P)^2}+(1+\varepsilon)\frac{\sigma_2^{jj}h_{jj}^2\langle X,\partial_j\rangle^2}{u^2} \\
=\ & (1+\varepsilon)[(1-\frac{n-3}{n-1}(1+\varepsilon))\frac{\sigma_2^{jj}h_{jj}^2\langle X,\partial_j\rangle^2}{u^2}+2\frac{n-3}{n-1}\frac{a\sigma_2^{jj}h_{jj}\langle X,\partial_j\rangle^2}{u}] \\
& -\frac{n-3}{n-1}a^2\sigma_2^{jj}\langle X,\partial_j\rangle^2. \\
\geqslant\ & -Ca^2\kappa_1.
\end{aligned} \tag{3.24}$$

The last inequality holds, provided ε is sufficiently small. Combining (3.23), (3.24) and (3.15), we obtain,

$$\sigma_2^{ii}\phi_{ii}\geqslant -C+(a-C)\kappa_1+\varepsilon\sigma_2^{ii}\kappa_i^2-Ca^2\kappa_1.$$

We further divide the case into two subcases to deal with the above inequality.
Case (B1): Suppose $\sigma_2^{22}\geqslant 1$. As $n\kappa_2 > \kappa_1$,

$$\begin{aligned}
\sigma_2^{ii}\phi_{ii} &\geqslant -C+(a-C)\kappa_1+\varepsilon\sigma_2^{22}\kappa_2^2-Ca^2\kappa_1 \\
&\geqslant -C+(a-C)\kappa_1+\frac{\varepsilon}{n^2}\kappa_1^2-Ca^2\kappa_1.
\end{aligned}$$

The above is nonnegative if κ_1 and a are sufficiently large.

Case (B2): Suppose $\sigma_2^{22} < 1$. In this subcase, we may assume that κ_1 is sufficiently large, then $\kappa_n < 0$. By the assumption, $1 \geqslant \kappa_1 + (n-2)\kappa_n$. This implies,

$$-\kappa_n \geqslant \frac{\kappa_1 - 1}{n-2}.$$

Since $\sigma_2^{nn} + \kappa_n = \kappa_1 + \sigma_2^{11}$, we have $\sigma_2^{nn} \geqslant \kappa_1$. Hence,

$$\begin{aligned} \sigma_2^{ii}\phi_{ii} &\geqslant -C + (a-C)\kappa_1 + \varepsilon\sigma_2^{nn}\kappa_n^2 - Ca^2\kappa_1 \\ &\geqslant -C + (a-C)\kappa_1 + \frac{\varepsilon}{(n-2)^2}\kappa_1(\kappa_1-1)^2 - Ca^2\kappa_1. \end{aligned}$$

The above is nonnegative, if a and κ_1 are sufficiently large. The proof of Theorem 12 is complete. □

We remark that the similar curvature estimate can be established for Dirichlet boundary problem of equation

$$(3.25) \qquad \begin{cases} \sigma_2[\kappa(x, u(x))] &= f(x, u, Du), \\ u|_{\partial\Omega} &= \phi, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Though such graph over Ω may not be starshaped. With the assumption of C^1 boundedness, one may shift the origin in $\mathbb{R}^{n+1}$ in the direction of $E_{n+1} = (0, \cdots, 0, 1)$ in appropriate way so that the surface is starshaped with respect to the new origin. Then the proof in this section yields the following theorem, which completely settles the regularity problem considered in Ivochkina [20, 19] when $k = 2$.

Theorem 16. *Suppose u is a solution of equation (3.25), then there is a constant C depending only on n, k, Ω, $\|u\|_{C^1}$, $\inf f$ and $\|f\|_{C^2}$, such that*

$$(3.26) \qquad \max_{x\in\Omega} |\nabla^2 u(x)| \leq C(1 + \max_{x\in\partial\Omega} |\nabla^2 u(x)|), \quad \forall i = 1, \cdots, n.$$

4. A global C^2 estimate for convex hypersurfaces

In this section, we consider the global C^2 estimates for convex solutions to curvature equation (1.1) in $\mathbb{R}^{n+1}$. We need further modify the test function constructed in the previous section.

Theorem 17. *Suppose $M \subset \mathbb{R}^{n+1}$ is a convex hypersurface satisfying curvature equation (1.1) for some positive function $f(X, \nu) \in C^2(\Gamma)$, where Γ is an open neighborhood of unit normal bundle of M in $\mathbb{R}^{n+1} \times \mathbb{S}^n$, then there is a constant C depending only on n, k, $\|M\|_{C^1}$, $\inf f$ and $\|f\|_{C^2}$, such that*

$$(4.1) \qquad \max_{X\in M, i=1,\cdots,n} \kappa_i(X) \leq C(1 + \max_{X\in\partial M, i=1,\cdots,n} \kappa_i(X)).$$

To proceed, consider the following test function,

$$(4.2) \qquad P(\kappa(X)) = \kappa_1^2 + \cdots + \kappa_n^2, \quad \phi = \frac{1}{2}\log P(\kappa(X)) - N\log u,$$

where N is a constant to be determined later. Note that,

$$\kappa_1^2 + \cdots + \kappa_n^2 = \sigma_1(\kappa(X))^2 - 2\sigma_2(\kappa(X)).$$

We assume that ϕ achieves its maximum value at $x_0 \in M$. By a proper rotation, we may assume that (h_{ij}) is a diagonal matrix at the point, and $h_{11} \geqslant h_{22} \cdots \geqslant h_{nn}$.

At x_0, differentiate ϕ twice,

$$
\begin{aligned}
\phi_i &= \frac{\sum_k \kappa_k h_{kki}}{P} - N\frac{u_i}{u} \\
&= \frac{\sum_k \kappa_k h_{kki}}{P} - N\frac{h_{ii}\langle \partial_i, X\rangle}{u} \quad = \quad 0,
\end{aligned} \tag{4.3}
$$

and,

$$
\begin{aligned}
0 &\geqslant \frac{1}{P}[\sum_k \kappa_k h_{kk,ii} + \sum_k h_{kki}^2 + \sum_{p\neq q} h_{pqi}^2] - \frac{2}{P^2}[\sum_k \kappa_k h_{kki}]^2 \\
&\quad -N\frac{u_{ii}}{u} + N\frac{u_i^2}{u^2} \\
&= \frac{1}{P}[\sum_k \kappa_k (h_{ii,kk} + (h_{ik}^2 - h_{ii}h_{kk})h_{ii} + (h_{ii}h_{kk} - h_{ik}^2)h_{kk}) \\
&\quad + \sum_k h_{kki}^2 + \sum_{p\neq q} h_{pqi}^2] - \frac{2}{P^2}[\sum_k \kappa_k h_{kki}]^2 - N\frac{\sum_l h_{ii,l}\langle X, \partial_l\rangle}{u} \\
&\quad -N\frac{h_{ii}}{u} + Nh_{ii}^2 + N\frac{h_{ii}^2\langle \partial_i, X\rangle^2}{u^2}.
\end{aligned} \tag{4.4}
$$

Now differentiate equation (1.1) twice,

$$
\sigma_k^{ii} h_{iij} \quad = \quad d_X f(X_j) + d_\nu f(\nu_j) \quad = \quad d_X f(\partial_j) + h_{jj} d_\nu f(\partial_j), \tag{4.5}
$$

$$
\begin{aligned}
&\sigma_k^{ii} h_{iijj} + \sigma_k^{pq,rs} h_{pqj} h_{rsj} \\
= \ & d_X f(X_{jj}) + d_X^2 f(X_j, X_j) + 2 d_X d_\nu f(X_j, \nu_j) + d_\nu^2 f(\nu_j, \nu_j) + d_\nu f(\nu_{jj}). \\
= \ & -h_{jj} d_X f(\nu) + d_X^2 f(\partial_j, \partial_j) + 2h_{jj} d_X d_\nu f(\partial_j, \partial_j) + h_{jj}^2 d_\nu^2 f(\partial_j, \partial_j) \\
& + \sum_k h_{kjj} d_\nu f(\partial_k) - h_{jj}^2 d_\nu f(\nu) \\
\geqslant \ & -C - C\kappa_j - C\kappa_j^2 + \sum_k h_{kjj} d_\nu f(\partial_k) \\
\geqslant \ & -C - C\kappa_j^2 + \sum_k h_{kjj} d_\nu f(\partial_k).
\end{aligned} \tag{4.6}
$$

It follows from (4.3) and (4.5),

$$
\frac{1}{P}\sum_{l,s} \kappa_l h_{sll} d_\nu f(\partial_s) - \frac{N\sigma_k^{ii}\sum_s h_{iis}\langle \partial_s, X\rangle}{u} \quad = \quad -\frac{N}{u}\sum_s d_X f(\partial_s)\langle \partial_s, X\rangle. \tag{4.7}
$$

We will also use

$$
-\sigma_k^{pq,rs} h_{pql} h_{rsl} \quad = \quad -\sigma_k^{pp,qq} h_{ppl} h_{qql} + \sigma_k^{pp,qq} h_{pql}^2,
$$

which follows from Lemma 7.

Denote

$$A_i = \frac{\kappa_i}{P}(K(\sigma_k)_i^2 - \sum_{p,q}\sigma_k^{pp,qq}h_{ppi}h_{qqi}), \quad B_i = 2\sum_j \frac{\kappa_j}{P}\sigma_k^{jj,ii}h_{jji}^2,$$

$$C_i = 2\sum_{j\neq i}\frac{\sigma_k^{jj}}{P}h_{jji}^2, \quad D_i = \frac{1}{P}\sum_j \sigma_k^{ii}h_{jji}^2, \quad E_i = \frac{2\sigma_k^{ii}}{P^2}(\sum_j \kappa_j h_{jji})^2.$$

Contracting with σ_k^{ii} in both side of inequality (4.4), it follows from (4.5)-(4.7),

(4.8)

$$\begin{aligned}
0 \quad \geqslant \quad & \frac{1}{P}[\sum_l \kappa_l(-C - C\kappa_l^2 - \sigma_k^{pq,rs}h_{pql}h_{rsl}) \\
& +\sigma_k^{ii}h_{ii}\sum_l \kappa_l^3 - \sigma_k^{ii}h_{ii}^2\sum_l \kappa_l^2 + \sum_l \sigma_k^{ii}h_{lli}^2 + \sigma_k^{ii}\sum_{p\neq q}h_{pqi}^2] - \frac{2\sigma_k^{ii}}{P^2}(\sum_j \kappa_j h_{jji})^2 \\
& -N\frac{\sigma_k^{ii}h_{ii}}{u} + N\sigma_k^{ii}h_{ii}^2 + N\frac{\sigma_k^{ii}h_{ii}^2\langle\partial_i, X\rangle^2}{u^2} \\
\geqslant \quad & \frac{1}{P}[\sum_l \kappa_l(-C - C\kappa_l^2 - K(\sigma_k)_l^2 + K(\sigma_k)_l^2 - \sigma_k^{pp,qq}h_{ppl}h_{qql} + \sigma_k^{pp,qq}h_{pql}^2) \\
& +kf\sum_l \kappa_l^3 - \sigma_k^{ii}h_{ii}^2\sum_l \kappa_l^2 + \sum_l \sigma_k^{ii}h_{lli}^2 + \sigma_k^{ii}\sum_{p\neq q}h_{pqi}^2] - \frac{2\sigma_k^{ii}}{P^2}(\sum_j \kappa_j h_{jji})^2 \\
& -N\frac{kf}{u} + N\sigma_k^{ii}h_{ii}^2 + N\frac{\sigma_k^{ii}h_{ii}^2\langle\partial_i, X\rangle^2}{u^2} - \frac{N}{u}\sum_s d_X f(\partial_s)\langle\partial_s, X\rangle \\
\geq \quad & \frac{1}{P}[\sum_l \kappa_l(-C - C\kappa_l^2 - K(\sigma_k)_l^2) + \sigma_k^{ii}h_{ii}\sum_l \kappa_l^3 - \sigma_k^{ii}h_{ii}^2\sum_l \kappa_l^2] \\
& -N\frac{kf}{u} + N\sigma_k^{ii}h_{ii}^2 + N\frac{\sigma_k^{ii}h_{ii}^2\langle\partial_i, X\rangle^2}{u^2} - \frac{N}{u}\sum_s d_X f(\partial_s)\langle\partial_s, X\rangle \\
& +\sum_i (A_i + B_i + C_i + D_i - E_i).
\end{aligned}$$

The main part of the proof is to deal with the third order derivatives. We divide it to two cases:

(1) $i \neq 1$;
(2) $i = 1$.

Lemma 18. *For each $i \neq 1$, if*

$$\sqrt{3}\kappa_i \quad \leqslant \quad \kappa_1,$$

we have,

$$A_i + B_i + C_i + D_i - E_i \quad \geqslant \quad 0.$$

Proof. By (2.4) in Lemma 8 (note that $\sigma_1^{pp,qq}=0$), when K is sufficiently large,

$$K(\sigma_k)_i^2-\sigma_k^{pp,qq}h_{ppi}h_{qqi} \quad \geqslant \quad \sigma_k(1+\frac{\alpha}{2})[\frac{(\sigma_1)_i}{\sigma_1}]^2 \quad > \quad 0, \tag{4.9}$$

so $A_i\geqslant 0$.

$$\begin{aligned}
&P^2(B_i+C_i+D_i-E_i) \\
= \quad & \sum_{j\neq i}P(2\kappa_j\sigma_k^{jj,ii}+2\sigma_k^{jj}+\sigma_k^{ii})h_{jji}^2+P\sigma_k^{ii}h_{iii}^2-2\sigma_k^{ii}(\sum_{j\neq i}\kappa_j^2h_{jji}^2+\kappa_i^2h_{iii}^2 \\
&+\sum_{m\neq l}\kappa_k\kappa_l h_{mmi}h_{lli}) \\
= \quad & \sum_{j\neq i}[P(3\sigma_k^{ii}+2\sigma_k^{jj}-2\sigma_{k-1}(\kappa|ij))-2\sigma_k^{ii}\kappa_j^2]h_{jji}^2+(P-2\kappa_i^2)\sigma_k^{ii}h_{iii}^2 \\
&-2\sigma_k^{ii}\sum_{m\neq l}\kappa_k\kappa_l h_{mmi}h_{lli} \\
= \quad & \sum_{j\neq i}(P+2(P-\kappa_j^2))\sigma_k^{ii}h_{jji}^2+(P-2\kappa_i^2)\sigma_k^{ii}h_{iii}^2-2\sigma_k^{ii}\sum_{m\neq l}\kappa_k\kappa_l h_{mmi}h_{lli} \\
&+2P\sum_{j\neq i}\kappa_i\sigma_k^{jj,ii}h_{jji}^2.
\end{aligned} \tag{4.10}$$

Note that, for each fixed i,

$$\begin{aligned}
2\sum_{j\neq i}\sum_{k\neq i,j}\kappa_k^2h_{jji}^2 \quad &= \quad \sum_{l\neq i}\sum_{k\neq i,l}\kappa_k^2h_{lli}^2+\sum_{k\neq i}\sum_{l\neq i,k}\kappa_l^2h_{kki}^2 \\
&\geqslant \quad 2\sum_{k\neq l;k,l\neq i}\kappa_k\kappa_l h_{kki}h_{lli}.
\end{aligned} \tag{4.11}$$

By $\sqrt{3}\kappa_i\leqslant\kappa_1$ or $\kappa_1^2\geqslant 3\kappa_i^2$,

$$\sum_{j\neq i,1}(\frac{2P}{3}+2\kappa_i^2)h_{jji}^2+\sum_{j\neq i,1}\kappa_j^2h_{iii}^2 \quad \geqslant \quad 2\kappa_i h_{iii}\sum_{j\neq i,1}\kappa_j h_{jji}. \tag{4.12}$$

Then (4.10) becomes,

$$\begin{aligned}
&P^2(B_i+C_i+D_i-E_i) \\
\geqslant \quad & (P+2\kappa_i^2)\sigma_k^{ii}h_{11i}^2+(\kappa_1^2-\kappa_i^2)\sigma_k^{ii}h_{iii}^2-4\sigma_k^{ii}\kappa_i h_{iii}\kappa_1 h_{11i} \\
&+\frac{P}{3}\sum_{j\neq 1,i}\sigma_k^{ii}h_{jji}^2+2P\sum_{j\neq i}\kappa_i\sigma^{ii,jj}h_{jji}^2 \\
\geqslant \quad & \sigma_k^{ii}[(\kappa_1^2+3\kappa_i^2)h_{11i}^2+(\kappa_1^2-\kappa_i^2)h_{iii}^2-4\kappa_1\kappa_i h_{iii}h_{11i}]+2P\sum_{j\neq i}\kappa_i\sigma^{ii,jj}h_{jji}^2.
\end{aligned} \tag{4.13}$$

The above is nonnegative, provided the following inequality holds,

$$\sqrt{\kappa_1^2+3\kappa_i^2}\sqrt{\kappa_1^2-\kappa_i^2} \quad \geqslant \quad 2\kappa_1\kappa_i. \tag{4.14}$$

Set $x=\kappa_i/\kappa_1$. Inequality (4.14) is equivalent to the following inequality,

$$3x^4+2x^2-1\leqslant 0.$$

This follows from the condition $\kappa_1\geqslant\sqrt{3}\kappa_i$. The proof is complete. $\square$

We need another Lemma.

Lemma 19. *For $\lambda=1,\cdots,k-1$, suppose there exists some positive constant $\delta\leqslant 1$, such that $\kappa_\lambda/\kappa_1\geqslant\delta$. Then there exits a sufficient small positive constant δ' depending on δ, such that, if $\kappa_{\lambda+1}/\kappa_1\leqslant\delta'$, we have*

$$A_i+B_i+C_i+D_i-E_i\geqslant 0,$$

for $i=1,\cdots,\lambda$.

Proof. By (4.10) and (4.11),

$$\begin{aligned}
&P^2(B_i+C_i+D_i-E_i)\\
=\ &\sum_{j\neq i}[P(3\sigma_k^{ii}+2\sigma_k^{jj}-2\sigma_{k-1}(\kappa|ij))-2\sigma_k^{ii}\kappa_j^2]h_{jji}^2+(P-2\kappa_i^2)\sigma_k^{ii}h_{iii}^2\\
&-2\sigma_k^{ii}\sum_{k\neq l}\kappa_k\kappa_l h_{kki}h_{lli}\\
\geqslant\ &\sum_{j\neq i}(P+2\kappa_i^2)\sigma_k^{ii}h_{jji}^2+(P-2\kappa_i^2)\sigma_k^{ii}h_{iii}^2-4\sigma_k^{ii}\kappa_i h_{iii}\sum_{j\neq i}\kappa_j h_{jji}\\
&+P\sum_{j\neq i}2(\sigma_{k-1}(\kappa|j)-\sigma_{k-1}(\kappa|ij))h_{jji}^2.
\end{aligned}\tag{4.15}$$

For $i=1$, the above inequality becomes,

$$\begin{aligned}
&P^2(B_i+C_i+D_i-E_i)\\
\geqslant\ &\sum_{j\neq 1}(3\kappa_1^2\sigma_k^{11}+\kappa_1^2\sigma_k^{jj})h_{jj1}^2+\sum_{j\neq 1}\kappa_j^2\sigma_k^{11}h_{111}^2-4\sigma_k^{11}\kappa_1 h_{111}\sum_{j\neq 1}\kappa_j h_{jj1}\\
&+P\sum_{j\neq 1}(\sigma_{k-1}(\kappa|j)-2\sigma_{k-1}(\kappa|1j))h_{jj1}^2-\kappa_1^2\sigma_k^{11}h_{111}^2\\
\geqslant\ &P\sum_{j\neq 1}(\sigma_{k-1}(\kappa|j)-2\sigma_{k-1}(\kappa|1j))h_{jj1}^2-\kappa_1^2\sigma_k^{11}h_{111}^2.
\end{aligned}\tag{4.16}$$

For $i\neq 1$, we replace the index $j\neq i,1$ with $j\neq i$ in (4.12), then

$$P^2(B_i+C_i+D_i-E_i)\geqslant\ P\sum_{j\neq i}2(\sigma_{k-1}(\kappa|j)-\sigma_{k-1}(\kappa|ij))h_{jji}^2-\kappa_i^2\sigma_k^{ii}h_{iii}^2.\tag{4.17}$$

By (2.4) in Lemma 8,

$$
\begin{aligned}
(4.18)\qquad A_i &\geqslant \frac{\kappa_i}{P}[\sigma_k(1+\frac{\alpha}{2})\frac{(\sigma_\lambda)_i^2}{\sigma_\lambda^2} - \frac{\sigma_k}{\sigma_\lambda}\sigma_\lambda^{pp,qq}h_{ppi}h_{qqi}] \\
&\geqslant \frac{\kappa_i\sigma_k}{P\sigma_\lambda^2}[(1+\frac{\alpha}{2})\sum_a(\sigma_\lambda^{aa}h_{aai})^2 + \frac{\alpha}{2}\sum_{a\neq b}\sigma_\lambda^{aa}\sigma_\lambda^{bb}h_{aai}h_{bbi} \\
&\quad + \sum_{a\neq b}(\sigma_\lambda^{aa}\sigma_\lambda^{bb} - \sigma_\lambda\sigma_\lambda^{aa,bb})h_{aai}h_{bbi}].
\end{aligned}
$$

For $\lambda = 1$, note that $\sigma_1^{aa} = 1$ and $\sigma_1^{aa,bb} = 0$. Hence,

$$
\begin{aligned}
(4.19)\qquad (1+\frac{\alpha}{2})\sum h_{aai}h_{bbi} &\geqslant 2(1+\frac{\alpha}{2})\sum_{a\neq 1}h_{aai}h_{11i} + (1+\frac{\alpha}{2})h_{11i}^2 \\
&\geqslant (1+\frac{\alpha}{4})h_{11i}^2 - C_\alpha\sum_{a\neq 1}h_{aai}^2
\end{aligned}
$$

In turn,

$$
\begin{aligned}
(4.20)\qquad P^2A_i &\geqslant \frac{P\kappa_i\sigma_k}{\sigma_1^2}(1+\frac{\alpha}{4})h_{11i}^2 - \frac{\kappa_i PC_\alpha}{\sigma_1^2}\sum_{a\neq 1}h_{aai}^2 \\
&\geqslant \frac{\kappa_i^2\sigma_k^{ii}}{(1+\sum_{j\neq 1}\kappa_j/\kappa_1)^2}(1+\frac{\alpha}{4})h_{11i}^2 - C_\alpha\kappa_i\sum_{a\neq 1}h_{aai}^2 \\
&\geqslant \kappa_i^2\sigma_k^{ii}h_{11i}^2 - C_\alpha\kappa_i\sum_{a\neq 1}h_{aai}^2.
\end{aligned}
$$

The last inequality comes from the fact

$$
(4.21)\qquad 1+\frac{\alpha}{4} \geqslant (1+(n-1)\delta')^2.
$$

For $\lambda \geqslant 2$, obviously, for $a \neq b$,

$$
\begin{aligned}
(4.22)\qquad & \sigma_\lambda^{aa}\sigma_\lambda^{bb} - \sigma_\lambda\sigma_\lambda^{aa,bb} \\
= \; & (\kappa_b\sigma_{\lambda-2}(\kappa|ab) + \sigma_{\lambda-1}(\kappa|ab))(\kappa_a\sigma_{\lambda-2}(\kappa|ab) + \sigma_{\lambda-1}(\kappa|ab)) \\
& -(\kappa_a\kappa_b\sigma_{\lambda-2}(\kappa|ab) + \kappa_a\sigma_{\lambda-1}(\kappa|ab) + \kappa_b\sigma_{\lambda-1}(\kappa|ab) + \sigma_\lambda(\kappa|ab))\sigma_{\lambda-2}(\kappa|ab) \\
= \; & \sigma_{\lambda-1}^2(\kappa|ab) - \sigma_\lambda(\kappa|ab)\sigma_{\lambda-2}(\kappa|ab) \\
\geqslant \; & 0,
\end{aligned}
$$

by the Newton inequality. It follows from (4.22),

$$
\begin{aligned}
(4.23)\qquad & \sum_{a\neq b;a,b\leqslant\lambda}(\sigma_\lambda^{aa}\sigma_\lambda^{bb}-\sigma_\lambda\sigma_\lambda^{aa,bb})h_{aai}h_{bbi}\\
\geqslant\ & -\sum_{a\neq b;a,b\leqslant\lambda}(\sigma_{\lambda-1}^2(\kappa|ab)-\sigma_\lambda(\kappa|ab)\sigma_{\lambda-2}(\kappa|ab))h_{aai}^2\\
\geqslant\ & -\sum_{a\neq b;a,b\leqslant\lambda}C_1(\frac{\kappa_{\lambda+1}}{\kappa_b})^2(\sigma_\lambda^{aa}h_{aai})^2\\
\geqslant\ & -\frac{C_2}{\delta^2}(\frac{\kappa_{\lambda+1}}{\kappa_1})^2\sum_a(\sigma_\lambda^{aa}h_{aai})^2 \quad\geqslant\quad -\epsilon\sum_a(\sigma_\lambda^{aa}h_{aai})^2.
\end{aligned}
$$

Here, we choose a sufficient small δ', such that,

$$
(4.24)\qquad \delta' \quad\leqslant\quad \delta\sqrt{\epsilon/C_2}.
$$

By (4.22),

$$
\begin{aligned}
(4.25)\qquad & 2\sum_{a\leqslant\lambda;b>\lambda}(\sigma_\lambda^{aa}\sigma_\lambda^{bb}-\sigma_\lambda\sigma_\lambda^{aa,bb})h_{aai}h_{bbi}\\
\geqslant\ & -2\sum_{a\leqslant\lambda;b>\lambda}\sigma_\lambda^{aa}\sigma_\lambda^{bb}|h_{aai}h_{bbi}|\\
\geqslant\ & -\epsilon\sum_{a\leqslant\lambda;b>\lambda}(\sigma_\lambda^{aa}h_{aai})^2-\frac{1}{\epsilon}\sum_{a\leqslant\lambda;b>\lambda}(\sigma_\lambda^{bb}h_{bbi})^2.
\end{aligned}
$$

Again by (4.22),

$$
\begin{aligned}
(4.26)\qquad \sum_{a\neq b;a,b>\lambda}(\sigma_\lambda^{aa}\sigma_\lambda^{bb}-\sigma_\lambda\sigma_\lambda^{aa,bb})h_{aai}h_{bbi} \quad&\geqslant\quad -\sum_{a\neq b;a,b>\lambda}\sigma_\lambda^{aa}\sigma_\lambda^{bb}|h_{aai}h_{bbi}|\\
&\geqslant\quad -\sum_{a\neq b;a,b>\lambda}(\sigma_\lambda^{aa}h_{aai})^2.
\end{aligned}
$$

Combining (4.18), (4.23), (4.25) and (4.26), by (4.18),

$$
(4.27)\qquad A_i \quad\geqslant\quad \frac{\kappa_i\sigma_k}{P\sigma_\lambda^2}[(1-2\epsilon)\sum_{a\leqslant\lambda}(\sigma_\lambda^{aa}h_{aai})^2-C_\epsilon\sum_{a>\lambda}(\sigma_\lambda^{aa}h_{aai})^2].
$$

Therefore,

$$
\begin{aligned}
(4.28)\qquad & P^2 A_i \\
\geqslant\ & \frac{P\kappa_i^2\sigma_k^{ii}}{\sigma_\lambda^2}(1-2\epsilon)\sum_{a\leqslant\lambda}(\sigma_\lambda^{aa}h_{aai})^2 - \frac{P\kappa_i\sigma_k C_\epsilon}{\sigma_\lambda^2}\sum_{a>\lambda}(\sigma_\lambda^{aa}h_{aai})^2 \\
\geqslant\ & \frac{P\kappa_i^2\sigma_k^{ii}}{\kappa_1^2}(1-2\epsilon)\sum_{a\leqslant\lambda}(\frac{\kappa_a\sigma_\lambda^{aa}}{\sigma_\lambda})^2 h_{aai}^2 - \frac{\kappa_1^2\kappa_i C_\epsilon}{\sigma_\lambda^2}\sum_{a>\lambda}(\sigma_\lambda^{aa}h_{aai})^2 \\
\geqslant\ & \kappa_i^2\sigma_k^{ii}(1-2\epsilon)(1+\delta^2)\sum_{a\leqslant\lambda}(1-\frac{C_3\kappa_{\lambda+1}}{\kappa_a})^2 h_{aai}^2 - \frac{\kappa_a^2\kappa_i C_\epsilon}{\delta^2\sigma_\lambda^2}\sum_{a>\lambda}(\sigma_\lambda^{aa}h_{aai})^2 \\
\geqslant\ & \kappa_i^2\sigma_k^{ii}(1-2\epsilon)(1+\delta^2)(1-\frac{C_3\kappa_{\lambda+1}}{\delta\kappa_1})^2\sum_{a\leqslant\lambda}h_{aai}^2 - \frac{\kappa_i C_\epsilon}{\delta^2}\sum_{a>\lambda}h_{aai}^2 \\
\geqslant\ & \kappa_i^2\sigma_k^{ii}\sum_{a\leqslant\lambda}h_{aai}^2 - \frac{\kappa_i C_\epsilon}{\delta^2}\sum_{a>\lambda}h_{aai}^2.
\end{aligned}
$$

In the above, we have used the fact that we may choose δ' and ϵ satisfying

$$
(4.29)\qquad \delta' C_3 \leqslant 2\epsilon\delta, \quad (1-2\epsilon)^2(1+\delta^2)\geqslant 1.
$$

By (4.16), (4.17), (4.20) and (4.28), for each i, we have,

$$
\begin{aligned}
(4.30)\qquad & P^2(A_i+B_i+C_i+D_i-E_i) \\
\geqslant\ & \sum_{j\neq i}(P\sigma_{k-1}(\kappa|j)-2P\sigma_{k-1}(\kappa|ij))h_{jji}^2 - C_{\alpha,\delta}\kappa_i\sum_{j>\lambda}h_{jji}^2.
\end{aligned}
$$

Now, for $j\leqslant\lambda$,

$$
\begin{aligned}
(4.31)\qquad \sigma_{k-1}(\kappa|j)-2\sigma_{k-1}(\kappa|ij) &= \kappa_i\sigma_{k-2}(\kappa|ij)-\sigma_{k-1}(\kappa|ij) \\
&\geqslant \frac{\kappa_1\cdots\kappa_k}{\kappa_j} - C\frac{\kappa_1\cdots\kappa_{k+1}}{\kappa_i\kappa_j} \\
&\geqslant \frac{\kappa_1\cdots\kappa_k}{\kappa_j}(1-C\frac{\kappa_{k+1}}{\delta\kappa_1}) \\
&\geqslant \frac{\varepsilon\sigma_k}{\kappa_j}(1-C_4\delta'/\delta).
\end{aligned}
$$

For $\lambda<j\leqslant k$, in a similar way, we have,

$$
(4.32)\qquad \sigma_{k-1}(\kappa|j)-2\sigma_{k-1}(\kappa|ij)-C_{\epsilon,\alpha}\frac{\kappa_i}{P} \geqslant \frac{\varepsilon\sigma_k}{\kappa_j}(1-C_4\delta'/\delta)-\frac{C_{\epsilon,\alpha}}{\kappa_1}.
$$

For $j > k$,

$$\begin{aligned}
(4.33)\qquad & \sigma_{k-1}(\kappa|j) - 2\sigma_{k-1}(\kappa|ij) - \frac{C_{\epsilon,\alpha}\kappa_i}{\kappa_1^2} \\
= \ & \kappa_i\sigma_{k-2}(\kappa|ij) - \sigma_{k-1}(\kappa|ij) - \frac{C_{\epsilon,\alpha}\kappa_i}{\kappa_1^2} \ \geqslant \ \frac{\kappa_1\cdots\kappa_k}{\kappa_k} - C\frac{\kappa_1\cdots\kappa_k}{\kappa_i} - \frac{C_{\epsilon,\alpha}}{\kappa_1} \\
\geqslant \ & \frac{\kappa_1\cdots\kappa_k}{\kappa_k}(1 - C\frac{\kappa_k}{\delta\kappa_1}) - \frac{C_{\epsilon,\alpha}}{\kappa_1} \ \geqslant \ \frac{\varepsilon\sigma_k}{\kappa_k}(1 - C_4\delta'/\delta) - \frac{C_{\epsilon,\alpha}}{\kappa_1}.
\end{aligned}$$

We may choose

$$\delta' \ \leqslant \ \delta/(2C_4),$$

so that (4.31) is nonnegative. We further impose that

$$\delta' \ \leqslant \ \varepsilon\sigma_k/(2C_{\epsilon,\alpha}).$$

Thus, both (4.32) and (4.33) are non-negative. The proof is complete. □

A directly corollary of Lemma 18 and Lemma 19 is the following.

Corollary 20. *There exists a finite sequence of positive numbers $\{\delta_i\}_{i=1}^k$, such that, if the following inequality holds for some $1 \leqslant i \leqslant k$,*

$$\frac{\kappa_i}{\kappa_1} \ \geqslant \ \delta_i,$$

then,

$$\begin{aligned}
(4.34)\qquad 0 \ \leqslant \ & \frac{1}{P}[\sum_l \kappa_l(K(\sigma_k)_l^2 - \sigma_k^{pp,qq}h_{ppl}h_{qql} + \sigma_k^{pp,qq}h_{pql}^2) + \sum_{p,q}\sigma_k^{ii}h_{pqi}^2] \\
& - \frac{2\sigma_k^{ii}}{P^2}(\sum_j \kappa_j h_{jji})^2.
\end{aligned}$$

Proof. We use induction to find the sequence $\{\delta_i\}_{i=1}^k$. Let $\delta_1 = 1/\sqrt{3}$. Then $\kappa_1/\kappa_1 = 1 > \delta_1$. The claim holds for $i = 1$ follows from the proof in the previous lemma. Assume that we have determined δ_i for $1 \leqslant i \leqslant k-1$. We want to search for δ_{i+1}. In Lemma 19, we may choose $\lambda = i$ and $\delta = \delta_i$. Then there is some δ'_{i+1} such that, if $\kappa_{i+1} \leqslant \delta'_{i+1}\kappa_1$, we have $A_j + B_j + C_j + D_j - E_j \geqslant 0$ for $1 \leqslant j \leqslant i$. Pick

$$\delta_{i+1} = \min\{\delta_1, \delta'_{i+1}\}.$$

If $\kappa_{i+1} \leqslant \delta_{i+1}\kappa_1$, by Lemma 18, $A_j + B_j + C_j + D_j - E_j \geqslant 0$ for $j \geqslant i+1$. We obtain (4.34) for $i+1$ case. □

Proof of Theorem 17. Again, the proof will be divided into two cases.

Case (A): There exists some $2 \leqslant i \leqslant k$, such that $\kappa_i \leqslant \delta_i \kappa_1$. By Corollary 20, (4.8),(4.5) and the Schwarz inequality,

$$\begin{aligned} 0 \geqslant\ & \frac{1}{P}[\sum_l \kappa_l(-C - C\kappa_l^2 - K(\sigma_k)_l^2) + kf\sum_l \kappa_l^3 - \sigma_k^{ii}h_{ii}^2\sum_l \kappa_l^2] - N\frac{kf}{u} \\ & + N\sigma_k^{ii}h_{ii}^2 + N\frac{\sigma_k^{ii}h_{ii}^2\langle\partial_i, X\rangle^2}{u^2} - \frac{N}{u}\sum_s d_X f(\partial_s)\langle\partial_s, X\rangle. \\ \geqslant\ & \frac{1}{P}[-C(K) - C(K)\sum_l \kappa_l^3] - \sigma_k^{ii}h_{ii}^2 + N\sigma_k^{ii}h_{ii}^2 - C(N) \\ \geqslant\ & -\frac{C(K)\kappa_1^3 + C(K)}{P} + (N-1)\varepsilon\sigma_k\kappa_1 - C(N), \end{aligned}$$

in the last inequality, we have used

$$\kappa_1\sigma_k^{11} \geqslant \frac{k}{n}\sigma_k.$$

If we choose

$$\varepsilon\sigma_k(N-1) \geqslant C(K) + 1,$$

an upper bound of κ_1 follows.
Case(B): If the Case(A) does not hold. That means $\kappa_k \geqslant \delta_k\kappa_1$. Since $\kappa_l \geqslant 0$, we have,

$$\sigma_k \geqslant \kappa_1\kappa_2\cdots\kappa_k \geqslant \delta_k^k\kappa_1^k.$$

This implies the bound of κ_1. $\square$

We have three remarks about the above C^2 estimate.

Remark 21. *Following the same arguments, we can establish similar C^2 estimates for convex solutions of σ_k-Hessian equation*

$$\sigma_k(\nabla^2 u) = f(x, u, \nabla u). \tag{4.35}$$

Remark 22. *The key in the proof of C^2 estimate is a good choice of test function P. Here we pick $P = \sum_j \kappa_j^2$. Our arguments can be adopted for $P = \sum_j \kappa_j^m$ for any $m \geqslant 2$.*

Remark 23. *The assumption of convexity of solutions can be weakened. Our proof works if the principal curvatures are bounded from below by some constant, with test function modified as $\log P + g(u) + a|X|^2$. The convexity assumption can also be weakened to $k+1$ convex.*

5. The prescribed curvature equations

The a priori estimates we establish in the previous sections may yield existence of solutions to the prescribing equation (1.1). By Theorem 1 and Theorem 4, we need to obtain C^1 bounds for solutions. The treatment of this section follows largely from Caffarelli-Nirenberg-Spruck [8]. We are looking for starshaped hypersurface M.

For $x \in \mathbb{S}^n$, let

$$X(x) = \rho(x)x,$$

be the position vector of the hypersurface M.

First is the gradient bound.

Lemma 24. *If the hypersurface X satisfies condition (1.7) and ρ has positive upper and lower bound, then there is a constant C depending on the minimum and maximum values of ρ, such that,*

$$|\nabla\rho| \leqslant C.$$

Proof. We only need to obtain a positive lower bound of u. Following [15], we consider

$$\phi = -\log u + \gamma(|X|^2).$$

Assume X_0 is the maximum value point of ϕ. If X is not parallel to the normal direction of X at X_0 , we may choose the local orthonormal frame $\{e_1, \cdots, e_n\}$ on M satisfying

$$\langle X, e_1\rangle \neq 0, \quad \text{and} \quad \langle X, e_i\rangle = 0, \quad i \geqslant 2.$$

Then, at X_0,

$$\begin{aligned} u_i &= 2u\gamma'\langle X, e_i\rangle, \qquad (5.1)\\ \phi_{ii} &= -\frac{1}{u}[h_{ii1}\langle X, e_1\rangle + h_{ii} - h_{ii}^2 u] + [(\gamma')^2 + \gamma''](|X|_i^2)^2 + \gamma'|X|_{ii}^2. \end{aligned}$$

Thus,

$$\begin{aligned} 0 \geqslant \sigma_k^{ii}\phi_{ii} &= -\frac{\langle X, e_1\rangle}{u}\sigma_k^{ii}h_{ii1} - \frac{\sigma_k^{ii}h_{ii}}{u} + \sigma_k^{ii}h_{ii}^2 + 4[(\gamma')^2 + \gamma'']\langle X, e_1\rangle^2\sigma_k^{11} \\ &\quad + \gamma'\sigma_k^{ii}[2 - 2uh_{ii}]. \end{aligned} \tag{5.2}$$

By (4.5),

$$\sigma_k^{ii}h_{ii1} = d_X f(e_1) + h_{11}d_\nu f(e_1).$$

Using (5.1) and $\langle X, e_1\rangle \neq 0$, we have

$$h_{11} = 2\gamma' u.$$

Hence, (5.2) becomes,

$$\begin{aligned} 0 \geqslant & -\frac{1}{u}[\langle X, e_1\rangle d_X f(e_1) + kf] + \sigma_k^{ii}h_{ii}^2 + 4[(\gamma')^2 + \gamma'']\langle X, e_1\rangle^2\sigma_k^{11} \\ & + \gamma'\sigma_k^{ii}[2 - 2uh_{ii}] - 2\gamma'\langle X, e_1\rangle d_\nu f(e_1). \end{aligned} \tag{5.3}$$

Condition (1.7) yields,

$$0 \geqslant \rho^{k-1}[kf + \rho\frac{\partial f(X,\nu)}{\partial\rho}] = \rho^{k-1}[kf + \rho d_X f(\frac{\partial X}{\partial\rho})] = \rho^{k-1}[kf + d_X f(X)].$$

Since in the local frame, $\langle X, e_i\rangle = 0$, for $i \geqslant 2$, so $X = \langle X, e_1\rangle e_1$. (5.3) becomes,

$$0 \geqslant \sigma_k^{ii}h_{ii}^2 + 4[(\gamma')^2 + \gamma'']\langle X, e_1\rangle^2\sigma_k^{11} + \gamma'\sigma_k^{ii}[2 - 2uh_{ii}] - 2\gamma' d_\nu f(X). \tag{5.4}$$

Choose

$$\gamma(t) = \frac{\alpha}{t},$$

for sufficient large α. Therefore,

$$4[(\gamma')^2+\gamma'']|X|^2\sigma_k^{11}+2\gamma'\sum_i\sigma_k^{ii}+\sigma_k^{ii}h_{ii}^2\geqslant C\alpha^2\sigma_k^{11},$$

and

$$\sigma_k^{11}\geqslant\sigma_{k-1}\geqslant\sigma_k^{\frac{k-1}{k}}=f^{\frac{k-1}{k}}.$$

(5.4) is simplified to

$$0 \geqslant C_0\alpha^2 f^{\frac{k-1}{k}}+\frac{2\alpha}{|X|^4}d_\nu f(X). \tag{5.5}$$

By the assumption on C^0 bound, we have $|d_\nu f(X)|\leqslant C$. Rewrite (5.5),

$$0\geqslant f^{\frac{k-1}{k}}\alpha(C_0\alpha+\frac{2}{k|X|^4}d_\nu f^{\frac{1}{k}})>0,$$

for sufficient large α, contradiction. That is, at X_0, X is parallel to the normal direction. Since u is the support function, $u=\langle X,\nu\rangle=|X|$. □

Theorem 25. *Suppose $k=2$, and f satisfies condition (1.6) and (1.7), equation* (1.1) *has only one admissible solution in $\{r_1<|X|<r_2\}$.*

Proof. We use continuity method to solve the existence result. For $0\leqslant t\leqslant 1$, according to [8], we consider the family of functions,

$$f^t(X,\nu)=tf(X,\nu)+(1-t)C_n^2[\frac{1}{|X|^k}+\varepsilon(\frac{1}{|X|^k}-1)],$$

where ε is sufficient small constant satisfying

$$0<f_0\leqslant\min_{r_1\leqslant\rho\leqslant r_2}(\frac{1}{\rho^k}+\varepsilon(\frac{1}{\rho^k}-1)),$$

and f_0 is some positive constant.

At $t=0$, we let $X_0(x)=x$. It satisfies $\sigma_2(\kappa(X_0))=C_n^2$. It is obvious that $f^t(X,\nu)$ satisfies the barrier condition in the Introduction (1) and (2) with strict inequality for $0\leqslant t<1$. Suppose that X_t is the solution of f^t. Then, at the maximum point of $\rho_t=|X_t|$, the outer normal direction ν_t is parallel to the position vector X_t. If that point touches the sphere $|X|=r_2$, then , at that point,

$$\frac{C_n^2}{r_2^2}\leqslant\sigma_2(\kappa(X_t))=f(X_t,\frac{X_t}{|X_t|})<\frac{C_n^2}{r_2^2}.$$

It is a contradiction. That is $\rho_t\leqslant r_2$. Similar argument yields that $\rho_t\geqslant r_1$. C^0 estimate follows.

Since the outer normal direction

$$\nu=\frac{\rho x-\nabla\rho}{\sqrt{\rho^2+|\nabla\rho|^2}},$$

replace ρ by $t\rho$, ν does not change. The same argument in [8] gives the openness for $0\leqslant t<1$.

In view of Evans-Krylov theory, we only need gradient and C^2 estimate to complete the closedness part. With the positive upper and lower bound for ρ, Lemma 24 gives the gradient estimate. The C^2 estimate follows from Theorem 4.

The proof of the uniqueness is same as in [8]. $\square$

Now we consider the existence of convex solutions of equation (1.1) for the general k.

Lemma 26. *For any strictly convex solution of equation* (1.1) *and* $f \in C^2(\Gamma \times \mathbb{S}^n)$*, if* ρ *have a upper bound, then the global* C^2 *estimate (1.2) holds.*

Proof. First, we will prove that each convex hypersurface satisfying equation (1.1) contains some small ball whose radius has a uniform positive lower bound. Since our hypersurface is convex with an upper bound, we only need to prove that the volume of the domain enclosed by M has a uniform lower positive bound. Let u be the support function of the hypersurface M. Since M is strictly convex, the support function u can be viewed a function on the unit sphere. Let,

$$V_k(M) = \int_{\mathbb{S}^n} \sigma_k(W_u).$$

Here we denote $(W_u)_{ij} = u_{ij} + u\delta_{ij}$. We can rewrite equation (1.1),

$$\sigma_{n-k}(W_u) = f\sigma_n(W_u) \leqslant C\sigma_n(W_u).$$

Hence,

$$\int_{\mathbb{S}^n} u\sigma_{n-k}(W_u) \leqslant C \int_{\mathbb{S}^n} u\sigma_n(W_u).$$

Therefore,

$$V_{n-k+1}(M) \leqslant CV_{n+1}(M).$$

Here V_{n+1} is the volume of the domain enclosed by the hypersurface M. By the isoperimetric type inequality of Alexsandrov-Frenchel,

$$V_{n+1}^{\frac{n-k+1}{n+1}}(M) \leqslant CV_{n-k+1}(M) \leqslant CV_{n+1}(M).$$

That is, the volume is bounded from below.

For any hypersurface M satisfying (1.1), we may assume that the center of the above unit ball is X_M. Let $X - X_M = \rho' y$, where y is another position vector of unit sphere. Obviously, ρ' has positive upper and lower bound. We can view M as a radial graph over the unit sphere centered at X_M. By the convexity assumption, $\nabla\rho'$ is bounded by $\max_{\mathbb{S}^n} \rho'$. This gives the C^1 bound for M. Theorem 1 yields global C^2 estimate of ρ'. Thus, C^2 estimate of ρ follows. $\square$

Proof of Theorem 5. The existence can be deduced by the degree theory as in [13]. Since the main arguments are the same, we only give an outline. Consider an auxiliary equation,

$$\sigma_k(\kappa(X)) = f^t(X,\nu), \tag{5.6}$$

where

$$f^t = \big(tf^{\frac{1}{k}}(X,\nu) + (1-t)(C_n^k[\frac{1}{|X|^k} + \varepsilon(\frac{1}{|X|^k} - 1)])^{\frac{1}{k}}\big)^k.$$

By the assumptions in Theorem 5, f^t satisfies the structural condition in the Constant Rank Theorem (Theorem 1.2 in [16]). This implies the convexity of solutions to equation (5.6). Lemma 26 gives C^2 estimates. The Evans-Krylov Theorem yields a priori $C^{3,\alpha}$ estimates. To establish the existence, we only need to compute the degree at $t = 0$. It is obvious that, in this case, $\rho \equiv 1$ is the solution. Then the same computation in [13] yields the degree in non-zero. Hence, we have the existence part of the theorem. The strictly convex follows from constant rank theorem in [16]. □

6. Some examples

Curvature estimate (1.2) is special for equation (1.1). It fails for convex hypersurfaces in $\mathbb{R}^{n+1}$ for another type of fully nonlinear elliptic curvature equations. We construct such examples for hypersurfaces satisfying the quotient of curvature equation,

$$\frac{\sigma_k(\kappa)}{\sigma_l(\kappa)} = f(X, \nu). \tag{6.1}$$

Choose a smooth function u defined on sphere such that the spherical Hessian

$$W_u = (u_{ij} + u\delta_{ij}) \in \Gamma_{n-1}$$

but $\sigma_n(W_u(y_0)) < 0$ at some point $y_0 \in \mathbb{S}^n$. The existence of such functions are well known (e.g., [1]). Set $\tilde{f} = \sigma_{n-1}(W_u)$, so f is a positive and smooth function. Set

$$u_t = (1-t) + tu.$$

We have $W_{u_t} \in \Gamma_{n-1}$ and

$$\tilde{f}_t = \sigma_{n-1}(W_{u_t}), \tag{6.2}$$

is smooth and positive. Obviously, when t is close to 0, W_{u_t} is positive definite. There is some $1 > t_0 > 0$, such that $W_{u_t} > 0$ for $t < t_0$, and

$$\det(W_{u_{t_0}}(x_0)) = 0,$$

for some $x_0 \in \mathbb{S}^n$. Denote Ω_u to the convex body determined by its support function u_t, $0 \le t < t_0$.

Claim: *for each $0 \le t < t_0$ after a proper translation of the origin, we have some positive constant c_0 independent of $t < t_0$ such that,*

$$u_t(x) \geqslant c_0 > 0 \quad \textit{for } \forall x \in \mathbb{S}^n \quad \textit{and } t < t_0. \tag{6.3}$$

That is each Ω_{u_t} contains a ball of fixed radius, $t < t_0$.

Let's first consider $k = n, l = 1$ in equation (6.1). For $0 \le t < t_0$, denote

$$M_t = \partial\Omega_{u_t}. \tag{6.4}$$

For each $0 \le t < t_0$, M_t is strictly convex. By (6.3), we have uniform C^1 estimate for the radial function ρ_t, where $M_t = \{\rho_t(z)z | z \in \mathbb{S}^n\}$. We can rewrite the equation (6.2),

$$\frac{\sigma_n}{\sigma_1}(\kappa_1, \cdots, \kappa_n) = \frac{1}{\tilde{f}_t(\nu)}. \tag{6.5}$$

Since $\sigma_n(W_{u_{t_0}}(x_0)) = 0$, the principal curvature of M_t will blow up at some points as $t \to t_0$. The uniform curvature estimate (1.2) for equation (6.5) can not hold.

We prove **claim**. Fix $0 \leq t < t_0$, after a proper translation, we may assume the origin is inside the convex body Ω_{u_t}. It follows from the construction,

$$\tilde{f}_t \geqslant c > 0,$$

for some constant $c > 0$ and for any $t < t_0, x \in \mathbb{S}^n$, and

$$\|u_t\|_{C^3(\mathbb{S}^n)} \leqslant C, \tag{6.6}$$

where constant C is independent of t. At the maximum value points x_0^t of functions u_t, we have,

$$W_{u_t}(x_0^t) \leqslant u_t(x_0^t) I.$$

Hence,

$$u_t(x_0^t) \geqslant \tilde{f}_t^{\frac{1}{n-1}}(x_0^t) \geqslant C > 0.$$

Estimate (6.6) implies that there is some uniform radius R such that on the disc $B_R(x_0^t)$ with center at x_0,

$$u_t(x) \geqslant \frac{C}{2} > 0, \forall x \in B_R(x_0^t).$$

By the Minkowski identity,

$$\int_{\mathbb{S}^n} \sigma_n(W_{u_t}) = c_n \int_{\mathbb{S}^n} u_t \sigma_{n-1}(W_{u_t}) = c_n \int_{\mathbb{S}^n} u_t f_t \geqslant c_n \int_{B_R(x_0^t) \cap \mathbb{S}^n} u_t \tilde{f}_t \geqslant \tilde{c} > 0.$$

Hence, there exists $y_0^t \in \mathbb{S}^n$ satisfying

$$\sigma_n(W_{u_t}(y_0^t)) \geqslant \frac{\tilde{c}}{\omega_n}.$$

By (6.6), there are some uniform radius $\tilde{R} > 0$, such that for $y \in \mathbb{S}^n \cap B_{\tilde{R}}(y_0^t)$, we have,

$$W_{u_t}(y) \geqslant \frac{\tilde{c}}{2\omega_n} > 0.$$

Hence, near the points $\nu^{-1}(y_0^t)$, the hypersuface M_t is pinched by two fixed paraboloids locally and uniformly. Thus, Ω_{u_t} contains a small ball whose radius has a uniform positive lower bound. Move the origin to the center of the ball, this yields (6.3). The claim is verified.

Proof of Theorem 2. We use the some sequence $\{M_t\}$ defined in (6.4) to construct f_t in (1.3). For any $m = 0, 1, \cdots, n-1$, for any $0 \leq t < t_0$, $\sigma_m(W_{u_t}) \in C^\infty(\mathbb{S}^n)$. By (6.2), (6.6) and Newton-MacLaurin inequality, there exists $c > 0$ independent of t,

$$c \leq \sigma_m(W_{u_t}) \leq \frac{1}{c}.$$

Since $\frac{\sigma_k}{\sigma_l}(\kappa_{M_t}) \equiv \frac{\sigma_{n-k}}{\sigma_{n-l}}(W_{u_t})$, there exists $a > 0$ independent of t, such that for any $1 \leq l < k \leq n$,

$$a \leq \frac{\sigma_k}{\sigma_l}(\kappa_{M_t}) \leq \frac{1}{a}.$$

M_t satisfies equation

$$\frac{\sigma_k}{\sigma_l}(\kappa_{M_t}) = \frac{\sigma_{n-k}}{\sigma_{n-l}}(W_{u_t}) = f_t(\nu),$$

$f_t, \frac{1}{f_t} \in C^\infty(\mathbb{S}^n)$ and the norms of $\|f_t\|_{C^3(\mathbb{S}^n)}$ and $\|\frac{1}{f_t}\|_{C^3(\mathbb{S}^n)}$ under control independent of $0 \le t < t_0$. That is, M_t satisfies conditions in theorem. The previous analysis on M_t indicates that estimate (1.2) fails and the principal curvature of M_t will blow up at some points when $t \to t_0$. $\square$

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