A PROOF OF THE ALEXANDEROV'S UNIQUENESS THEOREM FOR CONVEX SURFACES IN $\mathbb{R}^3$

PENGFEI GUAN, ZHIZHANG WANG, AND XIANGWEN ZHANG

ABSTRACT. We give a new proof of a classical uniqueness theorem of Alexandrov [4] using the weak uniqueness continuation theorem of Bers-Nirenberg [8]. We prove a version of this theorem with the minimal regularity assumption: the spherical hessians of the corresponding convex bodies as Radon measures are nonsingular.

We give a new proof of the following uniqueness theorem of Alexandrov, using the Weak Unique Continuation Theorem of Bers-Nirenberg [8].

Theorem 1 (Theorem 9 in [4]). *Suppose M_1 and M_2 are two closed strictly convex C^2 surfaces in $\mathbb{R}^3$, suppose $f(y_1, y_2) \in C^1$ is a function such that $\frac{\partial f}{\partial y_1}\frac{\partial f}{\partial y_2} > 0$. Denote $\kappa_1 \geq \kappa_2$ the principal curvatures of surfaces, and denote ν_{M_1} and ν_{M_2} the Gauss maps of M_1 and M_2 respectively. If*

$$f(\kappa_1(\nu_{M_1}^{-1}(x), \kappa_2(\nu_{M_1}^{-1}(x)) = f(\kappa_1(\nu_{M_2}^{-1}(x), \kappa_2(\nu_{M_2}^{-1}(x)), \quad \forall x \in \mathbb{S}^2, \tag{1}$$

then M_1 is equal to M_2 up to a translation.

This classical result was first proved for analytical surfaces by Alexandrov in [3], for C^4 surfaces by Pogorelov in [20], and Hartman-Wintner [14] reduced regularity to C^3, see also [21]. Pogorelov [22, 23] published certain uniqueness results for C^2 surfaces, these general results would imply Theorem 1 in C^2 case. It was pointed out in [19] that the proof of Pogorelov is erroneous, it contains an uncorrectable mistake (see page 301-302 in [19]). There is a counter-example of Martinez-Maure [15] (see also [19]) to the main claims in [22, 23]. The results by Han-Nadirashvili-Yuan [13] imply two proofs of Theorem 1, one for C^2 surfaces and another for $C^{2,\alpha}$ surfaces. The problem is often reduced to a uniqueness problem for linear elliptic equations in appropriate settings, either on $\mathbb{S}^2$ or in $\mathbb{R}^3$, we refer [4, 21]. Here we will concentrate on the corresponding equation on $\mathbb{S}^2$, as in [11]. The advantage in this setting is that it is globally defined.

If M is a strictly convex surface with support function u, then the principal curvatures at $\nu^{-1}(x)$ are the reciprocals of the principal radii λ_1, λ_2 of M, which are the eigenvalues of spherical Hessian $W_u(x) = (u_{ij}(x) + u(x)\delta_{ij})$ where u_{ij} are the covariant derivatives with respect to any given local orthonormal frame on $\mathbb{S}^2$. Set

$$\tilde{F}(W_u) =: f(\frac{1}{\lambda_1(W_u)}, \frac{1}{\lambda_2(W_u)}) = f(\kappa_1, \kappa_2). \tag{2}$$

Date: September 8, 2014.
1991 *Mathematics Subject Classification.* 53A05, 53C24.
Research of the first author was supported in part by an NSERC Discovery Grant.

In view of Lemma 1 in [5], if f satisfies the conditions in Theorem 1, then $\tilde{F}^{ij} = \frac{\partial \tilde{F}}{\partial w_{ij}} \in L^\infty$ is uniformly elliptic. In the case $n = 2$, it can be read off from the explicit formulas

$$\lambda_1 = \frac{\sigma_1(W_u) - \sqrt{\sigma_1(W_u)^2 - 4\sigma_2(W_u)}}{2}, \quad \lambda_2 = \frac{\sigma_1(W_u) + \sqrt{\sigma_1(W_u)^2 - 4\sigma_2(W_u)}}{2}.$$

As noted by Alexanderov in [5], $\tilde{F}^{ij}$ in general is not continuous if $f(y_1, y_2)$ is not symmetric (even f is analytic).

We want to address when Theorem 1 remains true for convex bodies in $\mathbb{R}^3$ with weakened regularity assumption. In the Bruun-Minkowski theory, the uniqueness of Alexandrov-Fenchel-Jessen [1, 2, 10] states that, if two bounded convex bodies in $\mathbb{R}^{n+1}$ have the same kth area measures on $\mathbb{S}^n$, then these two bodies are the same up to a rigidity motion in $\mathbb{R}^{n+1}$. Though for a general convex body, the principal curvatures of its boundary may not be defined. But one can always define the support function u, which is a function on $\mathbb{S}^2$. By the convexity, then $W_u = (u_{ij} + u\delta_{ij})$ is a Radon measure on $\mathbb{S}^2$. Also, by Alexandrov's theorem for the differentiability of convex functions, W_u is defined for almost every point $x \in \mathbb{S}^2$. Denote $\mathcal{N}$ to be the space of all positive definite 2×2 matrices, and let G be a function defined on $\mathcal{N}$. For a support function u of a bounded convex body Ω_u, $G(W_u)$ is defined for *a.e.* $x \in \mathbb{S}^2$. For fixed support functions u^l of $\Omega_{u^l}, l = 1, 2$, there is $\Omega \subset \mathbb{S}^2$ with $|\mathbb{S}^2 \setminus \Omega| = 0$ such that W_{u^1}, W_{u^2} are pointwise finite in Ω. Set $P_{u^1,u^2} = \{W \in \mathcal{N} | \exists x \in \Omega, W = W_{u^1}(x), or W = W_{u^2}(x)\}$, let $\mathcal{P}_{u^1,u^2}$ be the convex hull of P_{u^1,u^2} in $\mathcal{N}$.

We establish the following slightly more general version of Theorem 1.

Theorem 2. *Suppose Ω_1 and Ω_2 are two bounded convex bodies in $\mathbb{R}^3$. Let $u^l, l = 1, 2$ be the corresponding supporting functions respectively. Suppose the spherical Hessians $W_{u^l} = (u^l_{ij} + \delta_{ij} u^l)$ (in the weak sense) are two non-singular Radon measures. Let $G : \mathcal{N} \to \mathbb{R}$ be a $C^{0,1}$ function such that*

$$\Lambda I \geq (G^{ij})(W) := (\frac{\partial G}{\partial W_{ij}})(W) \geq \lambda I > 0, \quad \forall W \in \mathcal{P}_{\sqcap^\infty, \sqcap^\in},$$

for some positive constants Λ, λ. If

$$G(W_{u^1}) = G(W_{u^2}), \tag{3}$$

at almost every parallel normal $x \in \mathbb{S}^2$, then Ω_1 is equal to Ω_2 up to a translation.

Suppose u^1, u^2 are the support functions of two convex bodies Ω_1, Ω_2 respectively, and suppose $W_{u,}, l = 1, 2$ are defined and they satisfy equation (3) at some point $x \in \mathbb{S}^2$. Then, for $u = u^1 - u^2$, $W_u(x)$ satisfies equation

$$F^{ij}(x)(W_u(x)) = 0, \tag{4}$$

with $F^{ij}(x) = \int_0^1 \frac{\partial \tilde{F}}{\partial W_{ij}}(tW_{u^1}(x) + (1-t)W_{u^2}(x))dt$. By the convexity, $W_{u^l}, l = 1, 2$ exist almost everywhere on $\mathbb{S}^2$. If they satisfy equation (3) almost everywhere, equation (4) is verified almost everywhere. Note that u may *not be a solution (even in a weak sense) of partial differential equation (4)*. The classical elliptic theory (e.g., [16, 18, 8]) requires $u \in W^{2,2}$ in order to make sense of u as a weak solution of (4). A main step in the proof of Theorem 2 is to show that with the assumptions in the theorem, $u = u^1 - u^2$ is indeed in $W^{2,2}(\mathbb{S}^2)$. The proof will appear in the last part of the paper.

Let's now focus on $W^{2,2}$ solutions of differential equation (4), with general uniformly elliptic condition on tensor F^{ij} on $\mathbb{S}^2$:

$$\lambda|\xi|^2 \leq F^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \ \forall x \in \mathbb{S}^2, \xi \in \mathbb{R}^2, \tag{5}$$

for some positive numbers λ, Λ. The aforementioned proofs of Theorem 1 ([20, 14, 21, 13]) all reduce to the statement that any solution of (5) is a linear function, under various regularity assumptions on F^{ij} and u. Equation (4) is also related to minimal cone equation in $\mathbb{R}^3$ ([13]). The following result was proved in [13].

Theorem 3 (Theorem 1.1 in [13])**.** *Suppose $F^{ij}(x) \in L^\infty(\mathbb{S}^2)$ satisfies (5), suppose $u \in W^{2,2}(\mathbb{S}^2)$ is a solution of (4). Then, $u(x) = a_1x_1 + a_2x_2 + a_3x_3$ for some $a_i \in \mathbb{R}$.*

There the original statement in [13] is for 1-homogeneous $W^{2,2}_{loc}(\mathbb{R}^3)$ solution v of equation

$$\sum_{i,j=1}^{3} a^{ij}(X)v_{ij}(X) = 0. \tag{6}$$

These two statements are equivalent. To see this, set $u(x) = \frac{v(X)}{|X|}$ with $x = \frac{X}{|X|}$. By the homogeneity assumption, the radial direction corresponds to null eigenvalue of $\nabla^2 v$, the other two eigenvalues coincide the eigenvaules of the spherical Hessian of $W = (u_{ij} + u\delta_{ij})$. $v(X) \in W^{2,2}_{loc}(\mathbb{R}^3)$ is a solution to (6) if and only if $u \in W^{2,2}(\mathbb{S}^2)$ is a solution to (4) with $F^{ij}(x) = \langle e_i, Ae_j\rangle$, where $A = (a^{ij}(\frac{X}{|X|}))$ and (e_1, e_2) is any orthonormal frame on $\mathbb{S}^2$.

The proof in [13] uses gradient maps and support planes introduced by Alexandrov, as in [3, 20, 21]. We give a different proof of Theorem 3 using the maximum principle for smooth solutions and the unique continuation theorem of Bers-Nirenberg [8], working purely on solutions of equation (4) on $\mathbb{S}^2$.

Note that F in Theorem 2 (and Theorem 1) is not assumed to be symmetric. The weak assumption $F^{ij} \in L^\infty$ is needed to deal with this case. This assumption also fits well with the weak unique continuation theorem of Bers-Nirenberg. This beautiful result of Bers-Nirenberg will be used in a crucial way in our proof. If $u \in W^{2,2}(\mathbb{S}^2)$, $u \in C^\alpha(\mathbb{S}^2)$ for some $0 < \alpha < 1$ by the Sobolev embedding theorem. Equation (4) and $C^{1,\alpha}$ estimates for 2-d linear elliptic PDE (e.g., [16, 18, 8]) imply that u is in $C^{1,\alpha}(\mathbb{S}^2)$ for some $\alpha > 0$ depending only on $\|u\|_{C^0}$ and the ellipticity constants of F^{ij}. This fact will be assumed in the rest of the paper.

The following lemma is elementary.

Lemma 4. *Suppose $F^{ij} \in L^\infty(\mathbb{S}^2)$ satisfies (5), suppose at some point $x \in \mathbb{S}^2$, $W_u(x) = (u_{ij}(x) + u(x)\delta_{ij})$ satisfies (4). Then,*

$$|W_u|^2(x) \leq -\frac{2\Lambda}{\lambda}\det W_u(x).$$

Proof. At x, by equation (4),

$$\det W_u = -\frac{1}{F^{22}}\Big(F^{11}W_{11}^2 + 2F^{12}W_{11}W_{12} + F^{22}W_{12}^2\Big) \leq -\frac{\lambda}{\Lambda}\big(W_{11}^2 + W_{12}^2\big), \tag{7}$$

and similarly, $\det W_u \leq -\frac{\lambda}{\Lambda}\big(W_{22}^2 + W_{21}^2\big)$. Thus,

$$\big(W_{11}^2 + W_{12}^2 + W_{21}^2 + W_{22}^2\big) \leq -\frac{2\Lambda}{\lambda}\det W_u. \tag{8}$$

□

For each $u \in C^1(\mathbb{S}^2)$, set $X_u = \sum_i u_i e_i + u e_{n+1}$. For any unit vector E in $\mathbb{R}^3$, define

$$\phi_E(x) = \langle E, X_u(x)\rangle, \quad \text{and} \quad \rho_u(x) = |X_u(x)|^2, \tag{9}$$

where $\langle , \rangle$ is the standard inner product in $\mathbb{R}^3$. The function ρ was introduced by Weyl in his study of Weyl's problem [25]. It played important role in Nirenberg's solution of the Weyl's problem in [17]. Our basic observation is that there is a maximum principle for ρ_u and ϕ_E.

Lemma 5. *Suppose $U \subset \mathbb{S}^2$ is an open set, $F^{ij} \in C^1(U)$ is a tensor in U and $u \in C^3(U)$ satisfies equation (4), then there are two constants C_1, C_2 depending only on the C^1-norm of F^{ij} such that*

$$F^{ij}(\rho_u)_{ij} \geq -C_1|\nabla\rho_u|, \quad F^{ij}(\phi_E)_{ij} \geq -C_2|\nabla\phi_E| \quad \text{in } U. \tag{10}$$

Proof. Pick any orthonormal frame e_1, e_2, we have

$$(X_u)_i = W_{ij}e_j, \quad (X_u)_{ij} = W_{ijk}e_k - W_{ij}\vec{x}. \tag{11}$$

By Codazzi property of W and (4),

$$\frac{1}{2}F^{ij}(\rho_u)_{ij} = \langle X_u, F^{ij}W_{ijk}e_k\rangle + F^{ij}W_{ik}W_{kj} = -u_k F^{ij}_{,k}W_{ij} + F^{ij}W_{ik}W_{kj}.$$

On the other hand, $\nabla\rho_u = 2W\cdot(\nabla u)$. At the non-degenerate points (i.e., $\det W \neq 0$), $\nabla u = \frac{1}{2}W^{-1}\cdot\nabla\rho_u$, where W^{-1} denotes the inverse matrix of W. Now,

$$2u_k F^{ij}_{,k}W_{ij} = W^{kl}(\rho_u)_l F^{ij}_{,k}W_{ij} = (\rho_u)_l F^{ij}_{,k}\frac{A^{kl}W_{ij}}{\det W}. \tag{12}$$

where A^{kl} denote the co-factor of W_{kl}.

The first inequality in (10) follows (8) and (12).

The proof for ϕ_E follows the same argument and the following facts:

$$F^{ij}(\phi_E)_{ij} = -\langle E, e_k\rangle F^{ij}_{,k}W_{ij}, \quad \nabla\phi_E = W\cdot\langle E, e_k\rangle.$$

□

Lemma 5 yields immediately Theorem 1 in C^3 case, which corresponds to the Hartman-Wintner theorem ([14]).

Corollary 6. *Suppose $f \in C^2$ and symmetric, M_1, M_2 are two closed convex C^3 surfaces satisfy conditions in Theorem 1 , then the surfaces are the same up to a translation.*

Proof. Since $f \in C^2$ is symmetric, F^{ij} in (4) is in $C^1(\mathbb{S}^2)$ and $u \in C^3(\mathbb{S}^2)$. By Lemma 5 and the strong maximum principle, X_u is a constant vector. $\square$

To precede further, set

$$\mathcal{M} = \{p \in \mathbb{S}^2 \ : \ \rho_u(p) = \max_{q \in \mathbb{S}^2} \rho_u(q)\},$$

for each unit vector $E \in \mathbb{R}^3$,

$$\mathcal{M}_E = \{p \in \mathbb{S}^2 \ : \ \phi_E(p) = \max_{q \in \mathbb{S}^2} \phi_E(q)\}.$$

Lemma 7. *$\mathcal{M}$ and $\mathcal{M}_E$ have no isolated points.*

Proof. We prove the lemma for $\mathcal{M}$, the proof for $\mathcal{M}_E$ is the same. If point $p_0 \in \mathcal{M}$ is an isolated point, we may assume $p_0 = (0,0,1)$. Pick $\bar{U}$ a small open geodesic ball centered at p_0 such that $\bar{U}$ is properly contained in $\mathbb{S}^2_+$, and pick a sequence of smooth $2-$tensor $(F^{ij}_\epsilon) > 0$ which is convergent to (F^{ij}) in L^∞-norm in $\bar{U}$. Consider

$$\begin{cases} F^{ij}_\epsilon(u^\epsilon_{ij} + u^\epsilon \delta_{ij}) & = \quad 0 \ \text{ in } \bar{U} \\ \qquad u^\epsilon & = \quad u \ \text{ on } \partial\bar{U}. \end{cases} \tag{13}$$

Since $x_3 > 0$ in $\mathbb{S}^2_+$, one may write $u^\epsilon = x_3 v^\epsilon$ in $\bar{U}$. As $(x_3)_{ij} = -x_3\delta_{ij}$, it easy to check v^ϵ satisfies

$$F^{ij}_\epsilon v^\epsilon_{ij} + b_k v^\epsilon_k = 0, \quad \text{in } \bar{U}.$$

Therefore, (13) is uniquely solvable.

Since $p_0 \in \mathcal{M}$ is an isolated point, there are open geodesic balls $\bar{U}' \subset \bar{U}$ centered at p_0 and a small $\delta > 0$ such that

$$\rho_u(p_0) - \rho_u(p) \geq \delta \ \text{ for } \ \forall p \in \partial\bar{U}'. \tag{14}$$

By the $C^{1,\alpha}$ estimates for linear elliptic equation in dimension two and the uniqueness of the Dirichlet problem ([16, 8, 18]), $\exists \epsilon_k$ such that

$$\|u - u^{\epsilon_k}\|_{C^{1,\alpha}(\bar{U}')} \to 0, \quad \|\rho_u - \rho_{u^{\epsilon_k}}\|_{C^\alpha(\bar{U}')} \to 0.$$

Together with (14), if ϵ_k small enough, there is a local maximal point of $\rho_{u^{\epsilon_k}}$ in $\bar{U}' \subset \bar{U}$. Since $u^{\epsilon_k}, F^{ij}_\epsilon \in C^\infty(\bar{U}')$ satisfy (13), it follows from Lemma 5 and the strong maximum principle that $\rho_{u^{\epsilon_k}}$ must be constant in $\bar{U}'$, $\forall \epsilon_k$ in small enough. This implies ρ is constant in $\bar{U}'$. Contradiction. $\square$

We now prove Theorem 3.

Proof of Theorem 3. For any $p_0 \in \mathcal{M}$, if $\rho_u(p_0) = 0$, then $u \equiv 0$. We may assume $\rho_u(p_0) > 0$. Set $E := \frac{X_u(p_0)}{|X_u(p_0)|}$. Choose another two unit constant vectors β_1, β_2 with $< \beta_i, \beta_j >= \delta_{ij}, \beta_i \perp E$ for $i, j = 1, 2$. Under this orthogonal coordinates in $\mathbb{R}^3$,

$$X_u(p) = a(p)E + b_1(p)\beta_1 + b_2(p)\beta_2, \ \forall p \in \mathcal{M}_E. \tag{15}$$

On the other hand, $\phi_E(p) = \rho_u^{1/2}(p_0), \forall p \in \mathcal{M}_E$. Thus,

$$a(p) = \rho_u^{1/2}(p_0),\ b_1(p) = b_2(p) = 0,\ \forall p \in \mathcal{M}_E. \tag{16}$$

Consider the function $\tilde{u}(x) = u(x) - \rho_u^{1/2}(p_0)E \cdot x$. (15) and (16) yield, $\forall p \in \mathcal{M}_E$,

$$\nabla_{e_i}\tilde{u}(p) = \nabla_{e_i}u(p) - \rho_u^{1/2}(p_0)\langle E, e_i\rangle = \langle X_u(p), e_i\rangle - \rho_u^{1/2}(p_0)\langle E, e_i\rangle = 0. \tag{17}$$

Moreover, $\tilde{u}(x)$ also satisfies equation (4). As pointed out in [8], if $\tilde{u}$ satisfies an elliptic equation, $\nabla\tilde{u}$ satisfies an elliptic system of equations. Lemma 7, (17) and the Unique Continuation Theorem of Bers-Nirenberg (P. 13 in [7]) imply $\nabla\tilde{u} \equiv 0$. Thus, $\tilde{u}(x) \equiv \tilde{u}(p_0) = 0$ and $u(x)$ is a linear function on $\mathbb{S}^2$. □

Theorem 1 is a direct consequence of Theorem 3. We now prove Theorem 2.

Proof of Theorem 2. The main step is to show $u = u^1 - u^2 \in W^{2,2}(\mathbb{S}^2)$, using the assumption that $W_{u^l}, l = 1,2$ are non-singular Radon measures. It follows from the convexity, the spherical hessians $W_{u^l}, l = 1,2$ and W_u are defined almost everywhere on $\mathbb{S}^2$ (Alexandrov's Theorem). So, we can define $G(W_{u^l}), l = 1,2$ almost everywhere in $\mathbb{S}^2$. As $W_u^l, l = 1,2$ are nonsingular Radon measures, $W_{u^l} \in L^1(\mathbb{S}^2)$ (see [9]), we also have $W_u \in L^1(\mathbb{S}^2)$. Since u^1, u^2 satisfy $G(W_{u^1}) = G(W_{u^2})$ for almost every parallel normal $x \in \mathbb{S}^2$, there is $\Omega \subset \mathbb{S}^2$ with $|\mathbb{S}^2 \setminus \Omega| = 0$, such that W_u satisfies following equation *pointwise* in Ω,

$$G^{ij}(x)(u_{ij}(x) + u(x)\delta_{ij}) = 0,\ x \in \Omega,$$

where $G^{ij} = \int_0^1 \frac{\partial G}{\partial w_{ij}}(tW_u^1 + (1-t)W_u^2)dt$. By Lemma 4, we can obtain that

$$|W_u|^2 = W_{11}^2 + W_{12}^2 + W_{21}^2 + W_{22}^2 \le -\frac{2\Lambda}{\lambda}\det W_u, \qquad x \in \Omega.$$

On the other hand,

$$\det W_u \le \det W_{\tilde{u}},$$

where $\tilde{u} = u^1 + u^2$. Thus, to prove $u \in W^{2,2}(\mathbb{S}^2)$, it suffices to get an upper bound for $\int_{\mathbb{S}^2}\det W_{\tilde{u}}$.

Recall that $W_{u^l} \in L^1(\mathbb{S}^2)$, so $u^l \in W^{2,1}(\mathbb{S}^2), l = 1,2$ and the same for $\tilde{u}$. This allows us to choose two sequences of smooth convex bodies Ω_ϵ^l with supporting functions u_ϵ^l such that $||\tilde{u}_\epsilon - \tilde{u}||_{W^{2,1}(\mathbb{S}^2)} \to 0$ as $\epsilon \to 0$. By Fatou's Lemma and continuity of the area measures,

$$\int_{\mathbb{S}^2}\det W_{\tilde{u}} = \int_\Omega \det W_{\tilde{u}} \le \liminf_{\epsilon\to 0}\int_{\mathbb{S}^2}\det W_{\tilde{u}_\epsilon} \le V(\Omega^1) + V(\Omega^2) + 2V(\Omega^1, \Omega^2),$$

where $V(\Omega^1), V(\Omega^2)$ denote the volume of the convex bodies Ω^1 and Ω^2 respectively and $V(\Omega^1, \Omega^2)$ is the mixed volume.

It follows that $W_u \in L^2(\mathbb{S}^2)$ and thus, $u \in W^{2,2}(\mathbb{S}^2)$. This implies that u is a $W^{2,2}$ weak solution of the differential equation

$$G^{ij}(x)(u_{ij}(x) + u(x)\delta_{ij}) = 0, \quad \forall x \in \mathbb{S}^2.$$

Finally, the theorem follows directly from Theorem 3. □

Remark 8. *Alexanderov proved in* [3] *that, if* u *is a homogeneous degree* 1 *analytic function in* $\mathbb{R}^3$ *with* $\nabla^2 u$ *definite nowhere, then* u *is a linear function. As a consequence, Alexandrov proved in* [6] *that if a analytic closed convex surface in* $\mathbb{R}^3$ *satisfying the condition* $(\kappa_1 - c)(\kappa_2 - c) \le 0$ *at every point for some constant* c*, then it is a sphere. Martinez-Maure gave a* C^2 *counter-example in* [15] *to this statement, see also* [19]*. The counter-examples in* [15, 19] *indicate that Theorem 3 is not true if* F^{ij} *is merely assumed to be degenerate elliptic. It is an interesting question that under what degeneracy condition on* F^{ij} *so that Theorem 3 is still true, even in smooth case. This question is related to similar questions in this nature posted by Alexandrov* [4] *and Pogorelov* [21].

We shall wrap up this paper by mention a stability type result related with uniqueness. Indeed, by using the uniqueness property proved in Theorem 3, we can prove the following stability theorem via compactness argument.

Proposition 9. *Suppose* $F^{ij}(x) \in L^\infty(\mathbb{S}^2)$ *satisfies (5), and* $u(x) \in W^{2,2}(\mathbb{S}^2)$ *is a solution of the following equation*

$$F^{ij}(x)(W_u)_{ij} = f(x), \ \forall x \in \mathbb{S}^2. \tag{18}$$

Assume that $f(x) \in L^\infty(\mathbb{S}^2)$ *and there exists a point* $x_0 \in \mathbb{S}^2$ *such that* $\rho_u(x_0) = 0$ *(see (9) for the definition of* ρ_u*). Then,*

$$||u||_{L^\infty(\mathbb{S}^2)} \le C_3 ||f||_{L^\infty(\mathbb{S}^2)} \tag{19}$$

holds for some positive constant C_3 *only depends on the ellipticity constants* λ, Λ.

Proof. As mentioned above, we will prove this proposition by a compactness argument. Suppose the desired estimate (19) does not hold, then there exists a sequence of functions $\{f_n(x)\}_{n=1}^\infty$ on $\mathbb{S}^2$ with $||f||_{L^\infty(\mathbb{S}^2)} \le C_4$ and a sequence of points $\{x_n\}_{n=1}^\infty \subset \mathbb{S}^2$ such that $\rho_{u_n}(x_n) = 0$ and $K_n := \frac{||u||_{L^\infty(\mathbb{S}^2)}}{||f||_{L^\infty(\mathbb{S}^2)}} \longrightarrow +\infty$, where $u_n(x)$ is the solution of equation (18) with right hand side replaced by $f_n(x)$.

Let $v_n(x) = \frac{u_n(x)}{K_n ||f||_{L^\infty(\mathbb{S}^2)}}$, then $||v_n||_{L^\infty(\mathbb{S}^2)} = 1$ and $v_n(x)$ satisfies

$$F^{ij}(x)(W_{v_n})_{ij} = \tilde{f}_n := \frac{f_n(x)}{K_n ||f_n||_{L^\infty(\mathbb{S}^2)}}. \tag{20}$$

By the interior $C^{1,\alpha}$ estimates for linear elliptic equation in dimension two ([16, 8, 18]), we have

$$||v_n||_{C^{1,\alpha}(\mathbb{S}^2)} \le C_5 \left(||v_n||_{L^\infty(\mathbb{S}^2)} + ||\tilde{f}_n||_{L^\infty(\mathbb{S}^2)} \right) \le 2C_5$$

for some positive constant $C_5 = C_5(\lambda, \Lambda)$. In particular, this gives that $||\nabla v_n||_{L^\infty(\mathbb{S}^2)} \le C_6$. Now, apply the *a priori* $W^{2,2}$ estimate for linear elliptic equation in dimension two ([16, 8, 18, 12]), we see that $||v_n||_{W^{2,2}(\mathbb{S}^2)} \le C_7$ for some constant $C_7 = C_7(\lambda, \Lambda, C_6)$. It follows from this uniform estimate that, up to a subsequence, $\{v_n(x)\}_{n=1}^\infty$ converges to some function $v(x) \in W^{2,2}(\mathbb{S}^2)$ and $v(x)$ satisfies

$$F^{ij}(x)(W_v)_{ij} = 0, \ a.e. \ x \in \mathbb{S}^2.$$

Then, the previous uniqueness result Theorem 3 tells that $v(x)$ must be a linear function, i.e., there exists a constant vector $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ such that $v(x) = a_1x_1 + a_2x_2 + a_3x_3$.

On the other hand, recall that, by the assumption at the beginning, there exists $x_n \in \mathbb{S}^2$ such that $\rho_{v_n}(x_n) = 0$. Then, up to a subsequence, $x_n \to x_\infty \in \mathbb{S}^2$ and $\rho_v(x_\infty) = 0$. This together with the linear property of $v(x)$ imply that $v(x) \equiv 0$. However, this contradicts with the fact that $||v||_{L^\infty(\mathbb{S}^2)} = 1$ as $||v_n||_{L^\infty(\mathbb{S}^2)} = 1$.

□

As a direct corollary, we have the following stability property for convex surfaces.

Theorem 10. *Suppose M_1, M_2 and f satisfy the same assumptions as in Theorem 3. Define $\mu_1(x) := f(\kappa_1(\nu_{M_1}^{-1}(x), \kappa_2(\nu_{M_1}^{-1}(x))$ and $\mu_2(x) := f(\kappa_1(\nu_{M_2}^{-1}(x), \kappa_2(\nu_{M_2}^{-1}(x))$ for $\forall x \in \mathbb{S}^2$. If $||\mu_1 - \mu_2||_{L^\infty(\mathbb{S}^2)} < \epsilon$, then, module a linear translation, M_1 is very close to M_2. More precisely, suppose u_1, u_2 are the supporting functions of M_1 and M_2 after module the linear translation, then there exists a constant C such that*

$$||u_1 - u_2||_{L^\infty(\mathbb{S}^2)} \leq C||\mu_1 - \mu_2||_{L^\infty(\mathbb{S}^2)}. \tag{21}$$

Finally, it is worth to remark that there are many stability type results for convex surfaces proved in the literature (see [24]). However, almost all the proofs need to use the assumption that $f(\kappa_1, \kappa_2, \cdots, \kappa_n)$ satisfies divergence property. Here, we do not make such kind assumption in this dimension two case. There is one drawback in the above stability result: one could not get the sharp constant via the compactness argument. It would be an interesting question to derive a sharp estimate for (21).

Acknowledgement: The first author would like to thank Professor Louis Nirenberg for stimulation conversations. Our initial proof was the global maximum principle for C^3 surfaces Lemma 5 and Corollary 6 (we only realized the connection of the result of [13] to Theorem 1 afterward). It was Professor Louis Nirenberg who brought our attention to the paper of [15] and suggested using the unique continuation theorem of [8]. That leads to Theorem 2. We want to thank him for his encouragement and generosity.

References

[1] A.D. Alexandrov, *Zur Theorie der gemischten Volumina von konvexen korpern, II. Neue Ungleichungen zwischen den gemischten Volumina und ihre Anwendungen* (in Russian) Mat. Sbornik N.S. 2 (1937), 1205-1238.

[2] A.D. Alexandrov, *Zur Theorie der gemischten Volumina von konvexen korpern, III. Die Erweiterung zweeier Lehrsatze Minkowskis uber die konvexen polyeder auf beliebige konvexe Flachen* (in Russian) Mat. Sbornik N.S. 3, (1938), 27-46.

[3] A.D. Alexandrov, *Sur les théorèmes d'unicité pour les surfaces fermeès.* C. R. (Dokl.) Acad. Sci. URSS, N.S. 22 (1939), 99-102; translation in *Selected works.* Part I. Selected scientific papers, 149-153. Classics of Soviet Mathematics, 4. Gordon and Breach, Amsterdam, 1996.

[4] A.D. Alexandrov, *Uniqueness theorems for surfaces in the large I.* (Russian) Vestnik Leningrad. Univ. 11 (1956), no. 19, 5-17.

[5] A.D. Alexandrov, *Uniqueness theorems for surfaces in the large. II.* (Russian) Vestnik Leningrad. Univ. 12 (1957) no. 7, 15-44.

[6] A.D. Alexandrov, *On the curvature of surfaces* (Russian), Vestnik Leningrad. Univ. 21 (19) (1966) 5-11.

[7] L. Bers and L. Nirenberg, *On a representation theorem for linear elliptic system with discontinuous coefficients and its application*. Edizioni cremonese dells S.A. editrice perrella, Roma, 1954, 111-140.
[8] L. Bers and L. Nirenberg, *On linear and non-linear elliptic boundary value problems in the plane*. Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, 141-167.
[9] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press INC, 1992.
[10] W. Fenchel and B. Jessen, *Mengenfunktionen und konvexe korper*, Det. Kgl. Danske Videnskab. Selskab, Math.-fys. Medd. 16(3), (1938), 1-31.
[11] B. Guan and P. Guan, *Convex Hypersurfaces of Prescribed Curvatures*. Annual of Mathematics, Vol 256, No. 2 (2002), pp 655-673.
[12] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998.
[13] Q. Han, N. Nadirashvili and Y. Yuan, *Linearity of homogeneous order-one solutions to elliptic equations in dimension three*. Comm. Pure and App. Math. Vol. 56, (2003), 425-432.
[14] P. Hartman and A. Wintner, *On the third fundamental form of a surface*. Amer. J. Math. 75, (1953). 298-334.
[15] Y. Martinez-Maure, *Contre-example à une caracteérisation conjecturée de ;as sphére.*,C.R. Acad. Sci. Paris Sér. I Math., 332 (2001), 41-44.
[16] C. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*. Trans. Amer. Math. Soc. 43 (1938), no. 1, 126166.
[17] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*. Comm. Pure Appl. Math. 6, (1953). 337-394.
[18] L. Nirenberg, *On nonlinear elliptic partial differential equations and Hölder continuity*. Comm. Pure Appl. Math. 6, (1953). 103-156.
[19] G. Panina, *New counterexamples to A. D. Alexanderovs hypothesis*. Adv. Geom. 5 (2005), 301-317.
[20] A. V. Pogorelov, *Extension of a general uniqueness theorem of A. D. Aleksandrov to the case of nonanalytic surfaces*. (Russian) Doklady Akad. Nauk SSSR (N.S.) 62, (1948), 297-299.
[21] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*. Translated from the Russian by Israel Program for Scientific Translations. Translations of Mathematical Monographs, Vol. 35. American Mathematical Society, Providence, R.I., 1973.
[22] A. V. Pogorelov, *Solution of a problem of A. D. Aleksandrov*. (Russian) Dokl. Akad. Nauk 360 (1998), 317-319.
[23] A. V. Pogorelov, *Uniqueness theorems for closed convex surfaces*. (Russian) Dokl. Akad. Nauk 366 (1999), 602-604.
[24] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia Math. Appl.44, Cambridge Univ. Press, Cambridge,1993.
[25] H. Weyl, *Ueber die Bestimmung einer geschlossenen konvexen Flche durch ihr Linienelement*. Vierteljahrschrift Naturforsch. Gesell. Zurich , 3 : 2 (1916), 40-72

Department of Mathematics and Statistics, McGill University, Montreal, Canada
E-mail address: `guan@math.mcgill.ca`

Department of Mathematics, Fudan University, Shanghai, China
E-mail address: `zwang@math.mcgill.ca`

Department of Mathematics, Columbia University, New York, US
E-mail address: `xzhang@math.columbia.edu`