System Dynamics and the Lorenz Equations

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ABSTRACT. A chaotic system is characterized as one that exhibits aperiodic long-term behavior, coupled with a sensitive dependence on initial conditions. Our aim is to study the Lorenz equations, a simple example of a chaotic system. To do so, we build the necessary tools, starting from the basics of one-dimensional system dynamics, such as fixed points, bifurcations, etc. We then transition to linear two-dimensional systems, where we'll see classifications of fixed points and of linear systems through their eigensolutions. Next, we move to non-linear systems, limit cycles, and finish our study of two-dimensional systems with bifurcations in two dimensions. Finally, we'll apply these tools to the Lorenz equations.

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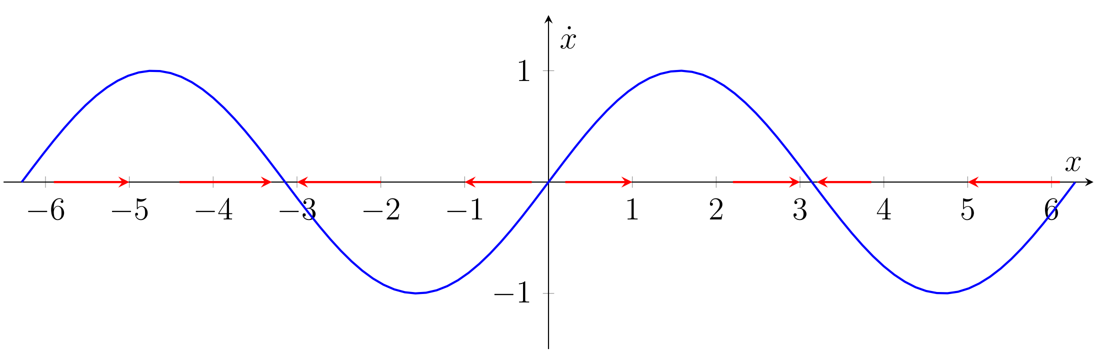
1 Introduction and One-Dimensional systems

In this paper we begin from the basics of system dynamics, then gradually evolve our study from one-dimensional systems to two dimensions, then three dimensions with the Lorenz Equations as our eventual goal. Unless stated otherwise, the content of this paper is based on *Nonlinear Dynamics and Chaos* by Steven H. Strogatz [Str18]. To begin, consider the following.

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, \dots, x_n) \\
\dot{x}_2 &= f_2(x_1, \dots, x_n) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \dots, x_n)
\end{aligned}$$

This is the most general form of a dynamical system in continuous time. Here, each variable, x_i, is really a function of time, t. We use $\dot{x}_i$ as shorthand for $\frac{dx_i}{dt}$. So, the i^{th} equation specifies the velocity in the x_i component. Further, we call the functions $f_1, f_2, ..., f_n$ the **evolution laws** of the system. Note also that throughout this paper, we use bold letters to denote vectors, i.e. for an n-dimensional system, $\mathbf{x} = (x_1, x_2, ..., x_n)$.

1.1 **Phase Portraits and Stability** In many cases, it's simply not possible to solve a system, or even if we could, we would not be able to read off any meaningful qualitative information from the solution. This dilemma motivates the idea of the **phase portrait**: a vector field showing the trajectory of a solution started from any given point. These are used to visualize the "flow", or the dynamics, of the system's solutions. An example in one dimension is given below for the system $\dot{x} = \sin(x)$. We also include the phase line, the graph of $\dot{x}$ versus x, as is typical in one dimension.



The red arrows on the x-axis are the phase portrait; they show the flow of the solution at any given point. We'll commonly refer to this flow as a trajectory of the system. They are dictated by the sign of $\dot{x}$ at that point. That is, for $\dot{x} > 0$, the solution, x, is increasing, so the flow is rightwards. Conversely, $\dot{x} < 0$ corresponds to a leftward flow. Finally, for $\dot{x} = 0$, the solution is not changing, and so it remains constant for all time. Now, consider the following definition.

Definition 1.1. *(fixed points)*
*A **fixed point** for an n-dimensional system, $\dot{x}_i = f_i(x_1, ..., x_n)$, $i \in [n]$, is a point, $\mathbf{x}_0 \in \mathbb{R}^n$, such that $f_i(\mathbf{x}_0) = 0 \; \forall i \in [n]$. That is, if for any $t \geqslant 0$, we have $\mathbf{x}(t) = \mathbf{x}_0$, then $\forall\, t' \geqslant t$, $\mathbf{x}(t') = \mathbf{x}_0$.*

Remark 1.2. In the example above, the fixed points are $x = 0$ and $x = k\pi$ for $k \in \mathbb{Z}$. At $x = 0$ and $x = k\pi$ for even k, the trajectory flows away from the fixed point, while trajectories flow towards $x = k\pi$ for odd k.

Definition 1.3. *(stable and unstable fixed points)*
*For a one-dimensional system, a fixed point is said to be **stable**, otherwise called a sink, if trajectories beginning sufficiently close (that is, with initial conditions sufficiently close) intersect the fixed point as $t \to \infty$. A fixed point is said to be **unstable**, or a source, if trajectories beginning sufficiently close to it grow further as t increases. Fixed points can also be half-stable and half-unstable if trajectories are drawn towards the fixed point from one side, and repelled on the other. We'll see an example of this later on in the section.*

We'll now derive a more concise condition for checking stability/instability of a fixed point.

Proposition 1.4. *(Linear Stability Analysis)*
Consider the one-dimensional system $\dot{x} = f(x)$, where $x = x(t)$, $t \geqslant 0, x \in \mathbb{R}$ as always. Suppose the system has a fixed point at $x^\star \in \mathbb{R}$. Then,

$f'(x^\star) > 0 \implies x^\star$ is an unstable fixed point
$f'(x^\star) < 0 \implies x^\star$ is a stable fixed point
If $f'(x^\star) = 0$, then our test is inconclusive.

Proof. Consider $\eta(t) = x(t) - x^\star$. From here on, assume x is near $x^\star$, so that η is a small perturbation away from $x^\star$. That is, it represents the distance between $x^\star$ and a trajectory started close to $x^\star$. We want to check whether the perturbation grows or decays over time. Now, note that

$$\dot{\eta} = \frac{d}{dt}(x - x^\star) = \dot{x} = f(x) = f(\eta + x^\star)$$

We can use Taylor's expansion on f to simplify:

$$f(x^\star + \eta) = f(x^\star) + \eta f'(x^\star) + \mathcal{O}(\eta^2)$$

Note that $f(x^\star) = 0$, so we have $\dot{\eta} = \eta f'(x^\star) + \mathcal{O}(\eta^2)$.
Again, since x is near $x^\star, \eta$ is very small. We can thus neglect the error term, so long as $f'(x^\star) \neq 0$. So,

$$\dot{\eta} \approx \eta f'(x^\star)$$

Solving this differential equation, we find $\eta(t) = \eta(0)e^{f'(x^\star)t}$. If $f'(x^\star) > 0$, the perturbation grows exponentially, meaning the trajectory grows further from the fixed point over time. This was precisely our definition for $x^\star$ being unstable. Conversely, if $f'(x^\star) < 0$, then our perturbation is decaying, meaning the distance from our trajectory to the fixed point is shrinking, and thus our fixed point is stable. $\square$

As one might expect, the possible behavior of a system in one dimension is very limited when compared to those in higher dimensions. One interesting result that highlights this fact is given below.

Theorem 1.5. *(Impossibility of Oscillations)*
Let $\dot{x} = f(x)$ be a one dimensional system where f is continuous. Then, trajectories can either diverge to infinity or converge to a fixed point.

Remark 1.6. This theorem means that solutions can never ”turn around”. Suppose a trajectory starts moving to the right, then it can only continue moving to the right (if no fixed point is on its path) or it can stop at a fixed point. For simplicity, we'll assume that all fixed points are either stable or unstable in the theorem below. However, the result still holds in the absence of this assumption.

Proof. .
Case 1: f has no roots.
Since f is continuous, it follows directly from intermediate value theorem that either $f(x) > 0$ or $f(x) < 0$ $\forall x \in \mathbb{R}$. So, $\dot{x}$ is positive or negative on $\mathbb{R}$, hence our solution is monotonically increasing or decreasing. We must have that every solution diverges to infinity.

Case 2: $f \equiv 0$.
Trivially, our solution is constant, so every point is a fixed point. Thus each trajectory converges to a fixed point.

Case 3: f has a countably infinite number of roots, say $x_1, x_2, x_3, ...$ where $x_k < x_{k+1}$ for all $k \in \mathbb{N}$.
Suppose we have an initial condition $x(t_0) = x_0$ such that $x_k < x_0 < x_{k+1}$ for some $k \in \mathbb{N}$. We have one of the following three scenarios:
- x_k, x_{k+1} are both stable
- x_k, x_{k+1} are both unstable
- One is stable, the other is unstable

Suppose both fixed points are stable. We know $f(x_k) = f(x_{k+1}) = 0$. Further, for x slightly greater than x_k, $f(x) < 0$, and for x slightly smaller than x_{k+1}, $f(x) > 0$. Again, since f is continuous, intermediate value theorem tells us that there must be a point, $x^\star$ such that $x_k < x^\star < x_{k+1}$ and $f(x^\star) = 0$. However, this is not

possible due to our assumption of countably many fixed points, all of which were listed above.
A similar argument holds for the second scenario. So, we cannot have two subsequent fixed points with the same stability.

In the third scenario, we simply have that trajectories move away from the unstable fixed point, and towards the stable fixed point.
So, for case 3, solutions always converge to a fixed point.

Case 4: f has finitely many roots, say $x_1, ..., x_n$, such that $x_k < x_{k+1}$ for all $k \in [n]$
The proof for this case is almost exactly the same as case 3, only with the caveat that stability of x_1 and x_n determine whether trajectories converge or diverge for trajectories beginning before x_1 and after x_n respectively. □

1.2 **Bifurcations** We'll now finish the section on one-dimensional systems with a brief introduction to bifurcations. This topic is the main reason why one-dimensional systems are studied, as they serve as a great model for higher dimensional bifurcations.

Definition 1.7. *(bifurcations in one dimension)*
A ***bifurcation*** *for a dynamical system is any process involving the variation of a parameter which results in a qualitative change in the phase portrait of the system. The* ***bifurcation point*** *is the specific value of the parameter at which the qualitative change occurs. For a one dimensional system on the real line, a bifurcation can only take the form of creation or destruction of fixed points, or the change of stability of a fixed point, as parameters of the system are varied.*

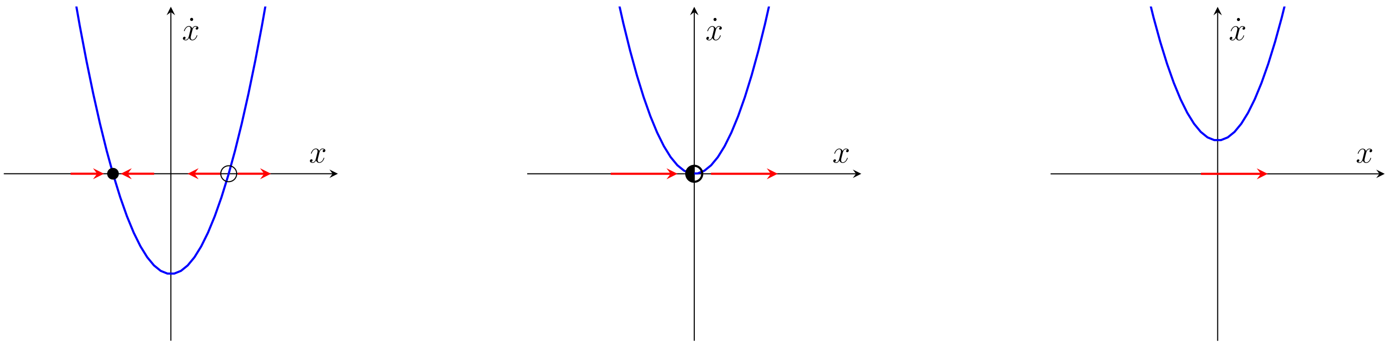
For instance, take

$$\dot{x} = A \ln x + B, \text{where } A, B \in \mathbb{R}$$

This is a system with parameters A and B. Changing their values can create and destroy fixed points, or change their stability. These are called bifurcations.

Bifurcations come in several different types. We'll now see the general form of the saddle-node bifurcation and the transcritical bifurcations, and some basic examples. By no means are these the only types of bifurcations that can occur in one dimension.

First, we'll study the **Saddle-Node Bifurcation**: the simplest type. This process entails two fixed points colliding and being destroyed. For example:
$\dot{x} = x^2 + r, r \in \mathbb{R}$.
The phase portraits and phase lines for $r < 0, r = 0, r > 0$ follow.



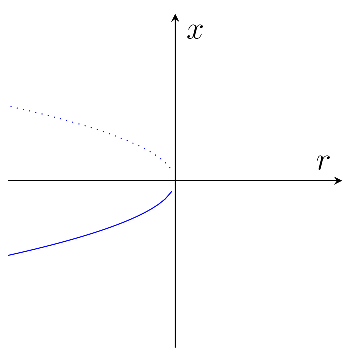
Note that in the diagrams above, and in general, a solid dot denotes a stable fixed point, while a hollowed dot is an unstable fixed point. It follows that a half-hollowed out, half-solid dot is a fixed point that is stable on the solid side (attracts trajectories), and unstable on the hollow side (repels trajectories).

We can clearly see from these graphs that the bifurcation point is in fact $r = 0$, since it marks a qualitative change in the phase diagram: for all smaller values of r, there are two fixed points, while for all larger values of r, there are no fixed points.

We'll now see a more compact way to represent bifurcations on a graph.

Definition 1.8. *(bifurcation diagrams) For a one-dimensional system $\dot{x} = f(x)$ with parameter r, a **bifurcation diagram diagram** for r is a graph with r on the horizontal axis, and x on the vertical axis which shows the value of the fixed points of the system for each value of r, along with their stability.*

Below is a bifurcation diagram for $\dot{x} = x^2 + r$, i.e. for the standard saddle-node bifurcation.



Interpretation: The dotted curve shows the unstable fixed points, while the solid curve shows the stable fixed points. This lines up perfectly with our phase diagrams. For $r < 0$, there's one stable and one unstable fixed point. For $r = 0$, there's one fixed point that is half-stable and half-unstable (hence the intersection of the dotted and solid curves), and for $r > 0$, there are no fixed points.

Now for **Transcritical Bifurcations**. This process involves two fixed points with different stability (one is stable, one is unstable), which are present for all values of the parameter. As the parameter is varied, the fixed points eventually collide and their stability types are swapped. The general form for such a bifurcation is

$$\dot{x} = rx - x^2$$

We clearly see that $x = 0$ and $x = r$ are fixed points for all $r \in \mathbb{R}$, and they collide at $r = 0$. To produce the bifurcation diagram, we'll need to check the stability of the fixed points for each value of the parameter. To do so, let $f(x) = rx - x^2$, then $f'(x) = r - 2x$.

For $r < 0$:

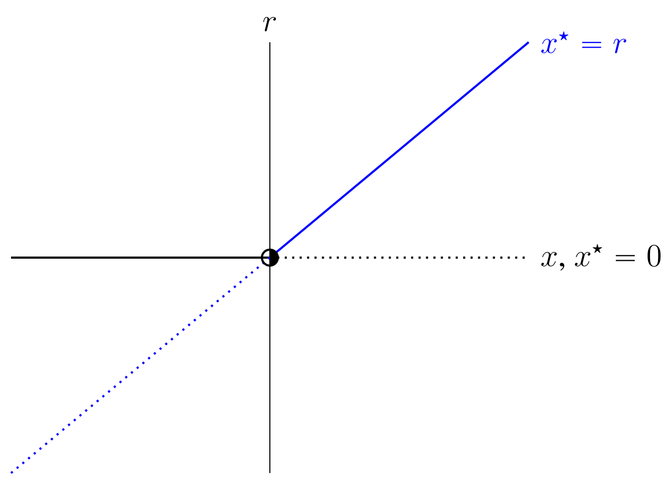
$$f'(0) = r < 0,\ f'(r) = -r > 0 \implies x = 0 \text{ is stable, and } x = r \text{ is unstable}$$

For $r > 0$:

$$f'(0) = r > 0,\ f'(r) = -r < 0 \implies x = 0 \text{ is unstable, and } x = r \text{ is stable}$$

At $r = 0$, we have $f(x) = -x^2$. Linear stability analysis is inconclusive. However, we observe that $\dot{x}$ is always negative. We thus have that trajectories with initial conditions left of the y-axis are monotonically decreasing, while those beginning on the right of the y-axis decrease to 0 and remain there for all future time, and so $x = 0$ is a half-stable, half-unstable fixed point.

Using this information, we can produce the bifurcation diagram below.



To finish the section, we'll do an example to prove the existence of such a bifurcation for a typical system.

Example : Show that $\dot{x} = x(1 - x^2) - a(1 - e^{-bx})$ undergoes transcritical bifurcation at $x = 0$, and find the bifurcation point.

Solution To prove a transcritical bifurcation, we must show that the system has two fixed points, one which persists for all parameter values, and another which "passes through" the first as the parameters evolve. Finally, the two must interchange their stability types after the bifurcation point.

We first note that $x^\star = 0$ is a fixed point for all $a, b \in \mathbb{R}$, so a transcritical bifurcation of this fixed point is plausible. Since bifurcations occur locally, we consider x to be near $x^\star = 0$ from here on. Applying the Taylor expansion to e^{-bx}, we get:

$1 - e^{-bx} = 1 - (1 - bx + \frac{1}{2}b^2x^2 + O(x^3)) = bx - \frac{1}{2}(bx)^2 + O(x^3)$

$\implies \dot{x} = (1 - ab)x + \frac{1}{2}(ab^2)x^2 + O(x^3)$

Using this form, we can find the second fixed point and deduce the bifurcation point/curve. Ignoring the error term of order x^3, we find

$\dot{x} = 0 \iff x = \frac{2(ab-1)}{ab^2}$ or $x = 0$

We'll set $\mu = \frac{2(ab-1)}{ab^2}$, so that our second fixed point is approximately $x = \mu$. This fixed point will pass through $x^\star = 0$ at the point $ab = 1$. It then stands to reason that $ab = 1$ would be our bifurcation point. To verify this, we'll analyze the stability of our fixed points for $ab < 1$ and $ab > 1$.

Set $f(x) = x(1 - x^2) - a(1 - e^{-bx})$ for our evolution law, then

$f'(0) = 1 - ab \implies f'(0) > 0$ for $ab < 1$ and $f'(0) < 0$ for $ab > 1$.

So, $x^\star = 0$ is unstable for $ab < 1$ and stable for $ab > 1$. To evaluate the stability of the second fixed point, we use the Taylor approximation $f(x) \approx (1 - ab)x + \frac{1}{2}(ab^2)x^2$.

By repeating the same process, one finds that $x = \mu$ is stable for $ab < 1$ and unstable for $ab > 1$ as desired.

The sufficient conditions are met, so we have a transcritical bifurcation between the two fixed points, $x^\star = 0$ and $x^\star \approx \mu$ at the bifurcation point $ab = 1$.

2 Two-Dimensional Systems

We now proceed to two-dimensional systems, using our study of one-dimensional systems as the general outline upon which we build our theory. However, as we'll see going forward, trajectories are harder to predict and classify in two-dimensions. We first analyze simple systems, in which the evolution laws are linear. Then, most of the developed theory can be transferred to the nonlinear case through the process of linearization.

2.1 **Linear Systems and Classification of Fixed Points** .

The most general form of a two-dimensional, linear system is

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

Taking $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, we can also represent the system as

$$\dot{\mathbf{x}} = A\mathbf{x}$$

A system of this form is called linear, not only because both of the evolution laws are linear, but also because if $\mathbf{x_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{x_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ solve the system, then any linear combination of the two vectors will also solve the system.

Definition 2.1. *(Fixed Point classes)*
Let $\mathbf{x}^\star \in \mathbb{R}^2$ be a fixed point to a linear, two-dimensional system, as given above. Below are the most important fixed point classes that we'll reference for the remainder of the paper.

*- We say $\mathbf{x}^\star$ is **attracting** if, for all initial conditions sufficiently close to $\mathbf{x}^\star$, we have $\mathbf{x} \to \mathbf{x}^\star$ as $t \to \infty$.*

*- We say $\mathbf{x}^\star$ is **Liapunov stable** if trajectories beginning sufficiently close to $\mathbf{x}^\star$ remain close to $\mathbf{x}^\star$. More precisely, if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|\mathbf{x}(0) - \mathbf{x}^\star| < \delta \implies |\mathbf{x}(t) - \mathbf{x}^\star| < \epsilon \ \ \forall t \in \mathbb{R}$$

then we say the fixed point is Liapunov stable.

- If, for any initial condition, $\mathbf{x}_0 \in \mathbb{R}^2$,

$$\mathbf{x}(0) = \mathbf{x}_0 \implies \mathbf{x}(t) \to \mathbf{x}^\star \textit{ as } t \to \infty,$$

*and $\mathbf{x}^\star$ is also Liapunov stable, then the fixed point is **globally asymptotically stable**.*

*- Finally, a fixed point is **unstable** if it is neither attracting, nor Liapunov stable.*

Now, we aim to classify systems as a whole, and attempt to plot the phase portraits of a solution. However, in the two-dimensional, linear case, we can actually solve any given system with relative ease. The method of eigensolutions is the most common way to solve such a system. A general outline on applying the method is given below, without proof.

Proposition 2.2. *(Method of Eigensolutions)*

If $\dot{\mathbf{x}} = A\mathbf{x}$, and A has two distinct eigenvalues, $\lambda_1, \lambda_2 \in \mathbb{R}$, with corresponding eigenvectors $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^2$, then the general solution is

$\mathbf{x} = c_1\mathbf{v_1}e^{\lambda_1 t} + c_2\mathbf{v_2}e^{\lambda_2 t}$ for constants c_1, c_2.

If A has only one eigenvalue and independent eigenvector, the general solution becomes

$\mathbf{x} = c_1 t \mathbf{v_1} e^{\lambda_1 t} + c_2 \mathbf{v_2} e^{\lambda_1 t}$, *where* $\mathbf{v_2}$ *is a generalized eigenvector for* λ_1.

To develop the rest of the theory regarding fixed points, we'll work through a series of examples.

Example

Consider the system $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x}$

The only fixed point of the system is the origin, and the eigenvectors of the matrix, A, are $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v_2} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -3$ respectively. Our general solution is

$\mathbf{x} = c_1 \mathbf{v_1} e^{2t} + c_2 \mathbf{v_2} e^{-3t}$

One prominent feature of this equation is that one eigenvalue is positive, and the other is negative. If we take an initial condition on the line spanned by $\mathbf{v_1}$, the second term in the solution vanishes, and we're left with

$\mathbf{x} = c_1 \mathbf{v_1} e^{2t}$

This means that solutions beginning on this line will grow exponentially, and diverge to infinity. For that reason, we call the line spanned by $\mathbf{v_1}$ the **unstable manifold**. Conversely, solutions beginning on the line spanned by $\mathbf{v_2}$ experience exponential decay, and are pulled into the origin. So, the line spanned by $\mathbf{v_2}$ is called the **stable manifold**. The origin is thus an unstable fixed point, but more specifically, it is called a **saddle point**, because it has one stable and one unstable manifold. The phase portrait of the system can then be deduced from plotting the stable and unstable manifolds.

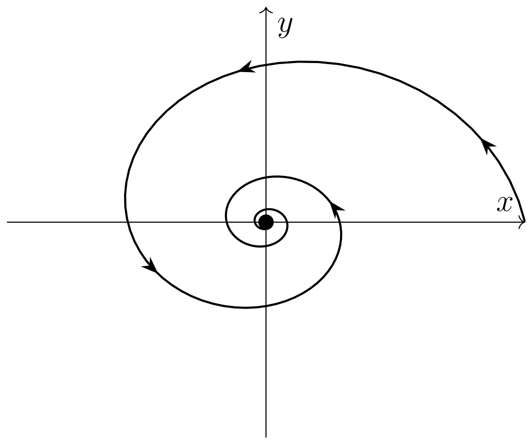
Example Take the system $\dot{\mathbf{x}} = A\mathbf{x}$. What if A has complex eigenvalues?

Recall that the eigenvalues of A are given by $\lambda_1 = \alpha + i\omega$, $\lambda_2 = \alpha - i\omega$, where $\alpha = \frac{1}{2}\text{trace}(A) \in \mathbb{R}$, and $\omega = \frac{1}{2}\sqrt{4 \cdot \det(A) - \text{trace}(A)^2}$. Since the eigenvalues are complex, we have $\omega \in \mathbb{R}\backslash\{0\}$.

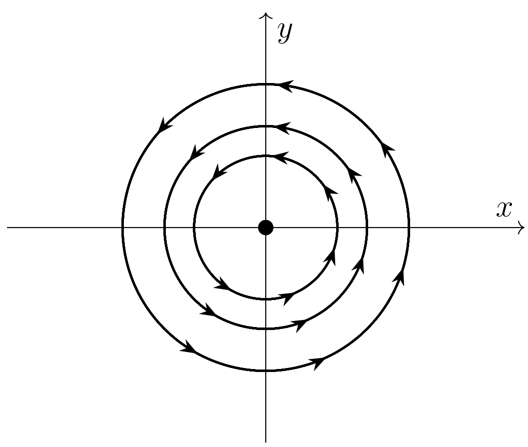
Using Euler's formula, we can write our solution as

$\mathbf{x} = e^{\alpha t}[c_1 \mathbf{v_1} \cos \omega t + c_2 \mathbf{v_2} \sin \omega t]$

Immediately, we can see that there will be exponential growth or decay, determined by the sign of α. There's clearly some oscillatory behavior, caused by the periodic sine and cosine terms. For $\alpha \neq 0$, this system's fixed points are called **spirals**. If $\alpha > 0$, the spiral is unstable, since trajectories are being pushed away from the fixed point at its center. Conversely, $\alpha < 0$ yields stable spirals. A phase portrait of a linear system with a stable spiral at the origin is given below.



Finally, if $\alpha = 0$, there ceases to be any exponential growth or decay. Solutions are periodic, so trajectories remain at the same distance from the origin as they began. In this case, the fixed points of the system are called **centers**, and the trajectories surrounding them are circular. A typical phase portrait for a system with a center is below.



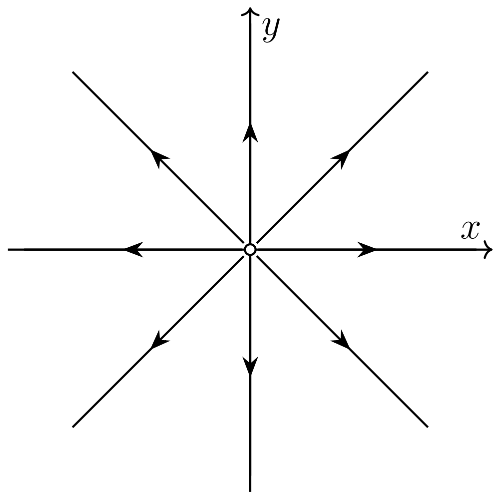
Example The final case we'll cover explicitly is that where the matrix, A, has only one eigenvalue, say λ.

Firstly, if λ has two independent eigenvectors, $\mathbf{v_1}$ and $\mathbf{v_2}$, the solution is

$\mathbf{x} = c_1\mathbf{v_1}e^{\lambda t} + c_2\mathbf{v_2}e^{\lambda t}$

Trivially, if $\lambda = 0$, then $\mathbf{x}$ is a constant, and the system becomes $\dot{\mathbf{x}} = 0$. So, every $x_0 \in \mathbb{R}^2$ is fixed.

The more eventful case is $\lambda \neq 0$. Note that since $\mathbf{v_1}, \mathbf{v_2}$ are independent, they form a basis for $\mathbb{R}^2$, and thus span the entire space. Consequently, every initial condition lies directly on a stable or unstable manifold, and so trajectories are straight lines that are either repelled from the origin or attracted directly to it. Here, the origin is called a **star**. An example of an unstable star, i.e. a case when $\lambda > 0$, is given below.



That is not all for the case of one eigenvalue. It's also possible that λ has only one independent eigenvector. In this case, any chosen initial condition will approach the fixed point parallel to said eigenvector. Fixed points are then called **degenerate nodes**.

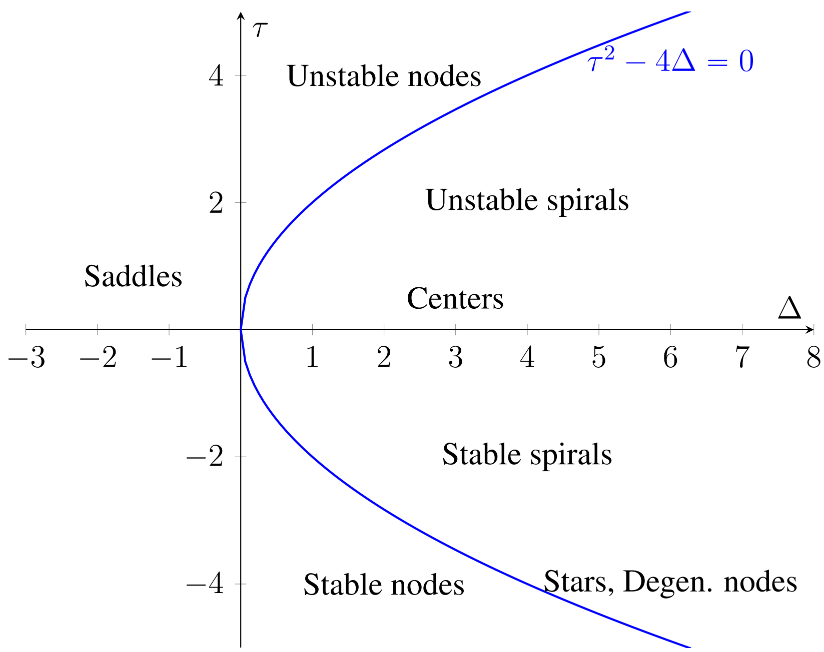
The only remaining types of fixed points are nodes, and we'll now provide a brief definition for each. Then, we move to a theorem which allows for faster classification of fixed points using all of our previous work.

Definition 2.3. *(Stable and Unstable Nodes)*

*Let $\dot{\mathbf{x}} = A\mathbf{x}$ be a two-dimensional system. If A has two distinct, real eigenvalues, $\lambda_1, \lambda_2 \in \mathbb{R}$, with the same sign, then its fixed points are either **stable nodes** or **unstable nodes**. A stable node occurs when both eigenvalues are negative. If $\lambda_1 < \lambda_2 < 0$, then the manifold of the eigenvector for λ_1 attracts solutions at a greater rate. The eigenvector for λ_1 is called the **fast eigendirection**, and that of λ_2 is the **slow eigendirection**.*

Conversely, if $\lambda_1 > \lambda_2 > 0$, then the nodes are unstable, and the manifold of the eigenvector for λ_1 repels trajectories at a greater rate. The same terminology of eigendirections applies in this case.

Theorem 2.4. *(Classification of Fixed Points)*
Let $\dot{\mathbf{x}} = A\mathbf{x}$, $\tau = trace(A)$, $\Delta = \det(A)$ Then the classification of the system's fixed points is determined fully by the expression ($\tau^2 - 4\Delta$). The following graph gives the detailed criterion for classification.



Note also that the case $\Delta = 0$ results in non-isolated fixed points, which are those which lie in a dense subset of fixed points.

Proof. The proof follows directly from standard facts about linear algebra, which tell us that if λ_1, λ_2 are the eigenvalues of the system's matrix, then

(1) $\lambda_{1,2} = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta})$

(2) $\Delta = \lambda_1\lambda_2$

(3) $\tau = \lambda_1 + \lambda_2$

Pairing this information with the examples produced in this section yields the desired results. Below are some proofs for a couple of the cases.

If $\Delta < 0$, then from (2) we must have $\lambda_1 > 0$, $\lambda_2 < 0$ (without loss of generality). As we've seen, the mismatch of signs of the two eigenvalues yields one stable manifold and one unstable manifold, so a saddle point.

If $\tau = 0$, $\Delta > 0$, then the eigenvalues are given by $\lambda_{1,2} = \pm i\sqrt{\Delta}$. As seen in example 2.4, when eigenvalues are purely complex, fixed points are centers.

The remaining cases are handled similarly. $\square$

2.2 **Non-linear systems and Linearization** .

The most general form of a dynamical system in two dimensions is

$\dot{x}_1 = f_1(x_1, x_2)$
$\dot{x}_2 = f_2(x_1, x_2)$

Or, as a vector field, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

In contrast to the linear case, finding exact solutions to non-linear systems can be near impossible. Instead, we deduce the system's trajectories by observing its fixed points and their stability, alongside what are called **closed orbits**, which are periodic solutions.

Theorem 2.5. *(Existence and Uniqueness theorem)*

Consider the n-dimensional initial value problem given by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$. Suppose now that $\mathbf{f}$ and all of its first order partial derivatives are continuous on some $D \subseteq \mathbb{R}^n$.

Then, if $\mathbf{x}_0 \in D$, the problem has a unique solution on some time interval of the form $]-\tau, \tau[$.

The proof of this theorem will not be covered, but it is generally seen in any first course in ODEs.

Corollary 2.6. *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be an n-dimensional system where $\mathbf{f}$ and all of its first order partial derivatives are continuous on some $D \subseteq \mathbb{R}^n$. Consider two initial conditions $\mathbf{x}_0, \mathbf{y}_0 \in D$, $\mathbf{x}_0 \neq \mathbf{y}_0$, and the two corresponding solutions, $\mathbf{x}(t)$ and $\mathbf{y}(t)$. These two solutions cannot intersect in D.*

Proof. .

Consider the system started from two distinct initial conditions

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x_0}$, and

$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y_0}$, where $\mathbf{x_0}, \mathbf{y_0} \in D$.

Suppose, for contradiction, that the two corresponding solutions intersect. That is, $\mathbf{x}(t_0) = \mathbf{y}(t_0) = \mathbf{z}_0$, for some $\mathbf{z}_0 \in D$.

Now, consider a third system $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$, $\mathbf{z}(0) = \mathbf{z_0}$. We then have two distinct solutions: $\mathbf{u}(t) = \mathbf{x}(t + t_0)$ and $\mathbf{v}(t) = \mathbf{y}(t + t_0)$. Both solve the appropriate differential equation and satisfy the same initial value condition. We have a contradiction: existence and uniqueness theorem guarantees exactly one solution to the problem. So, the solutions cannot intersect. $\square$

Corollary 2.7. *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be an n-dimensional system satisfying the usual differentiability and continuity conditions, as seen above. Now, suppose that $\mathbf{f}$ has a periodic solution. We then know that its corresponding trajectory is a closed orbit. Let C denote the closed curve traced out by this trajectory. Let $\mathbf{x}(0) = \mathbf{x_0}$, where $\mathbf{x_0}$ is interior to C, then $\mathbf{x}(t)$ is interior to C for all $t > 0$.*

Proof. Suppose for contradiction that the trajectory does escape the closed orbit. Then, since $\mathbf{f}$ is a continuous vector field, the trajectory must intersect the closed orbit, C. However, C is itself a trajectory. By the previous corollary, the two cannot intersect. Contradiction reached. □

We'll now cover the process of **linearization**, which allows us to classify fixed points using the same methods derived for linear systems.

To begin, take the two-dimensional system

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and suppose that $\mathbf{x}^\star = (x^\star, y^\star)$ is a fixed point of the system. That is, $f(\mathbf{x}^\star) = g(\mathbf{x}^\star) = 0$. Define $u = x - x^\star$, $v = y - y^\star$ so that, for (x, y) near $\mathbf{x}^\star$, (u, v) is a small perturbation away from the fixed point.

We then have $\dot{u} = \dot{x} = f(x, y) = f(x^\star + u, y^\star + v)$. Using a Taylor expansion about $\mathbf{x}^\star$, we get

$$\dot{u} = uf_x(\mathbf{x}^\star) + vf_y(\mathbf{x}^\star) + \mathcal{O}(u^2, v^2, uv)$$

And similarly, $\dot{v} = ug_x(\mathbf{x}^\star) + vg_y(\mathbf{x}^\star) + \mathcal{O}(u^2, v^2, uv)$, where subscripts are used to denote the appropriate partial derivatives. Again, since u and v are extremely small when (x, y) is near $(x^\star, y^\star)$, it can be appropriate to omit the error term, which leaves us with the **linearized system**:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

We call $A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ the **Jacobian matrix**.

Using the linearized system, we can classify almost all of a system's fixed points by evaluating the Jacobian at the appropriate fixed point, then applying the same methods as in the linear case. The caveat is that the prediction of a fixed point's class from the linearized system is only accurate in what are called **robust cases**, which are those where $\Re(\lambda) \neq 0$ for all eigenvalues of the Jacobian evaluated at the fixed point in question. Non-robust fixed points are called **marginal cases**.

So, this method can only correctly predict saddle points, nodes, or spirals, and not centers or non-isolated fixed points. For stars and degenerate nodes, the linearization can predict their stability, but may fail to predict their exact local geometry. In the interest of remaining on topic, we'll omit the proof for these facts.

To end the section, we'll give an example of a case where linearization fails, in order to show the type of abstract thinking required to classify a marginal fixed point.

Example

Let $\dot{x} = -y + ax(x^2 + y^2)$
$\dot{y} = x + ay(x^2 + y^2), \ a \in \mathbb{R}$

Note that the origin is a fixed point of the system. Show that, for $a > 0$, the linearization about the origin incorrectly predicts a center, when the origin is actually an unstable spiral.

Solution:

To begin, we'll linearize the system about the origin and obtain the Jacobian, given below.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So, $\tau = 0$, $\Delta = 1$, meaning the linearization predicts that the origin is a center.

Motivated by the $(x^2 + y^2)$ term in both of the evolution laws, we'll rewrite the system in polar coordinates.

$x^2 + y^2 = r^2 \implies 2x\dot{x} + 2y\dot{y} = 2r\dot{r} \implies \dot{r} = x(y + ax(x^2 + y^2)) + y(x + ay(x^2 + y^2)) = ar^3$

So, $\dot{r} = ar^3$. Using a similar proof, stemming from the fact that $\theta = \arctan(\frac{y}{x})$, we get $\dot{\theta} = 1$. This means that trajectories revolve around the origin at constant, unit angular velocity.

For $a > 0$, we have that whenever $r > 0$, trajectories grow further and further from the origin, while rotating around the origin with unit velocity. Thus, the origin is, in fact, an unstable spiral. The linearization thus incorrectly predicts the fixed point's class.

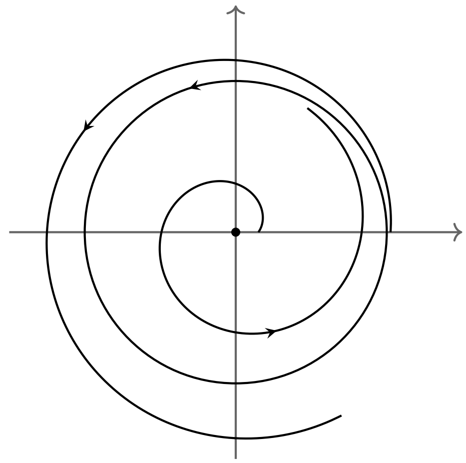
Further, we can also see that for $a = 0$, we have $\dot{r} = 0$, so the initial distance from the origin is maintained for all time, and we still have $\dot{\theta} = 1$. So, the linearization would correctly predict the origin as a center in this case. For $a < 0$, r is decreasing, so trajectories spiral inwards, meaning the origin is a stable spiral.

2.3 **Limit Cycles** Recall that a **closed orbit** is a periodic solution to a dynamical system. They appear in the phase portrait as closed curves.

Definition 2.8. *(Limit Cycle) A **limit cycle** is an isolated closed orbit. That is, its surrounding trajectories are not closed orbits; they either move towards or away from the limit cycle. Limit cycles can be stable if surrounding trajectories move towards it, unstable if surrounding trajectories travel away from it, or even half-stable.*

Remark 2.9. Limit cycles can only exist in non-linear systems, because if a linear system, $\dot{\mathbf{x}} = A\mathbf{x}$ has a solution, $\mathbf{y}(t)$, that is a closed orbit, then $c\mathbf{y}(t)$ is also a solution for all $c \in \mathbb{R}$. Trivially, integer multiples of a periodic function are also periodic, and so $c\mathbf{y}(t)$ is also a closed orbit $\forall c \in \mathbb{R}$, meaning $\mathbf{y}(t)$ is not isolated, and thus not a limit cycle.

To help visualize limit cycles, the phase portrait of a system with a half-stable limit cycle is included below. Trajectories beginning outside of the limit cycle are repelled, while those starting inside are drawn toward it.



This section will cover two powerful tools relating to limit cycles: the Poincaré-Bendixson theorem, which establishes the existence of a limit cycle, and Liapunov functions, used to rule out the existence of a closed orbit.

2.3.1 *Poincaré-Bendixson Theorem* The Poincaré-Bendixson theorem is a fundamental result in two-dimensional system dynamics. It's one of the most popular methods of establishing the existence of a closed orbit. In this section, we'll see the statement of the theorem and an example of how to apply it to a system.

Theorem 2.10. *(Poincaré-Bendixson)*
Consider the two-dimensional system given by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Suppose further that ...

(1) *R is a closed, bounded subset of the plane*

(2) *$\mathbf{f}$ is continuously differentiable on an open set containing R.*

(3) *R contains no fixed points*

(4) *There exists a trajectory, C, that is "confined" to R. That is, it starts in R and remains there for all time.*

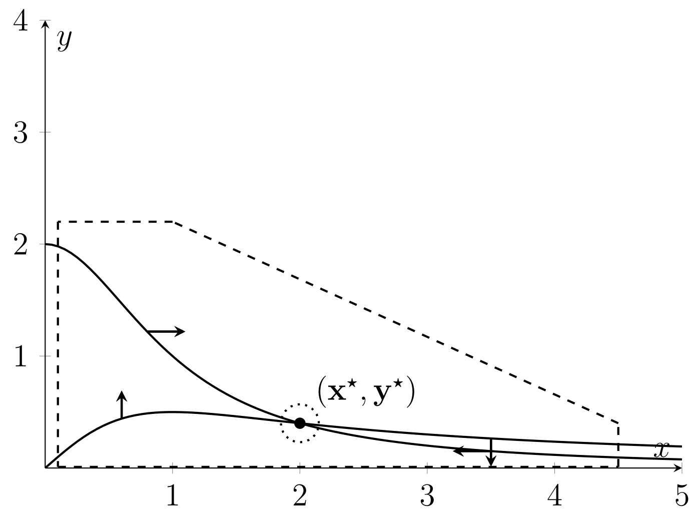
We then have that C is either a closed orbit, or moves toward a closed orbit as $t \to \infty$. In either case, R contains a closed orbit.

Remark 2.11. We call R a trapping region. It's a closed, connected set on which the vector field $\mathbf{f}(\mathbf{x})$ points inwards for all $\mathbf{x} \in R$.

Example 2.12. Consider the system below.

$\dot{x} = -x + ay + x^2y$
$\dot{y} = b - ay - x^2y$, $a, b \in \mathbb{R}$

We'll construct a trapping region for this system. First, consider the **nullclines** of the system. That is, the curves $\dot{x} = 0$ and $\dot{y} = 0$. These curves are $y = \frac{x}{x^2+a}$ and $y = \frac{b}{x^2+a}$ respectively. A sketch of these nullclines alongside the direction taken by the system on these curves is included below for $a, b > 0$.



The dotted area is our trapping region, and the dotted circle at the fixed point is an infinitesimal puncture. It can be shown that the fixed point is unstable, and so we have our trapping region. Further, our evolution law $\mathbf{f}(\mathbf{x})$ is continuously differentiable on $\mathbb{R}^2$, the region contains no fixed points. Conditions (1)-(4) of the Poincaré-Bendixson theorem are satisfied, thus there is a closed orbit in the dotted region.

2.3.2 *Liapunov Functions* The content of this section is based on the textbook *Nonlinear Ordinary Differential Equations : An Introduction for Scientists and Engineers* by D. W. Jordan and P. Smith (section 10) [JS07]

Definition 2.13. *Let* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *be a two-dimensional system with fixed point* $\mathbf{x}^\star \in \mathbb{R}^2$. *A* ***Liapunov function*** *for the fixed point is a continuously differentiable function,* $V : \mathbb{R}^2 \to \mathbb{R}$, *such that*

(1) $V(\mathbf{x}) > 0$ *for all* $\mathbf{x} \neq \mathbf{x}^\star$, *and* $V(\mathbf{x}^\star) = 0$.

(2) $\frac{d}{dt}V(\mathbf{x}) < 0$ *for all* $\mathbf{x} \neq \mathbf{x}^\star$.

Remark 2.14. The main claim of this section is that the existence of a Liapunov function for $\mathbf{x}^\star$ guarantees that it is globally asymptotically stable. In particular, the system can have no closed orbits, since if it did, then trajectories beginning outside the closed orbit would be unable to cross into the orbit and reach $\mathbf{x}^{\star}$ (Corollary 2.6), contradicting global asymptotic stability.

Seeing as how we'll use Liapunov functions in section 3, the proof of the above claim is covered here. We'll need to introduce some tools from Topology for the proof.

Definition 2.15. *(Topographic systems)*

Let $N \subset \mathbb{R}^2$ be a connected neighborhood of the origin, and let $V : \mathbb{R}^2 \to \mathbb{R}$ satisfy the following in N.

(1) *V(x,y), $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ are continuous in $N \backslash \{(0,0)\}$*

(2) *$V(0,0) = 0$, and $V(x,y) > 0$ elsewhere in N*

(3) *A value, $\mu > 0$, exists such that for every $0 < \alpha < \mu$, the equation $V(x,y) = \alpha$, $x, y \in N$ defines a simply connected domain, say T_α, contained in N, which surrounds the origin.*

*The family of curves, T_α, $0 < \alpha < \mu$, is called a **Topographic system**, denoted N_μ.*

Proposition 2.16. *A topographic system, as defined above, always satisfies the following:*

(1) *$V(x,y) < \alpha$ for (x,y) in the interior of T_α (the region defined by $V(x,y) = \alpha$).*

(2) *For every $0 < \alpha < \mu$, and each point, (x_0, y_0) interior to T_α, there exists $\beta < \alpha$ such that T_β passes through (x_0, y_0).*

(3) *$0 < \alpha_1 < \alpha_2 < \mu \implies T_{\alpha_1}$ is interior to T_{α_2}*

(4) *As α decreases to 0, T_α collapses into the origin.*

Definition 2.17. *A **positive half-path**, H, is a trajectory of a two-dimensional dynamical system which is only viewed forwards in time.*

Theorem 2.18. *Let N_μ be a topographic system in some connected neighborhood, N, of the origin, with corresponding topographic curves, T_α, given by $V(x,y) = \alpha$, $0 < \alpha < \mu$. Now consider the system defined for $(x,y) \in N$:*

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

Suppose that $\dot{V}(x,y) \leqslant 0$ for $(x,y) \in N_\mu$. Take any positive half-path, H, starting from a point, P, interior to T_α, then H can never cross the boundary of T_α.

Proof. Take some T_α, and a point, P, in the interior of T_α. Take a second point, B, exterior to T_α. By statement (2) of proposition 2.16, there exists $\alpha < \alpha' < \mu$ such that the curve $V(x, y) = \alpha'$, $(x, y) \in N$, passes through the point B.

Now suppose for contradiction that a trajectory of the system can take some path from P to B, then since $\alpha' > \alpha$, and V is continuous in N, there must be some part of the path on which the function is increasing. So, $\dot{V} > 0$ at some point on this trajectory. Contradiction: $\dot{V} \leqslant 0$ in N_μ. □

Theorem 2.19. *(Liapunov Stability at the origin)*

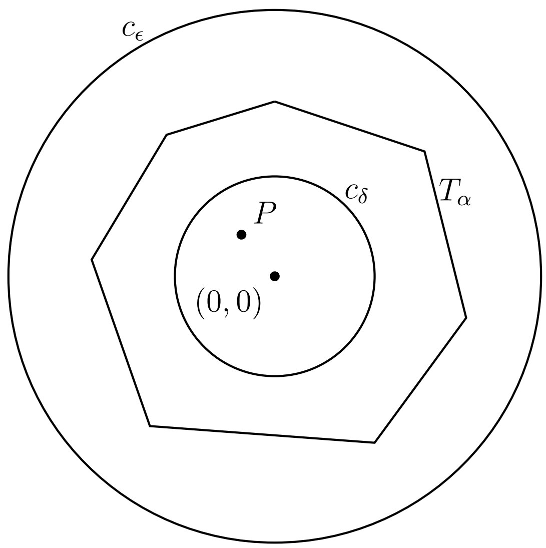
Let N be a connected neighbourhood of the origin, and $N_\mu \subset N$ a topographic system with topographic curves given by the function V as defined previously.

Suppose the two-dimensional system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a fixed point at the origin. Suppose further that V is also a Liapunov function for the origin, then the origin is Liapunov stable.

Proof. Consider the given figure, contained within N_μ. We have c_ϵ, a circle of radius $\epsilon > 0$ centered at the origin, where ϵ is sufficiently small so that the circle encloses no fixed points other than the origin. T_α is given by $V(x, y) = \alpha$, $(x, y) \in N$. Finally, c_δ is a circle of radius δ, where $0 < \delta < \epsilon$, that is centered at the origin and interior to T_α. Take any positive half-path of the system, H, starting from P at time $t = t_0$.

Since V is a Liapunov function, $\dot{V}(x, y) \leqslant 0$ in N_μ. So, by the previous theorem, H cannot cross T_α, and thus remains at a distance less than ϵ from the origin. We have the following:

Given $\epsilon > 0$, there exists $\delta > 0$ such that if $|\mathbf{x}(t_0) - (0, 0)| < \delta$, then $|\mathbf{x}(t) - (0, 0)| < \epsilon$ for all $t > 0$. This was precisely the definition of Liapunov stability at the origin.



□

Theorem 2.20. *Let N and N_μ be defined as usual with topographic curves defined by V. Suppose that*

$\dot{x} = f(x, y)$
$\dot{y} = g(x, y)$

in N_μ

If V is also a Liapunov function, then

(1) $R_\mu = N_\mu \backslash (0, 0)$ *contains no fixed points.*

(2) N_μ *does not contain any closed paths (fixed points or limit cycles)*

Proof. (1) Let $P \in R_\mu$ be a fixed point of the system. Then, $\dot{x}(P) = \dot{y}(P) = 0$. So,

$$\dot{V}(P) = V_x(P)\dot{x}(P) + V_y(P)\dot{y}(P) = 0$$

This contradicts the fact that $\dot{V} < 0$ in R_μ, from the definition of a Liapunov function. So, R_μ can contain no fixed points.

(2) Let G be any trajectory intersecting N_μ. Let A and B be points on G interior to N_μ such that G hits the point B after it hits A. Let T_A, T_B be the topographic curves intersecting the points A and B respectively, which are guaranteed to exist by proposition 2.7 statement (2). Since $\dot{V} < 0$ in N_μ, T_B must be interior to T_A. Thus, the path G is moving towards the origin.

We know from theorem 2.9 that once G enters T_B, it can never again cross its boundary. Thus, G cannot form a closed orbit in N_μ.

□

Theorem 2.21. *(Global Asymptotical Stability of the origin) Let N be a topographic system, with N_μ and V defined as usual. Take the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with continuously differentiable $\mathbf{f}$, and a fixed point at the origin. Suppose further that V is a Liapunov function for the origin in this system. Then, the origin is globally asymptotically stable.*

Proof. Let P be any point in N_μ and let T_P be the topographic curve through P. Suppose H is the positive half-path starting from P. Let R_P denote T_P together with its interior and with a puncture at the origin, then R_P is a closed region. By theorem 2.18, R_P is a trapping region, and by theorem 2.20, it contains no fixed points. Applying Poincaré-Bendixson theorem to R_P, we have that H either approaches a closed orbit or it is a closed orbit. By the previous theorem, $N_\mu \backslash (0, 0)$ contains no closed orbits, so we must have that H approaches the origin, which is a closed orbit of radius 0. Hence, the origin is globally asymptotically stable within N_μ. □

Remark 2.22. This proves everything that we set out to do in this subsection. However, we only considered the case where the fixed point is at the origin. This simplification was done for visual simplicity. Generalizing these results to an arbitrary fixed point is near-trivial. As it turns out, we'll only need these results in section 3 for a fixed point at the origin.

2.4 **Bifurcations in Two Dimensions** We now revisit a fundamental concept in system dynamics: bifurcations. In one dimension, these were relatively simple, albeit the most challenging part of the section. In two dimensions, the complexity increases immensely, with the addition of bifurcations of closed orbits. That is, two closed orbits can intersect one another and are either destroyed, exchange stability, etc. Note that this section will not entail an extensive study of every type of bifurcation, only those we'll study in the next section.

Definition 2.23. *(Bifurcations for two dimensions) We say that a* ***bifurcation*** *has occurred if, as a parameter in a system is varied, the topological structure of the system's phase portrait has changed.*

Seeing as how we've studied saddle-node bifurcations in one-dimension, and how they are relatively similar in two dimensions, we'll start with the **supercritical pitchfork bifurcation**. To begin, consider the system below.

$\dot{x} = \mu x - x^3$
$\dot{y} = -y$

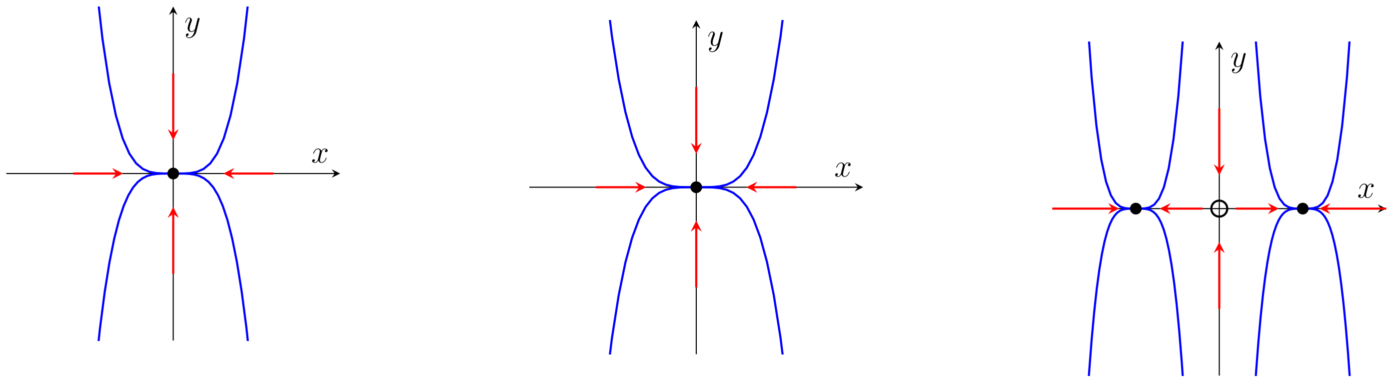
We'll plot the phase portraits for $\mu < 0$, $\mu = 0$, $\mu > 0$

First, for $\mu < 0$, the only fixed point is the origin. It can be shown by linearization about the origin that it is a stable node. The two stable manifolds are the x-axis and the y-axis.

At $\mu = 0$, the differential equation in x becomes $\dot{x} = -x^3$. The origin is still the only stable fixed point. However, trajectories slow down as they approach the origin, since $-x^3$ shrinks much slower than $\mu x - x^3$ for $\mu < 0$. This is called **critical slowing down**. The stable manifolds remain the same.

For $\mu > 0$, the origin becomes unstable and two new fixed points appear: $\mathbf{x}_1{}^\star, \mathbf{x}_2{}^\star = (\pm\sqrt{\mu}, 0)$. The origin becomes a saddle, and the two new fixed points are stable nodes. Now, the stable manifold of the origin is the y-axis, while the unstable manifold is the x-axis.

The phase portraits below are given in the order $\mu < 0$, $\mu = 0$, $\mu > 0$, and the blue curves are trajectories, with directions given by the arrows on the stable and unstable manifolds.



Here, the bifurcation occurs at $\mu = 0$, since it's the exact point at which we see the qualitative change in the phase portraits. In summary, a supercritical pitchfork bifurcation occurs when a stable node transforms into a saddle point, and produces two more stable nodes at the time of bifurcation.

Next is the **transcritical bifurcation**. The prototypical example is given by

$$\dot{x} = \mu x - x^2$$
$$\dot{y} = -y$$

We'll perform the same analysis of the system by plotting the phase portrait for $\mu < 0$, $\mu = 0$, and $\mu > 0$.

For $\mu < 0$, the origin is a stable node and $(\mu, 0)$ is a saddle point. The x-axis is the unstable manifold of the saddle, and the y-axis is the stable manifold.

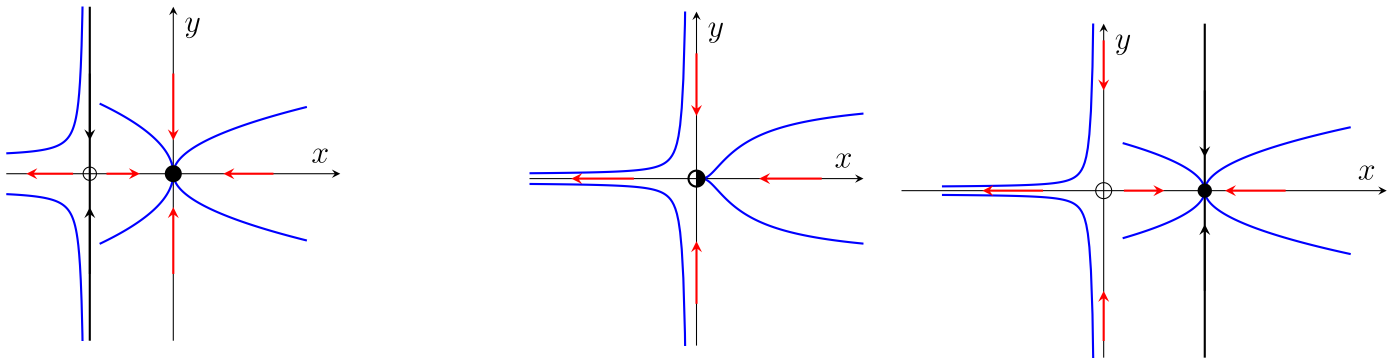
At $\mu = 0$, the system becomes

$$\dot{x} = -x^2$$
$$\dot{y} = -y$$

So the origin is the only fixed point. One finds that the method of linearization predicts a line of fixed points on the x-axis, which is clearly incorrect just from looking at the system. Instead, by simply choosing an initial condition and following the trajectory, we see that the x-axis is half-stable, with trajectories constantly flowing leftwards, and that the y-axis is fully stable.

Finally, for $\mu > 0$, the origin becomes a saddle point, with the x-axis as its unstable manifold, and the y-axis being stable. The point $(\mu, 0)$ is a stable node.

This is similar to what we saw in the one-dimensional transcritical bifurcation, where two fixed points collide and exchange stabilities. See below for a simple phase portrait of the system for $\mu < 0$, $\mu = 0$, then $\mu > 0$. The nullclines (including the x and y axes) are given in black, and trajectories in blue



Every transcritical bifurcation loosely follows this example: two fixed points collide, usually one saddle point and one stable or unstable node, and they exchange their stability types.

Now, for **Hopf bifurcations**. The general idea is that a system has a stable fixed point, and its associated eigenvalues in the linearization are λ_1 and λ_2, both of which have non-zero imaginary part, and are complex conjugates of one another. Since it's stable, $\Re(\lambda_1), \Re(\lambda_2) < 0$. The Hopf bifurcation itself occurs if the real parts of the eigenvalues become positive at the same time as a parameter is varied. The bifurcation itself would occur at the parameter value at which both eigenvalues are purely imaginary. We often visualize this as the two eigenvalues crossing the imaginary axis parallel to one another on the real-imaginary plane.

Consider the system given by

$$\dot{r} = \mu r + r^3$$
$$\dot{\theta} = \omega + br^2, \omega > 0$$

We'll use this example to study the geometry of its Hopf bifurcation at $\mu = 0$. First note that the origin is the only fixed point of the system. To recover the Jacobian, we'll first need to convert it to cartesian coordinates using the standard fact $x = r\cos\theta$, $y = r\sin\theta$. We find

$$\dot{x} = \mu x - \omega y + x^3 + xy^2 - bxy^2 - by^3$$
$$\dot{y} = \omega x + \mu y + bx^3 + bxy^2 + x^2y + y^3$$

And so the Jacobian is given by:

$$A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

So the classification of the origin is determined by the trace $\tau = 2\mu$ and determinant $\Delta = \mu^2 + \omega^2$. We have that $\Delta > 0$, since we're only considering $\omega > 0$. Also, $\Delta > \frac{1}{4}\tau^2$, so the only possibilities are stable spirals, centers, or unstable spirals depending on the sign of μ.

When $\mu < 0$, the origin is a stable spiral. We also observe directly from the differential equation in r that trajectories beginning at a radius of $r = \sqrt{-\mu}$ from the origin remain at that distance for all time. This must be a circular limit cycle. Further, we can conclude that this cycle is unstable, since the origin is a stable spiral. So, trajectories beginning at a distance less than $\sqrt{-\mu}$ from the origin spiral towards it.

As μ increases, the limit cycle shrinks, until $\mu = 0$, when it completely engulfs the origin, transforming it into an unstable fixed point. Finally, for $\mu > 0$, the origin is unstable, and the limit cycle disappears.

3 Lorenz Equations and Chaos

The Lorenz Equations form a three-dimensional system with three parameters. To study the system, we'll apply everything we've seen so far. We'll also briefly discuss strange attractors in this section. The system is given below.

$\dot{x} = \sigma(y - x)$
$\dot{y} = rx - y - xz$
$\dot{z} = xy - bz$, where $\sigma, r, b > 0$

We call σ the **Prandtl number** and r the **Rayleigh number**.

As a first observation, the system is invariant under the map $(x, y) \to (-x, -y)$, which means that if (x, y, z) solves the system, then so does $(-x, -y, z)$. This fact can be seen simply by replacing all x terms with $-x$, and doing the same for y, then observing that the system remains the same.

Theorem 3.1. *(Volume contraction)*
Volumes in phase space contract under flow. More specifically, take a closed surface, S, with volume, V, and consider trajectories of the Lorenz system with initial conditions on S at time t_0 (denote this surface $S(t_0)$ with volume $V(t_0)$). Then, after any arbitrary time $s > 0$, trajectories contort the surface into $S(t_0 + s)$ with volume $V(t_0 + s)$. We will then have $V(t_0 + s) < V(t_0)$

Proof. First, consider an arbitrary three-dimensional system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, and a closed surface, $S(t_0)$, with volume $V(t_0)$. Take trajectories with initial conditions on $S(t_0)$ at time $t = t_0$. Then, after an infinitesimal time, dt, the positions of the trajectories form the closed surface $S(t_0 + dt)$ with volume $V(t_0 + dt)$.

Use $\hat{n}$ to denote the outward unit normal vector to the surface S, then $(\mathbf{f} \cdot \hat{n})$ is the outward normal component of a trajectory's velocity. If, in time dt, trajectories sweep out an area dA, then the additional volume created is given by

$$\int_{S(t_0)} (\mathbf{f} \cdot \hat{n}\; dt)\; dA \implies V(t_0 + dt) = V(t_0) + \int_{S(t_0)} (\mathbf{f} \cdot \hat{n}\; dt)\; dA \implies \dot{V}(t_0) = \frac{V(t_0+dt)-V(t_0)}{dt} = \int_{S(t_0)} \mathbf{f} \cdot \hat{n}\; dA$$

Then, by Divergence theorem, $\dot{V}(t_0) = \int_{V(t_0)} \nabla \cdot \mathbf{f} dV$.
Applying this to the Lorenz system, $\nabla \cdot \mathbf{f} = -(\sigma + b + 1)$, which is negative and constant. So,

$\dot{V}(t) = -(\sigma + b + 1)V(t) \implies V(t) = V(0)e^{-(\sigma+b+1)t}$

Hence, the volume of the surface is decaying exponentially over time. We thus have the property of Volume contraction for the Lorenz system. □

Remark 3.2. In general, we say that a dynamical system that exhibits the property of Volume contraction is **dissipative**. Dissipative systems in continuous time are all eventually confined to a limiting set of volume 0, since the volume of the surface formed by their trajectories is constantly decaying.

Corollary 3.3. *The Lorenz system cannot have any repelling fixed points or repelling closed orbits.*

Proof. Suppose for contradiction that there is some object, either a fixed point or a closed orbit, that repels trajectories from all directions. Then, every trajectory beginning on a surface enclosing the repeller would be pushed away. Thus, the surface would see an increase in volume after some time. Clearly, the repeller contradicts the property of volume contraction. □

Remark 3.4. The term ”repeller” means an object (fixed point or limit cycle) that completely pushes trajectories away, no matter where the trajectory begins relative to the object. So, the Lorenz systems can still have unstable fixed points, like saddles, for example, which only push trajectories out in one direction, while drawing them in from other directions.

The next part of this section is a brief analysis of the fixed points, limit cycles, and bifurcations of the system. The Lorenz system famously sees a vast amount of bifurcations which dramatically change the behaviour of the system, so the discussion that follows is not extensive.

To begin, observe that the origin is a fixed point for all values of $\sigma, r, b > 0$. Also, for $r > 1$, there's a symmetric pair of fixed points that we'll call C^+ and C^-.

$$C^+ = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$$
$$C^- = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$$

Now, let's classify these fixed points and the bifurcation at $r = 1$.

Proposition 3.5. *The origin is a stable node for $r < 1$ and a saddle for $r > 1$ in the Lorenz system.*

Proof. First, we'll linearize the Lorenz equations simply by omitting nonlinear terms, as follows.

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y$$
$$\dot{z} = -bz$$

Here, the differential equation in z is decoupled, and easily solvable. The solution is $z = z(0)e^{-bt}$. This means that $z \to 0$ exponentially fast as $t \to \infty$ for all values of $r > 0$. Then, the two-dimensional Jacobian for the origin, omitting the z dimension, is:

$$A = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \implies \tau = -(\sigma + 1) < 0, \Delta = \sigma(1 - r)$$

For $r < 1$, we have $\tau < 0$, $\Delta > 0$, and
$\tau^2 - 4\Delta = (\sigma + 1)^2 - 4\sigma(1 - r) = \cdots = (\sigma - 1)^2 + 4\sigma r > 0$. So, in two-dimensions, we have a stable node. However, as we've seen, $z \to 0$ exponentially fast for any initial condition, so the z-axis must be a stable manifold. The linearization predicts a stable node, which is a robust case, and so the origin is, in fact, a stable node.

For $r > 1$, we have $\tau < 0$, $\Delta < 0$. This means that the origin is a saddle point in two dimensions. It has one stable manifold and one unstable manifold. If we include the z-dimension, there are two stable manifolds, and one unstable manifold, meaning the origin is a three-dimensional saddle point for $r > 1$. □

Proposition 3.6. *For $r < 1$, the origin is globally asymptotically stable; all trajectories go to the origin as $t \to \infty$.*

Proof. As we've seen in Theorem 2.21, it suffices to prove that there is a Liapunov function for the origin in this system.

Motivated by a proposition we'll see later in this section, take $V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$. These are concentric ellipsoids about the origin.

Clearly, $V(0, 0, 0) = 0$, and $V(x, y, z) > 0$ for $(x, y, z) \neq (0, 0, 0)$. Further,

$$\dot{V} = 2(\tfrac{1}{\sigma}x\dot{x} + y\dot{y} + z\dot{z}) \implies \tfrac{1}{2}\dot{V} = (r + 1)xy - x^2 - y^2 - bz^2$$
$$\implies \tfrac{1}{2}\dot{V} = -[x - \tfrac{r+1}{2}y]^2 - [1 - (\tfrac{r+1}{2})^2]y^2 - bz^2 < 0 \text{ for } (x, y, z) \neq (0, 0, 0) \text{ and } r < 1.$$

The function V is also continuously differentiable, and so it is a Liapunov function. This proves that the origin is globally asymptotically stable for $r < 1$. □

We'll now analyze the stability of the fixed points C^+, C^- for $r > 1$. The three-dimensional, linearized Jacbian is given by

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}.$$

The characteristic equation for determining the eigenvalues is then identical for both fixed points.

$$\det(A - \lambda I) = \lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r-1) = 0$$

We'll guess $\lambda = \pm i\omega$ to determine the value of r for which the eigenvalues are purely imaginary. Solving the resulting equation for r, we find $r_H = \sigma(\frac{\sigma+b+3}{\sigma-b-1})$. As we'll see later, the subscript, H, comes from the fact that a Hopf bifurcation occurs at this point.

Proposition 3.7. *The fixed points C^+ and C^- are attracting for $1 < r < r_H$.*

Proof. For the fixed points to be attracting, we need the real parts of the eigenvalues of the Jacobian to be negative. The **Routh-Hurwitz criterion** states that the roots of a third-order polynomial of the form

$$p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$

must all have negative real part if

(1) All coefficients are positive

(2) $a_2 a_1 > a_0$

Recall that the characteristic polynomial of the linearized Jacobian at the fixed points C^+, C^- was

$$det(A - \lambda I) = \lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r-1)$$

It is easily shown that, since $r > 1$ and $r < r_H$, the Routh-Hurwitz criterion are met, and so all the eigenvalues of the Jacobian have negative real part. So, the two fixed points are attracting. □

Proposition 3.8. *At $r = 1$, the fixed point at the origin bifurcates into the fixed points C^+ and C^- in a supercritical pitchfork.*

Proof. The work for this proof has been done in the previous propositions. For $r < 1$, we have a single, globally stable fixed point. Then, for $r > 1$, two new, stable, and symmetric fixed points appear, and the origin becomes a saddle point. This transition is precisely a supercritical pitchfork bifurcation at $r = 1$. □

Proposition 3.9. *Two subcritical Hopf bifurcations occur at $r = r_H$, rendering each C^+ and C^- into unstable spirals.*

Proof. We'll prove this statement for C^+, then the corresponding proof for C^- can be proven similarly. Note that we only need to show that two of the eigenvalues of the Jacobian for C^+ cross the imaginary axis parallel to one another, and that they do so with non-zero speed (that is, they have positive real part after bifurcating). Then, the typical geometry of a subcritical Hopf bifurcation will follow.

As we've seen, at $r = r_H$, two of the eigenvalues are complex conjugates of one another, so $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$ for some $\omega \in \mathbb{R}$. This confirms that the eigenvalues intersect the imaginary axis parallel to one another at $r = r_H$.

All that remains to show is that the two imaginary eigenvalues cross the axis with positive speed. To show this, consider λ as a function of r, and recall the characteristic equation:

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0$$
$$\implies 3\lambda^2\lambda' + 2(\sigma + b + 1)\lambda\lambda' + b\lambda + (r + \sigma)b\lambda' + 2b\sigma = 0$$

$$\implies \lambda'(r) = \frac{-b\lambda - 2b\sigma}{3\lambda^2 + 2(\sigma+b+1)\lambda + (r+\sigma)b}$$

We can then plug in $r = r_H$, $\lambda(r_H) = i\omega$, to find $\Re(\lambda'(r_H)) > 0$. We thus have a Hopf bifurcation at C^+ and C^-, meaning the two fixed points are transformed into unstable spirals after $r = r_H$. $\square$

Once this bifurcation occurs, there are no stable objects left in the entirety of the system. However, recall that Theorem 3.1 guarantees that all trajectories eventually enter a limiting set of volume zero. This set is called a **strange attractor**. We'll define the term in more detail now.

Definition 3.10. *An* ***attractor*** *is a closed set, A, that satisfies the following three properties.*

(1) *A is* ***invariant****, meaning that all trajectories starting in A remain there for all time.*

(2) *There exists an open set, U, such that if $\mathbf{x}(0) \in U$, then trajectories eventually reach A. In other words, A attracts some open set of initial conditions. The set U is called the* ***basin of attraction****.*

(3) *A is minimal. So, there does not exist any subset of A satisfying conditions (1) and (2).*

Definition 3.11. *A* ***strange attractor*** *is an attractor that exhibits sensitive dependence on initial conditions. What this means is that if we take a deterministic, dynamical system with a strange attractor, then trajectories of the system inside of the attractor do not depend continuously on initial data. So, changing initial data even slightly will result in large qualitative differences in the resulting trajectory.*

One fact about the Lorenz system's strange attractor that can be proven relatively easily is that it is contained entirely in an ellipse of the form $E = \{(x, y, z) \in \mathbb{R}^3 \mid rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leqslant C\}$ for some $C \in \mathbb{R}$.

Theorem 3.12. *Every trajectory of the Lorenz equation is eventually confined to an ellipsoid of the form*

$$rx^2 + \sigma y^2 + \sigma(z-2r)^2 \leqslant C$$

Proof. Let $V(x,y,z) = rx^2 + \sigma y^2 + \sigma(z-2r)^2$.

It can then be shown that $\dot{V} = -2\sigma(rx^2 + y^2 + b(z-r)^2 - br^2)$, by substituting the Lorenz equations into $\dot{x}$, $\dot{y}$, $\dot{z}$.
$\dot{V} = 0 \iff rx^2 + y^2 + b(z-r)^2 = br^2$.

Define $E_0 = \{(x,y,z) \in \mathbb{R}^3 \mid rx^2 + y^2 + b(z-r)^2 \leqslant br^2\}$. The region E_0 is then an ellipsoid with the property that $\dot{V} = 0$ on the boundary. Further, $\dot{V} > 0$ for points interior to E_0 and $\dot{V} < 0$ for points outside of E_0.

Define $E = \{(x,y,z) \in \mathbb{R}^3 \mid V(x,y,z) \leqslant C\}$ for a constant, C, large enough that $E_0 \subsetneq E$.

We then have the following:
Take any initial condition to the Lorenz equations, $\mathbf{x}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$. Then, one of the following cases holds:

(1) If $\mathbf{x}_0 \notin E_0$, then $\dot{V}(\mathbf{x}_0) < 0$. Hence trajectories move inwards towards the center of E. They'll enter the ellipsoid and continue moving toward its center, until they reach the boundary of E_0. Trajectories are unable to leave E, since that would require $\dot{V} > 0$ on ∂E.

(2) If $\mathbf{x}_0 \in E_0$, then trajectories are once again unable to cross through ∂E and exit the ellipsoid, since $\dot{V} < 0$ on the boundary and within the ellipsoid, excluding E_0.

In any case, trajectories are eventually confined to the ellipsoid E. □

For further exposition on the strange attractor and its properties, see Strogatz (2018).

4 Acknowledgements

I would like to thank my mentor, Vincent Painchaud, for aiding me throughout the semester. He was incredibly helpful and patient in finding the literature necessary to write this project, and in helping me plan my research and writing. I absolutely could not have completed the project without his help. Thank you also to the directors of the DRP: Hazem Hassan, Sasha Bell, and Catherine Fontaine for organizing the program, and for your amazing presentation on writing and presenting math literature.

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