A brief introduction to non-linear partial differential equations

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Abstract

In this paper, we present an introduction to partial differential equations, focusing on the method of characteristics and its application to linear and nonlinear equations. We develop the method in two dimensions before extending it to n-dimensional first-order nonlinear PDEs. We examine three important partial differential equations, the wave, diffusion, and Laplace equations, and present a derivation of their general solutions. We aim to provide a clear and concise exposition of these foundational topics in PDEs.

1 Introduction

The following introduction follows Section 1.1 and 1.4 of Strauss [1].

A partial differential equation (PDE) is an equation that relates the partial derivatives of a function of several variables. Consider the most general second-order partial differential equation in two dimensions

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{1}$$

a solution of (1) is a function $u = u(x, y)$ that satisfies the equation in some open set region $D \subseteq \mathbb{R}^2$.

There are three types of boundary conditions one can impose on a partial differential equation. Firstly, there is the Dirichlet condition, where the solution u is specified. Secondly, there is the Neumann condition, where the normal derivative $\partial u/\partial n$ is specified. Lastly, there is the Robin condition, where $\partial u/\partial n + au$ is specified, where a is a given function of x, y and of time t. Each condition holds for all $t \in (0, \infty)$ and all $(x, y) \in \partial D$, and the type of boundary condition used depends on the specific physical situation.

2 Method of characteristics in two dimensions

The following section follows Section 1.3 of Strauss [1].

Partial differential equations are generally much more difficult to solve directly than ordinary differential equations (ODEs), for which many solution techniques have been developed. Thus it is useful to turn a PDE into a system of ODEs using the *method of characteristics*.

Definition 1 *A characteristic curve is a curve* $(x(t), t)$ *along which* $\frac{du}{dt} = 0$, *so that* $u(x(t), t)$ *is constant.*

To illustrate this method, we provide an example. Consider the transport equation

$$u_t + cu_x = 0.$$

This equation is *linear* because it is linear in u and its derivatives, and it is a *first-order* equation because the highest order of derivatives in the equation is one. To solve the PDE, let $x(t)$ be a curve and consider the solution $u(x(t), t)$ along the curve. By the Chain Rule,

$$\frac{du}{dt} = u_t + \frac{dx}{dt} u_x = 0,$$

where the second equality holds if we set $\frac{dx}{dt} = c$. It follows that, the curves $x(t)$ such that $\frac{dx}{dt} = c$ are the *characteristic curves* and the solution u is constant along these curves. To solve the PDE, we now must just solve these two ODEs.

We can integrate to solve for the characteristic curves, $x = ct + x_0$ where x_0 is some constant. Thus $x - ct = x_0$. Since the solution is constant along each characteristic curve, it must only depend on $x - ct$. Thus, we can write

$$u(x, t) = f(x - ct)$$

where f is an arbitrary function of one variable. If we are given some initial data, we solve uniquely for f.

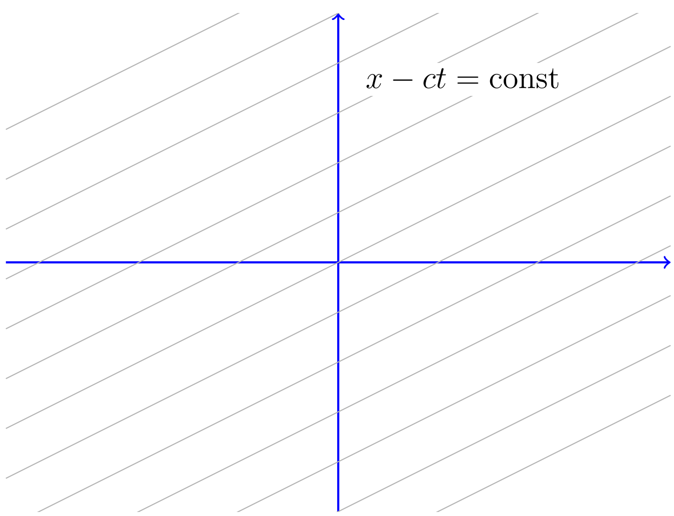

Figure 1: Characteristic curves for $u_t + cu_x = 0$

Figure 1 shows the xt plane decomposed into the characteristic curves, which are straight lines. Along each line, the PDE reduces to the ODE $\frac{du}{dt} = 0$.

3 Important partial differential equations

This section presents three important partial differential equations that arise in many real-world applications. They are the wave equation, the diffusion equation, and the Laplace equation.

3.1 Wave equation

The following section follows Section 2.1 of Strauss [1].

The wave equation is given as

$$u_{tt} = c^2 u_{xx} \quad \text{for } -\infty < x < +\infty. \tag{2}$$

It is the primary archetype of linear second-order hyperbolic PDEs. In order to determine the most general solution, we write the equation in the following way:

$$u_{tt} - c^2 u_{xx} = \Big(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\Big)u = 0. \tag{3}$$

To derive the general solution, we introduce the characteristic coordinates $\xi = x + ct$ and $\eta = x - ct$. By the Chain Rule, we have $\partial_x = \partial_\xi + \partial_\eta$ and $\partial_t = c\partial_\xi - c\partial_\eta$. Thus $\partial_t - c\partial_x = -2c\partial_\eta$ and $\partial_t + c\partial_x = 2c\partial_\xi$. Thus (3) becomes

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x) = (-2c\partial_\eta)(2c\partial_\xi)u = -4c^2 u_{\xi\eta} = 0.$$

Hence $u_{\xi\eta} = 0$ since $c \neq 0$. Thus the general solution to the wave equation is given by

$$u(x,t) = f(\xi) + g(\eta) = f(x+ct) + g(x-ct). \tag{4}$$

If we consider the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$, we can find the unique solution to this problem. Setting $t = 0$ in (4) gives that:

$$\phi(x) = f(x) + g(x), \tag{5}$$

and, by the Chain Rule, we can differentiate (4) with respect to t and set $t = 0$ to get

$$\psi(x) = cf'(x) - cg'(x). \tag{6}$$

Change the name of x to s for the moment. Differentiate (5) to get $\phi' = f' + g'$ and divide (6) by c to get $\frac{1}{c}\psi = f' - g'$. We can add and subtract these equations to get that

$$f' = \frac{1}{2}\Big(\phi' + \frac{\psi}{c}\Big) \quad \text{and} \quad g' = \frac{1}{2}\Big(\phi' - \frac{\psi}{c}\Big)$$

and integrate to get

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + A \quad \text{and} \quad g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + B$$

where A, B are integration constants. Adding these two equations gives $f(s) + g(s) = \phi(s) + A + B$, so by (5) we must have that $A + B = 0$. We can substitute $s = x + ct$ into the equation for f and $s = x - ct$ for g to get the solution

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

This is d'Alembert's solution to the 1D wave equation.

3.2 Diffusion equation

The following section follows Section 2.4 of Strauss [1].

We consider the following initial value problem to the diffusion equation:

$$u_t(x) = ku_{xx}(x) \quad \text{in } -\infty < x < \infty, 0 < t < \infty, \tag{7}$$

$$u(x,0) = \phi(x) \text{ for } t = 0. \tag{8}$$

The goal is to find the general solution for diffusion on the whole line.

We can construct a solution from first principles using the following five invariance properties of the diffusion equation (7):

(a) The translate $u(x-y,t)$ of any solution $u(x,t)$ is another solution for any fixed y.

(b) Any derivative of a solution $u(x,t)$, such as u_x, u_t, u_{xx}, etc, is another solution.

(c) A linear combination of solutions is a solution, by linearity.

(d) An integral of a solution is another solution.

(e) If $u(x,t)$ is a solution of (7), then so is the dilated function $u(\sqrt{a}x, at)$ for any $a > 0$. This can be verified using the Chain Rule: let $v(x,t) = u(\sqrt{a}x, at)$. Then $v_t = [\partial(at)/\partial(t)]u_t = au_t$ and $v_x = [\partial(\sqrt{a}x)/\partial(x)]u_x = \sqrt{a}u_x$ and $v_{xx} = \sqrt{a} \cdot \sqrt{a}u_{xx} = au_{xx}$. Thus $v_t = au_t = a(ku_{xx}) = k(au_{xx}) = kv_{xx}$ as desired.

Notably, if $S(x,t)$ is a solution of (7), then so is $S(x-y,t)$ by property (a) and so is

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy \tag{9}$$

for any function $g(y)$, as long as the improper integral converges, by property (d).

The procedure will be to find a particular solution of (7) and then construct all other solutions using (9). Denote the particular solution $Q(x,t)$ that satisfies

$$Q(x,0) = 1 \quad \text{for } x > 0, \quad Q(x,0) = 0 \quad \text{for } x < 0. \tag{10}$$

We will look for $Q(x,t)$ of the form:

$$Q(x,t) = g(p), \quad \text{where } p = \frac{x}{\sqrt{4kt}}. \tag{11}$$

where g is a function of one variable.

We expect Q to have this form because property (e) says that the equation (7) is unaffected by the dilation $x \to \sqrt{a}x, t \to at$. The special initial condition (10) is unaffected by the dilation since the change occurs at $t = 0$. Thus, we expect Q, the solution which is defined by (10), to not be affected by the dilation either. Indeed, if Q depends on x and t by $x/\sqrt{t}$, then the dilation takes $x/\sqrt{t}$ into $\sqrt{a}x/\sqrt{at} = x/\sqrt{t}$.

By the Chain Rule,

$$\begin{aligned} Q_t &= \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p) \\ Q_x &= \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p) \\ Q_{xx} &= \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4kt}g''(p), \end{aligned}$$

so then by (7), we obtain the following ODE:

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\left[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right].$$

Simplifying we get:

$$g'' + 2pg' = 0.$$

We may solve this ODE using the integrating factor $e^{\int 2pdp} = e^{p^2}$. We get $g'(p) = c_1 e^{-p^2}$, and thus

$$Q(x,t) = g(p) = c_1 \int_0^{x/\sqrt{4kt}} e^{-p^2}dp + c_2,$$

which is valid for $t > 0$. We investigate at $t = 0$ by taking the limit from the right:

$$\begin{aligned} &\text{If } x > 0, \quad 1 = \lim_{t\to 0^+} Q = c_1 \int_0^{+\infty} e^{-p^2}dp + c_2 = c_1\frac{\sqrt{\pi}}{2} + c_2. \\ &\text{If } x < 0, \quad 0 = \lim_{t\to 0^+} Q = c_1 \int_0^{-\infty} e^{-p^2}dp + c_2 = -c_1\frac{\sqrt{\pi}}{2} + c_2. \end{aligned}$$

Solving the system gives $c_1 = 1/\sqrt{\pi}$ and $c_2 = 1/2$. Therefore, we get the particular solution:

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-p^2}dp \tag{12}$$

for $t > 0$. Notice that this solution satisfies (7), (10), and (11).

Now we move to finding the general solution to (7). Define $S = \partial Q/\partial x$. By property (b), S is a solution of (1) since it is the derivative of a solution Q. Given any function ϕ, define

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \quad \text{for } t > 0. \tag{13}$$

This matches the form of (9), which is a solution of (7). We will show that u is the unique solution of (7), (8). First, we verify the validity of (8). Rewrite u in the following way:

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)dy \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\phi(y)dy \\ &= \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy - Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=+\infty} \end{aligned}$$

using integration by parts. Assume that $\phi(y) = 0$ for $|y|$ large, so the limits vanish. Thus, to verify (8):

$$\begin{aligned} u(x,0) &= \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)dy \\ &= \int_{-\infty}^{x} Q(x-y,0)\phi'(y)dy + \int_{x}^{\infty} Q(x-y,0)\phi'(y)dy \\ &= \int_{-\infty}^{x} \phi'(y)dy \quad \text{since } Q(x-y,0) = 0 \text{ for } y > x \text{ and } Q(x-y,0) = 1 \text{ for } y < x \\ &= \phi(y)\Big|_{-\infty}^{x} = \phi(x) \end{aligned}$$

using the initial condition for Q and the assumption that $\phi(-\infty) = 0$. Hence $u(x,t)$ satisfies the initial condition (8). We can thus conclude that (13) gives all of the solutions to (7), (8) for any initial condition function $\phi(x)$ provided, where

$$S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi k t}} e^{-x^2/4kt} \quad \text{for } t > 0$$

and hence

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)dy.$$

3.3 Laplace equation

The following section follows Sections 6.1 and 6.3 of Strauss [1].

The Laplace equation arises when a diffusion or wave process is independent of time, so that $u_t = u_{tt} \equiv 0$. From this, we obtain the equation:

$$\begin{aligned} u_{xx} &= 0 \quad \text{in one dimension} \\ \nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy} &= 0 \quad \text{in two dimensions} \\ \nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy} + u_{zz} &= 0 \quad \text{in three dimensions} \end{aligned}$$

Definition 2 *A solution to the Laplace equation is called a harmonic function.*

In one dimension, $u_{xx} = 0$, so the only harmonic functions are $u(x) = A + Bx$. The problem we aim to solve is Laplace's equation in a given domain D with some condition on ∂D. In this section, we will focus on the Dirichlet problem:

$$\Delta u = f \text{ in D}, u = h \text{ on } \partial D. \tag{14}$$

3.3.1 Maximum Principle and Rotational Invariance

Two important properties of the Laplace equation are the Maximum Principle, which states that a harmonic function reaches its largest and smallest values on the boundary, and rotational invariance.

Maximum Principle: Let D be a connected, bounded, open set in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let u be a harmonic function in D that is continuous on $\overline{D} = D \cup \partial D$. Then the maximum and minimum values of u are attained on ∂D and nowhere inside, unless $u \equiv$ constant.

Consider the Dirichlet problem $\Delta u = f$ in D, $u = h$ on ∂D. As a consequence of the Mean Value Property, we can show that the solution to the Dirichlet problem is unique. Suppose $\Delta u = f$ in D by linearity, $u = h$ on ∂D, $\Delta v = f$ in D, and $v = h$ on ∂D. Let $w = u - v$. Then $\Delta w = 0$ in D, and $w = 0$ on ∂D. Since the maximum and minimum values are attained on ∂D by the Maximum Principle, but $w = 0$ on ∂D, we have $0 = w(\mathbf{x}_m) \leq w(\mathbf{x}) \leq w(\mathbf{x}_M) = 0$. Thus $w \equiv 0$ and $u \equiv v$.

Invariance in Two Dimensions: The Laplace Equation is invariant under all rigid motions, which are motions in the plane consisting of translations and rotations. Invariance means that for transformations of $(x, y) \to (x', y')$, we have $u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}$.

A rotation in the plane through the angle α is given by

$$\begin{aligned} x' &= x \cos \alpha + y \sin \alpha \\ y' &= -x \sin \alpha + y \cos \alpha \end{aligned}$$

One can verify using the Chain Rule that invariance holds. The rotational invariance suggests that the two dimensional Laplacian $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ should take a simple form in polar coordinates $x = r\cos\theta, y = r\sin\theta$. This transformation has Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

with the inverse matrix

$$J^{-1} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \frac{-\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix}$$

By the Chain Rule, we have

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned}$$

The operators are squared to give

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \cos^2\theta \frac{\partial^2}{\partial r^2} - 2\left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial^2}{\partial r \partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} \\ \frac{\partial^2}{\partial y^2} &= \sin^2\theta \frac{\partial^2}{\partial r^2} + 2\left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial^2}{\partial r \partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} \end{aligned}$$

By adding these squared operators, the Laplace equation becomes:

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

It is natural to look for special harmonic functions that themselves are rotationally invariant. In two dimensions, use polar coordinates (r, θ) and look for solutions only depending on r and not on θ. The Laplace equation thus becomes $0 = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r$. This ODE can be solved:

$$(ru_r)_r = 0 \implies ru_r = c_1 \implies u = c_1 \ln(r) + c_2.$$

3.3.2 Poisson's Formula

The rotational invariance of Δ suggests that the circle is a natural shape for harmonic functions. Consider the Dirichlet problem for a circle

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{for } x^2 + y^2 < a^2 \\ u &= h(\theta) && \text{for } x^2 + y^2 = a^2 \end{aligned}$$

with radius a and any boundary data $h(\theta)$.

To solve the problem, consider solutions of the form $u = R(r)\Theta(\theta)$, expressed in terms of polar coordinates. If we plus this into the Laplace's equation given in polar coordinates, we find that:

$$\begin{aligned} 0 = u_{xx} + u_{yy} &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ &= R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' \end{aligned}$$

Dividing by $R\Theta$ and multiplying by r^2, we get

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0 \implies r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta}.$$

Since the left side depends only on r and the right side depends only on θ, both must be equal to the same constant λ. Thus set

$$\begin{aligned} \frac{R''}{R} + r\frac{R'}{R} &= \lambda \\ -\frac{\Theta''}{\Theta} &= \lambda \end{aligned}$$

and rearrange to get:

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 \\ r^2R'' + rR' - \lambda R &= 0. \end{aligned}$$

These are two ODEs that can be solved separately. Since $\Theta(\theta)$ must be 2π-periodic, the angular equation $\Theta'' + \lambda\Theta = 0$ has nontrivial periodic solutions only when

$$\lambda = n^2, n = 0, 1, 2, ...$$

For $n \geq 1$, the corresponding angular solutions are

$$\Theta_n(\theta) = A_n \cos n\theta + B_n \sin n\theta$$

Substituting $\lambda = n^2$ into the radial equation gives $r^2R'' + rR' - n^2R = 0$, which is a Cauchy-Euler equation that has the general solution

$$R_n(r) = C_n r^n + D_n r^{-n}.$$

Since we seek solutions that remain bounded at $r = 0$, we discard the singular terms r^{-n}. Thus each separated solution has the form

$$u_n(r, \theta) = r^n(A_n \cos n\theta + B_n \sin n\theta).$$

Since Laplace's equation is linear, any linear combination of solutions is again a solution. By the superposition principle, we obtain the general bounded solution

$$u(r, \theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)$$

We can use the inhomogeneous boundary conditions at $r = a$. By setting $r = a$ in the series above, we get that

$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n(A_n \cos n\theta + B_n \sin n\theta)$$

The right hand side is in the form of a Fourier series. If h can be expressed as a Fourier series, the coefficients will be given by:

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos(n\phi) d\phi$$
$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin(n\phi) d\phi$$

The series for u can be expressed in closed form by plugging A_n and B_n in the expression for u. u can be summed explicitly by plugging A_n and B_n:

$$\begin{aligned} u(r,\theta) &= \int_0^{2\pi} h(\phi) \frac{d\phi}{2\pi} + \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n} \int_0^{2\pi} h(\phi)\{\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\} d\phi \\ &= \int_0^{2\pi} h(\phi) \Big\{ 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \Big\} \frac{d\phi}{2\pi}. \end{aligned}$$

The term in braces can be written as a geometric series of complex numbers:

$$\begin{aligned} & 1 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)} \\ = 1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}} &= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2} \end{aligned}$$

This results in Poisson's formula, which expresses any harmonic function inside a circle in terms of its boundary values.

$$u(r,\theta) = (a^2 - r^2) \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \frac{d\phi}{2\pi}$$

Poisson's formula can also be written in a more geometric way. Write $\mathbf{x} = (x, y)$ as a point with polar coordinates (r, θ). Let $\mathbf{x}'$ be a point on the boundary, thus it has polar coordinates (a, θ). The origin and the points $\mathbf{x}$ and $\mathbf{x}'$ form a triangle with sides $r = |\mathbf{x}|$, $a = |\mathbf{x'}|$, and $|\mathbf{x} - \mathbf{x}'| = a^2 + r^2 - 2ar\cos(\theta - \phi)$ by the law of cosines. The arc length element on the circumference is $ds' = a\ d\phi$. Thus, we can write Poisson's formula as

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{x}'|=a} \frac{u(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} ds'$$

for $\mathbf{x} \in D$, where we write $u(\mathbf{x}') = h(\theta)$.

Theorem 1 *Let $h(\theta) = u(\boldsymbol{x}')$ be any continuous function on the circle $C = bdy\ D$. Then the Poisson formula provides the only harmonic function in D for which*

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} u(\boldsymbol{x}) = h(\boldsymbol{x}_0) \quad \textit{for all } \boldsymbol{x}_0 \in C$$

This means that $u(\boldsymbol{x})$ is a continuous function on $\overline{D} = D \cup C$. It is also differentiable to all orders inside D.

4 Method of characteristics in n dimensions

The following section follows Section 3.2 of Evans [2].

Now we will look at general nonlinear first-order partial differential equations of the form

$$F(Du, u, x) = 0, \tag{15}$$

where $x = (x_1, x_2, ..., x_n) \in U$ and U is an open subset of $\mathbb{R}^n$. Here, $F : \mathbb{R}^n \times \mathbb{R} \times \bar{U} \to \mathbb{R}$ is given, where $\bar{U} = U \cup \partial U$, $u : \bar{U} \to \mathbb{R}, u = u(x)$ is the unknown, and Du is the gradient vector $Du = (u_{x_1}, ..., u_{x_n})$.

For notation purposes, write $F = F(p, z, x) = F(p_1, ..., p_n, z, x_1, ..., x_n)$ for $p \in \mathbb{R}^n, z \in \mathbb{R}, x \in U$. Here, p is the variable for which we substitute the gradient $Du(x)$, and z is the variable for which we substitute $u(x)$. Assume F is smooth and set $D_pF = (F_{p_1}, ..., F_{p_n}), D_zF = F_z$, and $D_xF = (F_{x_1}, ..., F_{x_n})$.

We aim to discover solutions u to (15) subject to the boundary condition $u = g$ on Γ, where $\Gamma \subseteq \partial U$ and $g : \Gamma \to \mathbb{R}$ is given.

4.1 Derivation of the characteristic ODEs

As mentioned in Section 2, the method of characteristics aims to solve (15) by converting the PDE into a system of ODEs. If u is a solution to the problem and x is a point in U, we will calculate $u(x)$ by finding some curve lying within U that connects x with a point $x^0 \in \Gamma$. Since $u = g$ on Γ, we know the value of u at the one end x^0. We then want to be able to calculate u all along the curve, including at x.

Let the curve described above be written as $\mathbf{x}(s) = (x^1(s), ..., x^n(s))$, where s is a parameter lying in a subinterval $I \subseteq \mathbb{R}$. Assume u is a C^2 solution of (1). Then, define

$$z(s) := u(\mathbf{x}(s)) \tag{16}$$

to be the values of u along the curve, and

$$\mathbf{p}(s) := Du(\mathbf{x}(s)) = (p^1(s), ..., p^n(s)) \tag{17}$$

to be the values of the gradient Du, where

$$p^i(s) = u_{x_i}(\mathbf{x}(s)) \quad (i = 1, ..., n). \tag{18}$$

Differentiate (18) with respect to s:

$$\frac{dp^i(s)}{ds} = \dot{p}^i(s) = \sum_{j=1}^{n} u_{x_i x_j}(\mathbf{x}(s))\dot{x}^j(s) \tag{19}$$

and differentiating the PDE (15) with respect to x_i yields:

$$\sum_{j=1}^{n} F_{p_j}(Du, u, x)u_{x_j x_i} + F_z(D_u, u, x)u_{x_i} + F_{x_i}(Du, u, x) = 0 \tag{20}$$

which can be simplified if we set

$$\dot{x}^j(s) = F_{p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) = D_pF \quad (j = 1, ..., n). \tag{21}$$

Assume (21) holds and evaluate (20) at $x = \mathbf{x}(s)$.

$$\sum_{j=1}^{n} F_{p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s))u_{x_jx_i}(\mathbf{x}(s)) + F_z(\mathbf{p}(s), z(s), \mathbf{x}(s))p^i(s) + F_{x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) = 0 \tag{22}$$

Substituting (21) into (19) gives:

$$\begin{aligned} \dot{p}^i(s) &= \sum_{j=1}^{n} u_{x_ix_j}(\mathbf{x}(s))\dot{x}^j(s) \\ &= \sum_{j=1}^{n} u_{x_ix_j}(\mathbf{x}(s))F_{p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \quad \text{by (8)} \end{aligned}$$

which finally, by substituting (22), gives

$$\dot{p}^i(s) = -F_{x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) - F_z(\mathbf{p}(s), z(s), \mathbf{x}(s))p^i(s) \quad (i = 1, ..., n). \tag{23}$$

Moreover, differentiate (16) with respect to s:

$$\dot{z}(s) = \sum_{j=1}^{n} u_{x_j}(\mathbf{x}(s))\dot{x}^j(s) = \sum_{j=1}^{n} p^j(s)F_{p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)). \tag{24}$$

We have now derived the characteristic by equations, a system of $2n+1$ first-order ODEs that allows us to solve the PDE (15). We can write them in vector notation:

$$\begin{cases} (a) \quad \dot{\mathbf{p}}(s) = -D_xF(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_zF(\mathbf{p}(s), z(s), \mathbf{x}(s)) \ \mathbf{p}(s) \\ (b) \quad \dot{z}(s) = D_pF(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) \\ (c) \quad \dot{\mathbf{x}}(s) = D_pF(\mathbf{p}(s), z(s), \mathbf{x}(s)) \end{cases} \tag{25}$$

Furthermore, $F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \equiv 0$.

Theorem 2 *Let $u \in C^2$ solve the nonlinear first-order partial differential equation (1) in U. Assume $\boldsymbol{x}(\cdot)$ solves the ODE (c) above, where $\boldsymbol{p}(\cdot). = Du(\boldsymbol{x}(\cdot))$, $z(\cdot)u(\boldsymbol{x}(\cdot))$. Then $\boldsymbol{p}(\cdot)$ solves the ODE (a) and $z(\cdot)$ solves the ODE (b), for those s such that $\boldsymbol{x}(s) \in U$.*

4.2 Compatibility conditions on boundary data

Assume that for a point $x_0 \in \Gamma$ and Γ is flat near x_0 lying in the plane $x_n = 0$. We want to find the following initial conditions to be able to solve a PDE: $\mathbf{x}(0) = x^0$, $\mathbf{p}(0) = p^0$, $z(0) = z^0$. If $\mathbf{x}(0) = x^0$ then $z^0 = g(x^0)$, and near x^0 $u(x_1, ..., x_{n-1}, 0) = g(x_1, ..., x_{n-1})$.

Differentiate so that $u_{x_i}(x^0) = g_{x_i}(x^0)$ $(i = 1, ..., n-1)$. We need for $p^0 = (p_1^0, ..., p_n^0)$ to satisfy:

$$\begin{cases} p_i^0 = g_{x_i}(x^0) \quad (i = 1, ..., n-1) \\ F(p^0, z^0, x^0) = 0. \end{cases}$$

We then have a system of n equations for the n quantities $p^0 = (p_1^0, ..., p_n^0)$, these are the *compatibility conditions*, a triple $(p^0, z^0, x^0) \in \mathbb{R}^{2n+1}$ verifying these conditions is called *admissible*. z^0 is uniquely determined by the boundary condition, while p^0 may not exist or may not be unique.

Assume that $x^0 \in \Gamma$, that Γ near x^0 lies in the plane and that the triple (p^0, z^0, x^0) is admissible, then integrating the characteristic ODE, we can construct a solution u. We will perturb (p^0, z^0, x^0) to keep the compatibility conditions.
If there is a point $y = (y_1, ..., y_{n-1}, 0) \in \Gamma$, with y close to x^0, we need to solve the characteristic ODE (25), with the initial conditions

$$\mathbf{p}(0) = \mathbf{q}(y), z(0) = g(y), \mathbf{x}(0) = y. \tag{26}$$

So we have to find a function $\mathbf{q}(\cdot) = (q^1(\cdot), ..., q^n(\cdot))$ so that $\mathbf{q}(x^0) = p^0$ and the triple $(\mathbf{q}(y), g(y), y)$ is admissible such that:

$$\begin{cases} q^i(y) = g_{x_i}(y) \quad (i = 1, ..., n-1) \\ F(\mathbf{q}(y), g(y), y) = 0 \end{cases}$$

hold $\forall y \in \Gamma$ close to x^0.

Lemma 1 (Noncharacteristic boundary conditions): There exists a unique solution $\mathbf{q}(\cdot)$ of the two previous equations for all $y \in \Gamma$ sufficiently close to x^0, provided that $F_{p_n}(p^0, z^0, x^0) \neq 0$. The admissible triple is called *noncharacteristic* if the previous equation holds.

Lemma 2 (Local invertibility). Assume we have the noncharacteristic condition $F_{p_n}(p^0, z^0, x^0) \neq 0$. Then there exist an open interval $I \subseteq \mathbb{R}$ containing 0, a neighbourhood W of x^0 in $\Gamma \subset \mathbb{R}^{n-1}$, and a neighbourhood V of x^0 in $\mathbb{R}^n$, such that for each $x \in V$ there exist unique $s \in I, y \in W$ such that $x = \mathbf{x}(y, s)$. The mappings $x \mapsto s, y$ are C^2.

Select a point $x^0 \in \Gamma$ and assume that near x^0 the surface Γ is flat, lying in the plane $\{x_n = 0\}$. Suppose that (p^0, z^0, x^0) is an admissible triple of boundary data, which is noncharacteristic. According to **Lemma 1**, there is a function $\mathbf{q}$ so that $p^0 = \mathbf{q}(x^0)$ and the triple $(\mathbf{q}(y), g(y), y)$ is admissible, for all y sufficiently close to x^0. Given any such point $y = (y_1, ..., y_{n-1}, 0)$, we solve the characteristic ODE (25) subject to the initial conditions (25). Write:

$$\begin{cases} \mathbf{p}(s) = \mathbf{p}(y, s) = \mathbf{p}(y_1, ..., y_{n-1}, s) \\ z(s) = z(y, s) = z(y_1, ..., y_{n-1}, s) \\ \mathbf{x}(s) = \mathbf{x}(y, s) = \mathbf{x}(y_1, ..., y_{n-1}, s) \end{cases} \tag{27}$$

to display the dependence of the solution on both s and y. Regard x^0 as lying in $\mathbb{R}^{n-1}$, then:

Theorem 3 *(Local Existence Theorem). The function u defined above is C^2 and solves the PDE*

$$F(Du(x), u(x), x) = 0 \quad (x \in V)$$

with the boundary condition

$$u(x) = g(x) \quad (x \in \Gamma \cap V).$$

4.3 Example

This example is taken from Section 3.2 of Evans [2].

Consider the following PDE:

$$\begin{cases} u_{x_1}u_{x_2} = u & \text{in } U = \{x_1 > 0\} \\ u = x_2^2 & \text{on } \Gamma = \{x_1 = 0\} \end{cases}$$

Then, using the notation, we have that:

$$\begin{cases} F_{p_1} = p_2 \\ F_{p_2} = p_1 \\ F_z = -1 \\ F_{x_1} = F_{x_2} = 0. \end{cases}$$

This yields the characteristic ODEs:

$$\begin{cases} \dot{x}_1 = p_2 \\ \dot{x}_2 = p_1 \\ \dot{z} = p_1F_{p_1} + p_2F_{p_2} = 2p_1p_2 \\ \dot{p}_1 = -F_{x_1} - p_1F_z = p_1 \\ \dot{p}_2 = -F_{x_2} - p_2F_z = p_2. \end{cases}$$

Integrating this:

$$\begin{cases} x_1(s) = p_2^0(e^s - 1) \\ x_2(s) = x^0 + p_1^0(e^s - 1) \\ z(s) = z^0 + 2p_1^0p_2^0(e^{2s} - 1) \\ p_1 = p_1^0e^s \\ p_2 = p_2^0e^s, \end{cases}$$

with $z^0 = (x^0)^2$, $x^0 \in \mathbb{R}$, and $s \in \mathbb{R}$.

Now using the boundary condition: $p_2^0 = u_{x_2}(0, x^0) = 2x^0$, and $p_1^0p_2^0 = z^0 = (x^0)^2 \Rightarrow p_1^0 = \frac{x^0}{2}$, so we then have:

$$\begin{cases} x_1(s) = 2x^0(e^s - 1) \\ x_2(s) = \frac{x^0}{2}(e^s + 1) \\ z(s) = (x^0)^2e^{2s} \\ p_1 = \frac{x^0}{2}e^s \\ p_2 = 2x^0e^s. \end{cases}$$

Fixing a point, selecting s and x^0 so that $(x_1, x_2) = (x_1(s), x_2(s)) = (2x^0(e^s - 1), \frac{x^0}{2}(e^s + 1)) \in U$, then we have that: $x^0 = \frac{4x_2 - x_1}{4}$ and $e^s = \frac{x_1 + 4x_2}{4x_2 - x_1}$, giving:

$$u(x) = z(s) = (x^0)^2 e^{2s} = \frac{(x_1 + 4x_2)^2}{16}.$$

References

[1] Walter A. Strauss, *Partial Differential Equations: An Introduction*, 2nd ed., Wiley, 2007.

[2] Lawrence C. Evans, *Partial Differential Equations*, 2nd ed., Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, Providence, RI, 2010.