Poisson Geometry, Lie Groups and Geometric Mechanics

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Contents

1 Introduction **1**

2 Review of Linear Algebra and Differential Geometry **2**

2.1 Linear Algebra 2

2.2 Manifolds 4

2.3 Tangent Vectors and Vector Fields 5

2.4 Differential Forms 7

3 Poisson Manifolds **11**

3.1 General Setting 11

3.2 Case of the Cotangent Bundle 11

4 Lie Groups and Lie Algebras **18**

4.1 Abstract Theory 18

4.2 Matrix Lie Groups 22

4.3 Lie Group Actions 26

5 The Lie-Poisson Bracket **30**

5.1 Cartan Frames 30

5.2 The Canonical 1-Form and Symplectic Form of Lie Groups 31

5.3 The Canonical Poisson Bracket on Lie Groups 33

5.4 Lie-Poisson Reduction and the Lie-Poisson Bracket 35

6 Rigid Motion **41**

6.1 The Lie Algebra of Rotations 41

6.2 Incorporating Translations 43

7 Acknowledgements **48**

1 Introduction

In an introductory mechanics course one learns to describe basic physical systems (e.g. particle in space) using the mathematical machinery of real calculus. Later on, in more advanced mechanics courses, other formulations of mechanics are presented; although they are equivalent, they are built upon different mathematical frameworks. This report focuses on the general mathematical framework underpinning Hamiltonian mechanics–Poisson geometry–where problems are solved by exploiting symmetries and conserved quantities of the physical system under investigation.
When solving a problem using the Hamiltonian framework, we work with the system's phase space: the space of all possible states of the system. A given point (state) in phase space is the object's position and momentum (this is all the necessary information required to describe the system's state). Given an initial state, the most common and perhaps the most important question physicists will then ask, is how the system will evolve in time. The problem is that taking time derivatives in this mathematical framework is not exactly straight forward. This is because the phase space will not necessarily be flat. Calculus in curved spaces is provided by the theory of differential geometry, which we go over in the initial part of the report, along with some linear algebra which is necessary in order to define the pivotal concepts of tensors and vector fields.

Connecting these geometric and algebraic concepts leads us to the core concept of Poisson geometry: the Poisson manifold. A Poisson manifold is a space equipped with a mathematical tool called a Poisson bracket, which dictates how any observable quantity of a physical system evolves over time–which is the initial physical motivation we mentioned.

We then turn to some algebraic concepts, specifically Lie groups and their associated Lie algebras. These structures provide the rigorous mathematical language required to describe continuous symmetries, such as the rotational invariance of a ball, or the translational freedom of a free-floating object. Understanding how symmetries operate algebraically allows us to "reduce" the complex dynamics of our system, which is the concern of the final couple of sections.

2 Review of Linear Algebra and Differential Geometry

As the title suggests, this section consists of an overview of mathematical concepts which will be used extensively throughout this text.

2.1 Linear Algebra

Definition 2.1 Let $\mathfrak{g}$ be a vector space over $\mathbb{R}$, a linear function $\mu : \mathfrak{g} \to \mathbb{R}$ is called a **linear functional** on $\mathfrak{g}$. The set $\mathfrak{g}^*$ of linear functionals on $\mathfrak{g}$ is called the **dual space** of $\mathfrak{g}$. We often denote

$$\langle \mu, g \rangle := \mu(g)$$

for all $\mu \in \mathfrak{g}^*$ and $g \in \mathfrak{g}$.

Proposition 2.2 Let $\mathfrak{g}$ be vector space over $\mathbb{R}$, then $\mathfrak{g}^*$ is itself a vector space over $\mathbb{R}$.

Proof. The axioms of vector spaces are satisfied by linearity of the functionals and the usual addition and scalar multiplication of functions.

Definition 2.3 Let $\mathfrak{g}$ be a finite dimensional vector space over $\mathbb{R}$ with basis $\{e_i\}_{i=1}^n$. Then, **the dual basis** $\{e^i\}_{i=1}^n$ of $\{e_i\}_{i=1}^n$ is the set of functionals on $\mathfrak{g}$ satisfying

$$e^i(e_j) = \delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Proposition 2.4 Let $\mathfrak{g}$ be a finite dimensional vector space over $\mathbb{R}$ with basis $\{e_i\}_{i=1}^n$. Then, the dual basis of $\{e_i\}_{i=1}^n$ forms a basis of $\mathfrak{g}^*$.

Proof. By linearity of e^i and the linear independence of e_j, taking any element f of $\mathfrak{g}^*$ and a vector $v = \sum_{i=1}^n v^i e_i \in \mathfrak{g}$ we may compute

$$f(v) = f\left(\sum_{i=1}^n v^i e_i\right) = \sum_{i=1}^n v^i f(e_i)$$

Defining $\alpha_i := f(e_i)$, we also have

$$\begin{aligned}\sum_{i=1}^{n}\alpha_i e^i(v) &= \sum_{i=1}^{n}\alpha_i e^i\left(\sum_{j=1}^{n}v^j e_j\right)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i v^j e^i(e_j)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i v^j \delta^i_j\\ \\ &= \sum_{i=1}^{n}\alpha_i v^i\\ &= \sum_{i=1}^{n} v^i f(e_i)\\ &= f(v)\end{aligned}$$

Since this holds for all v, we conclude that there is a unique set of $\alpha_i = f(e_i) \in \mathbb{R}$ such that

$$f = \sum_{i=1}^{n}\alpha_i e^i$$

Thus the dual basis $\{e^i\}_{i=1}^n$ forms a basis of $\mathfrak{g}^*$. □

Proposition 2.5 Let $\mathfrak{g}$ be a finite dimensional vector space over $\mathbb{R}$, then $(\mathfrak{g}^*)^* \cong \mathfrak{g}$.

Proof.

For any $g \in \mathfrak{g}$, the function $\tau_g : \mathfrak{g}^* \mapsto \mathbb{R}$ defined by

$$\tau_g(f) := f(g) \;\; \forall f \in \mathfrak{g}^*$$

is linear since, for any $\alpha, \beta \in \mathbb{R}$ and $f, h \in \mathfrak{g}^*$, we have

$$\begin{aligned}\tau_g(\alpha \cdot f + \beta \cdot h) &= (\alpha \cdot f + \beta \cdot h)[g]\\ &= \alpha \cdot f(g) + \beta \cdot h(g)\\ &= \alpha \cdot \tau_g(f) + \beta \cdot \tau_g(h)\end{aligned}$$

Therefore $\tau_g \in (\mathfrak{g}^*)^*$. Furthermore, if $\{e_i\}$ forms a basis of $\mathfrak{g}$ and $\{e^i\}$ its dual basis of $\mathfrak{g}^*$, then

$$\tau_{e_i}(e^j) = e^j(e_i) = \delta^j_i$$

Hence $\{\tau_{e_i}\}$ forms the dual basis of $\{e^i\}$ on $(\mathfrak{g}^*)^*$. Note that the map $g \mapsto \tau_g$ is linear since, letting $u, v \in \mathfrak{g}$,

$\alpha, \beta \in \mathbb{R}$ and $f \in \mathfrak{g}^*$, we have

$$\tau_{\alpha u+\beta v}(f) = f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) = \alpha \tau_u(f) + \beta \tau_v(f)$$

So we conclude that the map $g \mapsto \tau_g$ is a linear bijection $\mathfrak{g} \longleftrightarrow (\mathfrak{g}^*)^*$. That is $\mathfrak{g}$ and $(\mathfrak{g}^*)^*$ are isomorphic. □

Remark The strength of this isomorphism is that it defines a **canonical isomorphism** between $\mathfrak{g}$ and $(\mathfrak{g}^*)^*$ since it is independent of the choice of basis on $\mathfrak{g}$. Hence we can identify any element of $(\mathfrak{g}^*)^*$ with one in $\mathfrak{g}$ while keeping the structure of operations. This is particularly useful in reducing the number of spaces we work over.

2.2 Manifolds

With this in hand, we can introduce concepts of differential geometry. The main object studied in this field is that of a smooth manifold. We build up to this with the following couple of definitions.

Definition 2.6 Let $\mathcal{P}$ be a Hausdorff, second-countable topological space. A **chart** on $\mathcal{P}$ is a pair (U, φ), where U is an open subset of $\mathcal{P}$ and $\varphi : U \to \hat{U} \subseteq \mathbb{R}^n$ is a homeomorphism.

Definition 2.7 Two charts $(U, \varphi), (V, \psi)$ on $\mathcal{P}$ are said to be **smoothly compatible** if either $U \cap V = \varnothing$, or if the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

Definition 2.8 An **atlas** A on a topological manifold $\mathcal{P}$ is a collection of smoothly compatible charts whose domains cover $\mathcal{P}$. Furthermore, we call the atlas **maximal** if any chart that is smoothly compatible with every chart in A, is in A.

Definition 2.9 A **smooth manifold** is a pair $(\mathcal{P}, A)$, where $\mathcal{P}$ is a Hausdorff, second-countable topological space and A is a maximal smooth atlas on $\mathcal{P}$.

Remark Since we will only be concerned with smooth manifolds in what follows, from here on, when we write "manifold", we implicitly mean "smooth manifold".

Remark We note that any vector space $\mathfrak{g}$ over $\mathbb{R}$ of dimension n is an n-manifold as it admits a coordinate chart from its coordinate function. That is, if $\{g_i\}$ forms a basis of $\mathfrak{g}$, then for any $v \in \mathfrak{g}$, there exists unique $\alpha_i \in \mathbb{R}$ such that $v = \sum_{i=1}^n \alpha_i g_i$ and so the map $\varphi : \mathfrak{g} \to \mathbb{R}^n$ that acts as $\varphi\left(\sum_{i=1}^n \alpha_i g_i\right) = \sum_{i=1}^n \alpha_i e_i$ is bijective and smooth and thus defines a chart. Since this maps works for $U = \mathfrak{g}$, we say that φ is a **global** chart.

Definition 2.10 Let $\mathcal{P}$ and $\mathcal{Q}$ be smooth manifolds of dimension m and n respectively. A mapping $f : \mathcal{P} \to \mathcal{Q}$ is said to be **smooth** (or $\boldsymbol{C^\infty}$) if for every $p \in \mathcal{P}$ there exists a chart (U, φ) on $\mathcal{P}$ containing p, and another chart (V, ψ) on $\mathcal{Q}$ containing $f(p)$, such that $f(U) \subseteq V$, and the coordinate representation of

f, namely

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^m$$

is smooth in the sense of real calculus.

2.3 Tangent Vectors and Vector Fields

Definition 2.11 Let $\mathcal{P}$ be a manifold. A **smooth curve** γ on $\mathcal{P}$ is a C^∞ map $I \to \mathcal{P}$ where $I \subset \mathbb{R}$ is an interval.

Definition 2.12 Let $\mathcal{P}$ be a n-manifold and $p \in \mathcal{P}$. A **tangent vector** v to $\mathcal{P}$ at p is an equivalence class of smooth curves on $\mathcal{P}$ defined in the following way. Given a local chart $\varphi : U \subset \mathcal{P} \to \mathbb{R}^n$ with $p \in U$. If γ_1, γ_2 are smooth curves on $\mathcal{P}$ satisfying $\gamma_i(0) = p$. We define the equivalence relation

$$\gamma_1 \sim_p \gamma_2 \iff (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0) \text{ in } \mathbb{R}^n$$

Then, a tangent vector v is an equivalence class with chosen representative curve γ_v

$$v = \{\gamma : [-\epsilon, \epsilon] \to \mathcal{P} \mid \epsilon > 0,\ \gamma(0) = p,\ \gamma \sim_p \gamma_v\}$$

Definition 2.13 Let $\mathcal{P}$ be a n-manifold and $p \in \mathcal{P}$. The **tangent space** to $\mathcal{P}$ at p denoted $T_p\mathcal{P}$ is defined as the set of tangent vectors to $\mathcal{P}$ at p.

Proposition 2.14 Let $\mathcal{P}$ be a n-manifold and $p \in \mathcal{P}$. Then $T_p\mathcal{P}$ forms an n-dimensional vector space with scalar multiplication and vector addition defined as follows. Let u and v be tangent vectors at p with representative curves γ_u, γ_v respectively, let φ be a local chart and let $\alpha \in \mathbb{R}$. Then, if

$$(\varphi \circ \gamma_u)'(0) = a, \quad (\varphi \circ \gamma_v) = b, \ \text{ for some } a, b \in \mathbb{R}^n$$

We define $u + v$ as the equivalence class of curves γ_{u+v} such that

$$(\varphi \circ \gamma_{u+v})'(0) = a + b \in \mathbb{R}^n$$

Similarly, $\alpha \cdot u$ is the equivalence class of curves $\gamma_{\alpha \cdot u}$ such that

$$(\varphi \circ \gamma_{\alpha \cdot u})'(0) = \alpha \cdot a \in \mathbb{R}^n$$

Proof. The proof can be found in [1] section 3.

Definition 2.15 Let $\mathcal{P}$ be a manifold and $p \in \mathcal{P}$. A **derivation** on $\mathcal{P}$ at p is a function $D_p : \mathcal{F}(\mathcal{P}) \to \mathbb{R}$

which satisfies

$$D_p(fg) = f(p)D_p(g) + D_p(f)g(p) \;\; \forall f, g \in \mathcal{F}(\mathcal{P})$$

We denote the set of derivations on $\mathcal{P}$ at p by $\mathrm{Der}_p(\mathcal{F}(\mathcal{P}))$.

The proof of the following theorem can be found in [1] section 3.

Theorem 2.16 Let $\mathcal{P}$ be a manifold and $p \in \mathcal{P}$. There is an isomorphism between $T_p\mathcal{P}$ and $\mathrm{Der}_p(\mathcal{F}(\mathcal{P}))$. In local coordinates $(x_1, \ldots, x_n)$, this simplifies to

$$\mathrm{Der}_p(\mathcal{F}(\mathcal{P})) = \mathbb{R}\,\mathrm{Span}\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p \right\}$$

Remark By this isomorphism, we may now write any tangent vector simply as $v = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}$ for some unique set of $\alpha_i \in \mathbb{R}$. We will assume implicitly that the derivatives are evaluated at the base point p on the manifold.

Definition 2.17 Let $\mathcal{P}$ be a n-manifold, its **tangent bundle** $T\mathcal{P}$ is defined as

$$T\mathcal{P} := \{(p, v) \mid p \in \mathcal{P},\ v \in T_p\mathcal{P}\} \tag{1}$$

in other words, it is the set of all tangent vectors with respect to any basepoint p on the manifold.

Remark It follows from the properties of $\mathcal{P}$ and the fact that $T_p\mathcal{P}$ is a n dimensional vector space that $T\mathcal{P}$ is a $2n$-manifold. For the explicit construction, see proposition 3.18 of [1].

Proposition 2.18 Let $\mathcal{P}$ and $\mathcal{Q}$ be manifolds of dimensions n and m respectively. Then $\mathcal{P} \times \mathcal{Q}$ is an $(n+m)$-manifold. Furthermore, let $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, then we have

$$T_{(p,q)}(\mathcal{P} \times \mathcal{Q}) = T_p\mathcal{P} \times T_q\mathcal{Q}$$

Proof. First, since $\mathcal{P}$ and $\mathcal{Q}$ are manifold, they have charts $\varphi_\mathcal{P} : p \to (p_1, \ldots, p_n) \in \mathbb{R}^n$ and $\varphi_\mathcal{Q} : q \to (q_1, \ldots, q_m) \in \mathbb{R}^m$. It follows that $\mathcal{P} \times \mathcal{Q}$ has a chart

$$\varphi : (p, q) \mapsto (\varphi_\mathcal{P}(p), \varphi_\mathcal{Q}(q)) = (p_1, \ldots, p_n, q_1, \ldots, q_m) \in \mathbb{R}^{m+n}$$

The other manifold conditions are also inherited from those of $\mathcal{P}$ and $\mathcal{Q}$.

Then, let $\gamma(t) = (p(t), q(t))$ be a curve in $\mathcal{P} \times \mathcal{Q}$ such that $\gamma(0) = (p, q)$. Then, $\gamma'(0) = (p'(0), q'(0)) \in T_{(p,q)}(\mathcal{P} \times \mathcal{Q})$. If we look at the two components, we notice that $p'(0) \in T_p\mathcal{P}$ and $q'(0) \in T_q\mathcal{Q}$ and thus $\gamma'(0) \in T_p\mathcal{P} \times T_q\mathcal{Q}$ The reverse inclusion uses the exact same argument. □

We may now generalize the concept of a vector field from vector calculus to manifolds.

Definition 2.19 A **vector field** X on a manifold $\mathcal{P}$ is a function $X : \mathcal{P} \to T\mathcal{P}$ that acts as $X(p) = (p, v(p))$ for any $p \in \mathcal{P}$. The set of vector fields on $\mathcal{P}$ is denoted as $\mathfrak{X}(\mathcal{P})$.

Remark We say that X is smooth vector field if $X : \mathcal{P} \to T\mathcal{P}$ is smooth in the sense of definition 2.10.

Definition 2.20 Let $\mathcal{P}$ be a manifold, γ be a smooth curve defined on an an interval $I \subset \mathbb{R}$ and X be a smooth vector field, both on $\mathcal{P}$. We say that γ is an **integral curve** of X if the following holds

$$\gamma'(t) = X \circ \gamma(t) \; \forall t \in I$$

The following statement is a generalization of the *theorem of existence and uniqueness* of ordinary differential equations in $\mathbb{R}^n$. It turns out that since it holds in local coordinates, it generalizes to manifolds. The proof can be found in [2] section 4.3.

Theorem 2.21 Let X be a smooth vector field on a manifold $\mathcal{P}$ and $p \in \mathcal{P}$. Then there exists an interval I containing 0 and a unique integral curve γ of X such that

$$\begin{cases} \gamma(0) = p \\ \gamma'(t) = X \circ \gamma(t) \; \forall t \in I \end{cases}$$

Remark We note that if we view the manifold $\mathcal{P}$ as embedded in Euclidean space then an solution to the above equations will remain inside of $\mathcal{P}$ for all $t \in I$

2.4 Differential Forms

Definition 2.22 Let $(x_1, \ldots, x_n)$ be local coordinates on $\mathcal{P}$ with $p \in \mathcal{P}$, we define $(dx_1, \ldots, dx_n)$ to be the dual basis of $T_p\mathcal{P}$. That is, for any $i, j \in \{1, \ldots, n\}$, we have

$$dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_i^j$$

The dual space of $T_p\mathcal{P}$ is thus $T_p^*\mathcal{P} := (T_p\mathcal{P})^* = \text{Span}\{dx_1, \ldots dx_n\}$ and is called the **cotangent space** of $\mathcal{P}$ at p.

Definition 2.23 Similar to definition 2.17, given a n-manifold $\mathcal{P}$, its **cotangent bundle** is defined as

$$T^*\mathcal{P} := \left\{ (p, \theta) \mid p \in \mathcal{P}, \; \theta \in T_p^*\mathcal{P} \right\}$$

Again, $T^*\mathcal{P}$ forms a $2n$-manifold.

Definition 2.24 Le $\mathcal{P}$ be a n-manifold and let $k = 0, 1, \ldots, n$. A **k-form** on $\mathcal{P}$ is an assignment $\alpha : p \mapsto \alpha_p$ to each point $p \in \mathcal{P}$ such that α_p is a k-multilinear skew symmetric map $(T_p\mathcal{P})^k \to \mathbb{R}$. We denote the set of k-forms on $\mathcal{P}$ by $\Omega^k(\mathcal{P})$.

Remark From the above definition, a 0-form is simply a function $f \in \mathcal{F}(\mathcal{P})$.

Definition 2.25 Let $\mathcal{P}$ be a manifold and $f \in \mathcal{F}(\mathcal{P})$. The **exterior derivative** df of f is the 1-form defined by

$$df_p(v) = v[f]$$

For any $p \in \mathcal{P}$ and $v \in T_p\mathcal{P}$ where v is viewed as a derivation.

Proposition 2.26 Let $\mathcal{P}$ be a manifold with local coordinates $(x_1, \ldots, x_n)$ and $f \in \mathcal{F}(\mathcal{P})$. The exterior derivative of f is locally described by

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot dx_i$$

Proof. Since $(x_1, \ldots, x_n)$ are local coordinates, $T_p\mathcal{P}$ has a basis $\left\{\frac{\partial}{\partial x_i}\right\}$ And from the proof of proposition 2.4, we know that, since $\{dx_i\}$ forms its dual basis, we have

$$df_p = \sum_{i=1}^{n} df_p\left(\frac{\partial}{\partial x_i}\right) dx_i = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[f] dx_i = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot dx_i$$

Which is the desired result. □

Definition 2.27 Let $\mathcal{P}$ be a manifold, X be a vector on $\mathcal{P}$ and $f \in \mathcal{F}(\mathcal{P})$. The **Lie derivative** of f along X is another function on $\mathcal{P}$ defined by

$$\mathcal{L}_X f := df(X)$$

which in local coordinates $\{x^i\}_{i=1}^n$ with $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$ gives

$$\mathcal{L}_X f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \left(\sum_{j=1}^{n} X^j \frac{\partial}{\partial x_j}\right) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \sum_{j=1}^{n} X^j \delta_i^j = \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x^i} = X[f]$$

Definition 2.28 Let $\mathcal{P}$ be a manifold and $X, Y \in \mathfrak{X}(\mathcal{P})$. The **Lie derivative** of Y along X gives another vector field $\mathcal{L}_X Y$ on $\mathcal{P}$ defined by its action on functions $f \in \mathcal{F}(\mathcal{P})$

$$\mathcal{L}_X Y(f) := X[Y[f]] - Y[X[f]]$$

Where, as usual, we consider X, Y as derivations. We sometimes also denote it in terms of the **Jacobi-Lie bracket** of X and Y that is

$$\mathcal{L}_X Y = [X, Y]$$

In local coordinates $\{x^i\}_{i=1}^n$ we obtain

$$\mathcal{L}_X Y(f) = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right) \frac{\partial}{\partial x^j} = (X \cdot \nabla)Y - (Y \cdot \nabla)X$$

Remark A straightforward computation can show that the Jacobi-Lie bracket obeys the Jacobi identity.

Proposition 2.29 Let $\mathcal{P}$ be a manifold, $X, Y \in \mathfrak{X}(\mathcal{P})$ and $h \in \mathcal{F}(\mathcal{P})$. Then we have

$$[\mathcal{L}_X\,, \mathcal{L}_Y]\,(h) = \mathcal{L}_{[X\,,Y]}(h)$$

Proof. We compute

$$\begin{aligned}[\mathcal{L}_X\,, \mathcal{L}_Y]\,(h) :=& \mathcal{L}_X(\mathcal{L}_Y(h)) - \mathcal{L}_Y(\mathcal{L}_X(h)) \\ =& X(Y(h)) - Y(X(h)) \\ =& [X\,, Y]\,(h) \\ =& \mathcal{L}_{[X\,,Y]}(h)\end{aligned}$$

□

Definition 2.30 Let $\mathcal{P}$ be a manifold and θ be a 1-form on $\mathcal{P}$. Its exterior derivative $d\theta$ is then the 2-form defined by

$$d\theta(X, Y) := X[\theta(Y)] - Y[\theta(X)] - \theta([X, Y]) \;\; \forall X, Y \in \mathfrak{X}(\mathcal{P})$$

Definition 2.31 Let $\mathcal{P}$ be a manifold and $\alpha, \beta \in \Omega^1(\mathcal{P})$. We define the **wedge product** $\wedge : \Omega^1(\mathcal{P}) \times \Omega^1(\mathcal{P}) \to \Omega^2(\mathcal{P})$ by its action

$$(\alpha \wedge \beta)(u, v) := \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

It is associative, bilinear and antisymmetric.

Remark This definition can be extended to product of k-forms with l-forms, but we will only use this simpler case. We also highlight the fact that by antisymmetry, for any 1-form α, $\alpha \wedge \alpha = 0$

Definition 2.32 Let $\mathcal{P}$ be a manifold, $X \in \mathfrak{X}(\mathcal{P})$ and $\alpha \in \Omega^k(\mathcal{P})$. Then, we define the interior product $\iota_X\alpha \in \Omega^{k-1}$ of X and α by

$$(\iota_X\alpha)_m(v_2, \ldots, v_k) := \alpha_m(X(m), v_2, \ldots, v_k) \;\; \forall m \in \mathcal{P},\, v_i \in T_m\mathcal{P}$$

Definition 2.33 Let $\mathcal{P}$ be a manifold, and let $\omega \in \Omega^2(\mathcal{P})$. We say that ω is non-degenerate if the following holds for any $X \in \mathfrak{X}(\mathcal{P})$

$$\iota_X\omega = 0 \implies X = 0$$

The proofs for the following propositions can be found in [2] (ch.4).

We now give a local formula for the exterior derivative of a 1-form.

Proposition 2.34 Let $\mathcal{P}$ be a manifold and $\alpha \in \Omega^1(\mathcal{P})$. If, locally, $\mathcal{P}$ admits local coordinates (x^i) and

there exists $\alpha_i \in \mathcal{F}(\mathcal{P})$ such that for $m \in \mathcal{P}$,

$$\alpha_m = \sum_{i=1}^{n} \alpha^i(m) dx^i$$

Then, we have a local formula

$$d\alpha_m = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial \alpha^i}{\partial x^j} dx^j \wedge dx^i$$

Proposition 2.35 Let $\mathcal{P}$ be a manifold and $\alpha \in \Omega^{k+1}(\mathcal{P})$. If there exists $\beta \in \Omega^k(\mathcal{P})$ such that $\alpha = d\beta$, then $d\alpha = 0$. Furthermore, we say that α is **exact**.

Proposition 2.36 Let $\mathcal{P}$ be a manifold and$X \in \mathfrak{X}(\mathcal{P})$. Then, for any $\alpha \in \Omega^k(\mathcal{P})$, Cartan's formula holds that is

$$\mathcal{L}_X \alpha = d(\iota_X \alpha) + \iota_X (d\alpha)$$

Proposition 2.37 Let $\mathcal{P}$ be a manifold, $X, Y \in \mathfrak{X}(\mathcal{P})$ and $\alpha \in \Omega^k(\mathcal{P})$. Then, the following identity holds

$$\iota_{[X,Y]} = \mathcal{L}_X(\iota_Y \alpha) - \iota_Y(\mathcal{L}_X \alpha)$$

Proposition 2.38 Let $\mathcal{P}$ be a manifold, $X \in \mathfrak{X}(\mathcal{P})$ and $f \in \mathcal{F}(\mathcal{P})$. Then the exterior derivative and the Lie derivative along X commute. That is

$$d(\mathcal{L}_X f) = \mathcal{L}_X(df)$$

3 Poisson Manifolds

3.1 General Setting

The main object studied in Poisson geometry is the Poisson Manifold which is defined as follows.

Definition 3.1 Let $\mathcal{P}$ be an n-manifold and $\mathcal{F}(\mathcal{P}) = C^\infty(\mathcal{P})$ be the (vector) space of infinitely differentiable functions $F : \mathcal{P} \to \mathbb{R}$. A map $\{\cdot, \cdot\} : \mathcal{F}(\mathcal{P}) \times \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{P})$ satisfying the following properties (**PB1-PB4**) is called a **Poisson bracket** on $\mathcal{P}$:

$$\begin{array}{ll} \text{Bilinearity} & \mathbf{PB1} : \{F, G\} \text{ is linear in F and G} \\ \text{Antisymmetry} & \mathbf{PB2} : \{F, G\} = -\{G, F\} \\ \text{Jacobi identity} & \mathbf{PB3} : \{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \\ \text{Leibniz rule} & \mathbf{PB4} : \{FG, H\} = F\{G, H\} + \{F, H\} G \end{array} \tag{2}$$

and the pair $(\mathcal{P}, \{., .\})$ is called a **Poisson manifold**.

Definition 3.2 Let $(\mathcal{P}, \{., .\})$ be a Poisson manifold and $H : \mathcal{P} \to \mathbb{R}$ a function (called the Hamiltonian). We call the **Hamiltonian vector** field associated with H, denoted by X_H, the vector field given by

$$\langle dF, X_H \rangle = dF \cdot X_H = \{F, H\}$$

for any $F \in C^\infty(\mathcal{P})$.

An important concept is that of a flow of a vector field, and particularly one of a Hamiltonian vector field.

Definition 3.3 The flow of a vector field, in particular a Hamiltonian vector field X_H, is the map $\phi_t : \mathcal{P} \to \mathcal{P}$ such that for all $p \in \mathcal{P}$

$$\frac{d}{dt}\phi_t(p) = X_H(\phi_t(p))$$

3.2 Case of the Cotangent Bundle

When considering the motion of a particle in space, we often describe its position using coordinates $(q_1, \ldots, q_n)$. The set of all accessible positions Q forms a manifold which is called the **configuration space** of the system. While a free particle in space will have $Q = \mathbb{R}^3$ which is a vector space, if constraints are enforced it may not be the case. For example, a pendulum consists of a particle attached by a rigid rod to a fixed point and thus has coordinates lying on the sphere, so $Q = \mathbb{S}^2$ which is a manifold but not a vector space, which justifies the use of differential geometry.

While the position of the particle partially describes its dynamics, its momentum $(p_1, \ldots, p_n)$ is also needed for a full analysis. Momentum is naturally defined by assigning a numerical value to each velocity vector. Since the particle always remains in Q, its velocity v at a point $q \in Q$ must be tangent to Q and so

$v \in T_qQ$. Momentum is viewed as a way to assign a numerical value to any velocity and so so p_i must be a functional on T_qQ that is $p_i \in T_q^*Q$. From this heuristic, it is natural to consider the cotangent bundle T^*Q of Q as the geometric space of the particle. In physics, we call T^*Q, which is the set of achievable (q_i, p_i), the **phase space** of the particle.

The rest of this section concerns itself in prescribing a Poisson bracket on any cotangent bundle.

Definition 3.4 Let Q be an n-manifold and $(q,p) \in T^*Q$. Define the **projection** $\pi : T^*Q \to Q$ by

$$\pi(q,p) := q \ \ \forall (q,p) \in T^*Q$$

We then define the **canonical 1-form** θ on T^*Q by

$$\theta_{(q,p)}(v) := \langle p,\, d\pi_{(q,p)}(v) \rangle$$

Where $d\pi_{(q,p)} : T_{(q,p)}(T^*Q) \to T_qQ$ is the differential of π at $(q,p) \in T^*Q$. which acts as

$$d\pi_{(q,p)}(v_q, v_p) = v_q, \ \text{ where } (v_q, v_p) \in T_{(q,p)}(T^*Q),\ v_q \in T_qQ, \text{ and } v_p \in T_q^*Q$$

In local coordinates (q_i, p_i), taking $v = \sum_{i=1}^n (v_q^i \frac{\partial}{\partial q^i} + v_p^i \frac{\partial}{\partial p^i}) \in T_{(q,p)}(T^*Q)$ and $(q,p) = (q_1, \ldots, q_n, p_1, \ldots, p_n) \in T^*Q$ we obtain

$$\begin{aligned}
\theta_{(q,p)}(v) &= \left\langle p,\, d\pi_{(q,p)} \left(\sum_{i=1}^n \left(v_q^i \frac{\partial}{\partial q^i} + v_p^i \frac{\partial}{\partial p^i} \right) \right) \right\rangle \\
(p \in T_q^*Q) \ &= \left\langle \sum_{j=1}^n p_j dq_j,\, \sum_{i=1}^n v_q^i \frac{\partial}{\partial q^i} \right\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n p_j v_q^i \left\langle dq_j,\, \frac{\partial}{\partial q^i} \right\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n p_j v_q^i \delta_j^i \\
&= \sum_{i=1}^n p_i v_q^i
\end{aligned}$$

We compare with the action of the 1-form $\tilde{\theta}$ locally defined by $\tilde{\theta} = \sum_{i=1}^n p_i dq_i$ and obtain

$$\tilde{\theta}(v) = \sum_{i=1}^n p_i dq_i \left(\sum_{j=1}^n \left(v_q^j \frac{\partial}{\partial q^j} + v_p^j \frac{\partial}{\partial p^j} \right) \right) = \sum_{i=1}^n p_i v_q^i$$

And since they have the same action, locally we have $\theta = \tilde{\theta} = \sum_{i=1}^n p_i dq_i$

Definition 3.5 Given a n-manifold Q, the **canonical symplectic form** on T^*Q is the 2-form ω defined

by

$$\omega := -d\theta$$

where θ is the canonical 1-form. From definition 3.4 and proposition 2.34, we obtain the local formula

$$\omega = -d\theta = -d\sum_{i=1}^{n} p_i dq^i = -\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial p_i}{\partial q^j} dq^j \wedge dq^i + \frac{\partial p_i}{\partial p_j} dp_j \wedge dq^i\right)$$

Since the coordinates are independent, we get $\frac{\partial p_i}{\partial q^j} = 0$ and $\frac{\partial p_i}{\partial p_j} = \delta_i^j$ and so

$$\omega = -\sum_{i=1}^{n} dp_i \wedge dq^i = \sum_{i=1}^{n} dq^i \wedge dp_i$$

Proposition 3.6 Let Q be an n-manifold, then the canonical 2-form ω on T^*Q is non degenerate.

Remark A manifold $\mathcal{P}$ paired with a non degenerate 2-form ω on $\mathcal{P}$ is called a **symplectic manifold** if ω is closed, that is $d\omega = 0$. It is the object studied in symplectic geometry which is a field of mathematics related to Poisson geometry since any symplectic manifold is a Poisson manifold as we will soon uncover.

Proof. Suppose $X \in \mathfrak{X}(T^*Q)$ satisfying $\iota_X\omega = 0$. Locally, we may write

$$X = \sum_{i=1}^{n} A^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial p_i}$$

Evaluating at a point $m \in T^*Q$ then gives for any $v \in T_m(T^*Q)$

$$\begin{aligned}
(\iota_X\omega)_m(v) &= \omega_m(X(m), v) \\
&= \sum_{i=1}^{n} dq^i \wedge dp_i(X(m), v) \\
&= \sum_{i=1}^{n} dq^i(X(m))dp_i(v) - dq^i(v)dp_i(X(m)) \\
&= \sum_{i=1}^{n} A^i dp_i(v) - B^i dq^i(v)
\end{aligned}$$

By assumption, we must have $(\iota_X\omega)_m(v) = 0 \quad \forall v \in T_m(T^*Q)$ and since $\{dq^i, dp_i\}$ forms a basis for $T^*_m(T^*Q)$, the unique coefficients for this to hold are $A^i = B^i = 0$ which gives $X(m) = 0$. Since this must hold at every point in m, we conclude that $X = 0 \in \mathfrak{X}(T^*Q)$ and thus, by definition 2.33, ω is non-degenerate. □

Corollary 3.7 Let Q be a n-manifold, then, for any $f \in \mathcal{F}(T^*Q)$, there exists a unique vector field $\tilde{X}_f$ over T^*Q such that

$$\iota_{\tilde{X}_f}\omega = df$$

We call $\tilde{X}_f$ the **symplectic vector field** of f.

Remark The symplectic vector field of f is in fact the Hamiltonian vector field of f with respect to the canonical poisson bracket to be defined in theorem 3.8.

Proof. Consider the map $\omega^\flat : \mathfrak{X}(T^*Q) \to \Omega^1(T^*Q)$ such that at any point $m \in T^*Q$ and $X \in \mathfrak{X}(T^*Q)$ we have

$$\omega^\flat_m(X) := (\iota_X \omega)_m$$

By proposition 3.6, we know that $\ker(\omega^\flat) = \{0\}$ and so $\omega^\flat$ is injective. This implies that at any point $\omega^\flat_m : T_m(T^*Q) \to T^*_m(T^*Q)$ is an injection and, since both vector spaces have a dimension of $2n$, it is an isomorphism. Thus at each point $\omega^\flat_m$ admits an inverse denoted $\omega^\sharp_m$

Thus, given a function $f \in \mathcal{F}(T^*Q)$, at any point m, we may define

$$\tilde{X}_f(m) := \omega^\sharp_m(df_m)$$

Which concludes existence of $\tilde{X}$. Furthermore, suppose that $\tilde{X}_f$ and $\tilde{Y}_f$ are two solutions, then

$$\iota_{\tilde{X}_f}\omega - \iota_{\tilde{Y}_f}\omega = \omega(\tilde{X}_f, .) - \omega(\tilde{Y}_f, .) = \omega(\tilde{X}_f - \tilde{Y}_f, .) = \iota_{\tilde{X}_f - \tilde{Y}_f}\omega$$

However, if both are solutions, then

$$\iota_{\tilde{X}_f}\omega - \iota_{\tilde{Y}_f}\omega = df - df = 0$$

And by non degeneracy, $\tilde{X}_f - \tilde{Y}_f = 0$ Which proves uniqueness. □

Theorem 3.8 Let Q be a n-manifold. Then, the function $\{.,.\} : \mathcal{F}(T^*Q) \times \mathcal{F}(T^*Q) \to \mathcal{F}(T^*Q)$ defined by

$$\{f\,, g\} := \omega(\tilde{X}_f, \tilde{X}_g) \;\; \forall f, g \in \mathcal{F}(T^*Q)$$

defines a Poisson bracket on T^*Q and is called the **canonical Poisson Bracket** on the cotangent bundle of Q.

Proof. We consider $f, g, h \in \mathcal{F}(T^*Q)$. First, by skew symmetry of 2-forms, we get

$$\{f\,, g\} = \omega\left(\tilde{X}_f, \tilde{X}_g\right) = -\omega\left(\tilde{X}_g, \tilde{X}_f\right) = -\{g\,, f\}$$

so antisymmetry is satisfied. Similarly, bilinearity is satisfied by the properties of differential forms and of the isomorphism $\omega^\flat$.

We now consider $\tilde{X}_{fg}$. By definition, it satisfies

$$\tilde{X}_{fg} = \omega^\sharp(d(fg)) = \omega^\sharp(df \cdot g + f \cdot dg) = g\omega^\sharp(df) + f\omega^\sharp(dg) = g\tilde{X}_f + f\tilde{X}_g$$

Where $\omega^\sharp$ is the function defined in the proof of corollary 3.7. And thus

$$\{fg\,,h\} = \omega\left(\tilde{X}_{fg},\,\tilde{X}_h\right) = \omega\left(g\tilde{X}_f + f\tilde{X}_g,\,\tilde{X}_h\right) = g\cdot\omega\left(\tilde{X}_f,\,\tilde{X}_h\right) + f\cdot\omega\left(\tilde{X}_g,\,\tilde{X}_h\right) = f\{g\,,h\} + g\{f\,,h\}$$

Which shows that the Leibniz rule is satisfied.

We now proceed with a few computations first considering

$$\mathcal{L}_{\tilde{X}_f}\omega = d(\iota_{\tilde{X}_f}\omega) + \iota_{\tilde{X}_f}(d\omega) = d(df) + \iota_{\tilde{X}_f}d(-d\theta) = 0$$

Where we used proposition 2.36, the definition of $\tilde{X}_f$ and the fact that df and $d\theta$ are exact. Then, applying proposition 2.37, we get

$$\iota_{[\tilde{X}_f\,,\tilde{X}_g]}\omega = \mathcal{L}_{\tilde{X}_f}\left(\iota_{\tilde{X}_g}\omega\right) - \iota_{\tilde{X}_g}\left(\mathcal{L}_{\tilde{X}_f}\omega\right)$$

By the previous computation, the last term vanishes and using our definition for $\tilde{X}_g$, we get

$$\iota_{[\tilde{X}_f\,,\tilde{X}_g]}\omega = \mathcal{L}_{\tilde{X}_f}(dg) = d(\mathcal{L}_{\tilde{X}_f}g) = d(dg(\tilde{X}_f)) = d(\iota_{\tilde{X}_g}\omega(\tilde{X}_f)) = d(\omega(\tilde{X}_g,\,\tilde{X}_f)) = d(-\{f\,,g\})$$

And so by the definition of $\tilde{X}$, we obtain that

$$\left[\tilde{X}_f\,,\tilde{X}_g\right] = \tilde{X}_{-\{f\,,g\}}$$

Now, we show a useful property

$$\mathcal{L}_{\tilde{X}_f}(h) = \tilde{X}_f(h) = df(\tilde{X}_f) = \iota_{\tilde{X}_h}\omega(\tilde{X}_f) = \omega(\tilde{X}_h,\tilde{X}_f) = \{h\,,f\}$$

Therefore, if we apply the identity from proposition 2.29, we get

$$\begin{aligned}
&\left[\mathcal{L}_{\tilde{X}_f}\,,\mathcal{L}_{\tilde{X}_g}\right](h) = \mathcal{L}_{[\tilde{X}_f\,,\tilde{X}_g]}(h)\\
\iff &\mathcal{L}_{\tilde{X}_f}\left(\mathcal{L}_{\tilde{X}_g}(h)\right) - \mathcal{L}_{\tilde{X}_g}\left(\mathcal{L}_{\tilde{X}_f}(h)\right) = \mathcal{L}_{\tilde{X}_{-\{f\,,g\}}}(h)\\
\iff &\mathcal{L}_{\tilde{X}_f}\left(\{h\,,g\}\right) - \mathcal{L}_{\tilde{X}_g}\left(\{h\,,f\}\right) = -\{h\,,\{f\,,g\}\}\\
\iff &\{\{h\,,g\}\,,f\} - \{\{h\,,f\}\,,g\} = -\{h\,,\{f\,,g\}\}\\
\iff &\{f\,,\{g\,,h\}\} + \{g\,,\{h\,,f\}\} + \{h\,,\{f\,,g\}\} = 0
\end{aligned}$$

Thus Jacobi's identity is satisfied which concludes the proof that it does indeed consist of a Poisson bracket for any cotangent bundle. □

We end this section with a local computation of the canonical Poisson bracket.

Example 3.9 Let Q be a n-manifold, (q^i, p_i) be local coordinates on T^*Q and $f, g \in \mathcal{F}(T^*Q)$. If we let

$\tilde{X}_f = \sum_{i=1}^n A^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial p_i}$, we compute

$$\begin{aligned}
\iota_{\tilde{X}_f}\omega &= \omega(\tilde{X}_f, .) \\
&= \sum_{i=1}^n dq^i \wedge dp_i \left(\sum_{j=1}^n A^j \frac{\partial}{\partial q^j} + B^j \frac{\partial}{\partial p_j} \right) \\
&= \sum_{i=1}^n dq^i \left(\sum_{j=1}^n A^j \frac{\partial}{\partial q^j} + B^j \frac{\partial}{\partial p_j} \right) dp_i - dp_i \left(\sum_{j=1}^n A^j \frac{\partial}{\partial q^j} + B^j \frac{\partial}{\partial p_j} \right) dq^i \\
&= \sum_{i=1}^n -B^i dq^i + A^i dp_i
\end{aligned}$$

Enforcing that it satisfies the definition in corollary 3.7, we obtain

$$\iota_{\tilde{X}_f}\omega = df = \sum_{i=1}^n \frac{\partial f}{\partial q^i} dq^i + \frac{\partial f}{\partial p_i} dp_i$$

Matching components, we get

$$\tilde{X}_f = \sum_{i=1}^n \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}$$

And so we finally get a local expression for the canonical Poisson bracket

$$\begin{aligned}
\{f\,, g\} &= \omega\left(\tilde{X}_f, \tilde{X}_g\right) \\
&= \sum_{i=1}^n dq^i \wedge dp_i \left(\sum_{j=1}^n \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial f}{\partial q^j}\frac{\partial}{\partial p_j}, \sum_{k=1}^n \frac{\partial g}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial g}{\partial q^k}\frac{\partial}{\partial p_k} \right) \\
&= \sum_{i=1}^n dq^i \left(\sum_{j=1}^n \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial f}{\partial q^j}\frac{\partial}{\partial p_j} \right) dp_i \left(\sum_{k=1}^n \frac{\partial g}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial g}{\partial q^k}\frac{\partial}{\partial p_k} \right) \\
&\quad - \sum_{i=1}^n dp_i \left(\sum_{j=1}^n \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial f}{\partial q^j}\frac{\partial}{\partial p_j} \right) dq^i \left(\sum_{k=1}^n \frac{\partial g}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial g}{\partial q^k}\frac{\partial}{\partial p_k} \right) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}
\end{aligned}$$

Remark This last expression for the canonical Poisson bracket is often the definition of the Poisson bracket given in a first course in Hamiltonian mechanics. It is particularly useful since, as stated earlier, any physical system can be described using a cotangent bundle and computations are far simpler with this computation than with the canonical symplectic form.

Remark Consider a physical system with phase space T^*Q and total energy $H \in C^\infty(T^*Q)$. Recall that this defines the (Hamiltonian) vector field X_H with integral curves γ satisfying

$$\gamma'(t) = X_H \circ \gamma(t)$$

Writing this out in local coordinates we get

$$(\dot{q}_1(t), ..., \dot{q}_n(t), \dot{p}_1(t), ..., \dot{p}_n(t)) = \Big(\sum_{i=1}^{n} \frac{\partial H}{\partial p^i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p^i}\Big)(q_1(t), ..., q_n(t), p_1(t), ..., p_n(t))$$

and using $\frac{\partial q_i}{\partial p_j} = \frac{\partial p_i}{\partial q_j} = 0$ and $\frac{\partial q_i}{\partial q_j} = \frac{\partial p_i}{\partial p_j} = \delta_{ij}$ we obtain

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

which are Hamilton's equations of motion: the fundamental postulates of Hamiltonian mechanics. Thus, dynamics are given by the integral curves of the Hamiltonian vector field.

Furthermore, let $f \in C^\infty(T^*Q)$ be an observable quantity for the system. Using definition 2.27 along with theorem 3.8 we arrive at

$$\dot{f} = X_H[f] = \mathcal{L}_{X_H} f = \{f, H\}$$

Here we arrive at the fact that the time-evolution of any observable quantity of a physical system is given by the Poisson bracket.

4 Lie Groups and Lie Algebras

The study of symmetries in mechanical systems has been extensively used to simplify its dynamics. For example, translational symmetry of the center of mass, which occurs when no external forces are applied, allows a two-body system to be studied in the center of mass frame and essentially reduces the problem to that of a single body in a potential. A particularly useful mathematical framework for the study of symmetries is that of **Lie groups** and their **Lie algebra**. In this section we will introduce the main concepts which will then be applied in the next section to reduce the dimensionality of physical systems.

4.1 Abstract Theory

Definition 4.1 A **Lie group** is a group $(G, \cdot)$ which has a manifold structure in the sense of definition 2.9 and for which both

1. $(g_1, g_2) \mapsto g_1 \cdot g_2$
2. $g_1 \mapsto (g_1)^{-1}$

are smooth operations in the sense of definition 2.10.

Example 4.2 A particularly useful Lie group is that of linear invertible transformations on $\mathbb{R}^n$. The associated group has a representation $\mathrm{GL}_n(\mathbb{R}) = \{A \in \mathcal{M}_n(\mathbb{R}) \mid \det(A) \neq 0\}$ where $\mathcal{M}_n(\mathbb{R})$ denotes the set of $n \times n$ matrices with real entries, with composition of transforms as its binary operation which is equivalent to matrix multiplication. The manifold structure comes from mapping the matrix

$$A = (a_{ij})_{n\times n} \mapsto (a_{11}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nn}) \in \mathbb{R}^{n^2}$$

which is a global chart. Then, as

$$(A \cdot B)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

the binary operation is smooth by smoothness of polynomials in $\mathbb{R}^{n^2}$. Similarly, for the inverse, we get that

$$A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}(A)$$

is smooth since the determinant is never zero and the adjoint of the matrix has entries that are polynomials in a_{ij}.

Remark Most Lie groups we will use in this work are **matrix Lie groups** which are subgroups of $\mathrm{GL}_n(\mathbb{R})$.

Definition 4.3 Let $\mathfrak{g}$ be a vector space over $\mathbb{R}$. A function $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is $\mathbb{R}$-bilinear and satisfies

1. Skew-symmetry: $[f\,, g] = -\,[g\,, f]$
2. Jacobi Identity : $[f\,, [g\,, h]] + [g\,, [h\,, f]] + [h\,, [f\,, g]] = 0$

for all $f, g, h \in \mathfrak{g}$ is called a **Lie bracket** on $\mathfrak{g}$ and the pair $(\mathfrak{g}, [.,.])$ is called a **Lie Algebra**.

Remark For a manifold $\mathcal{P}$, the set of vector fields on it paired with the Jacobi-Lie bracket from definition 2.28, that is $(\mathfrak{X}(\mathcal{P}), [.,.])$, forms a Lie algebra.

Definition 4.4 Let $(\mathfrak{g}, [.,.])$ be a Lie algebra and $\{e_i\}_{i=1}^n$ be a basis of $\mathfrak{g}$. We define the **structure constants** c_{jk}^i of $\mathfrak{g}$ with respect to $\{e_i\}$ to be the unique numbers such that

$$[e_j, e_k] = \sum_{i=1}^{n} c_{jk}^i e_i$$

Definition 4.5 Let G be a Lie group and $g \in G$, we define **left translation** by g as $L_g : G \to G$ such that

$$L_g(h) := gh \ \ \forall h \in G$$

Proposition 4.6 Let G be a Lie group and $g \in G$, then L_g is a diffeomorphism whith inverse $L_{g^{-1}}$.

Proof. Since G is a group, if $g \in G$, then there exists $g^{-1} \in G$ such that

$$g \cdot g^{-1} = g^{-1} \cdot g = e$$

So we may compute directly for any $h \in G$,

$$L_{g^{-1}}\left(L_g(h)\right) = L_{g^{-1}}(g \cdot h) = (g^{-1} \cdot g) \cdot h = e \cdot h = h$$

Thus, $L_{g^{-1}} \circ L_g = \mathrm{Id}_G$. That is L_g has $L_{g^{-1}}$ as its inverse. Furthermore, by the definition of a Lie group, inversion and multiplication are smooth operations so L_g and $L_{g^{-1}}$ are smooth which implies that L_g is a diffeomorphism. □

Definition 4.7 Let G be a Lie group and $X \in \mathfrak{X}(G)$. We say that X is a **left invariant vector field** if

$$D_h L_g[X(h)] = X(gh) \ \ \forall g, h \in G$$

Where $D_h L_g : T_h G \to T_{gh} G$ is the derivative of L_g at h. We denote the set of left invariant vector fields on G by $\mathfrak{X}_L(G)$.

Remark This definition is a bit contrived, so here is some intuition. The left hand side essentially transports the vector assigned h through the infinitesimal action of L_g to its target point gh while the right hand side evaluates the vector field at the target point. Hence, for a left invariant vector field, those two actions are equivalent. This is why we say it is left invariant : acting on the vector field by L_g does not change it globally.

Proposition 4.8 Let G be a Lie group. Then, for every $v \in T_e G$, there exists a unique left invariant

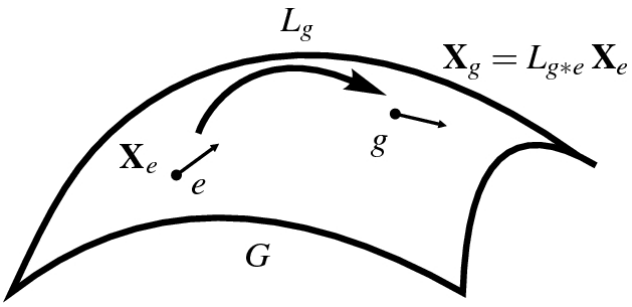


Figure 1: Schematic of a left invariant vector field. The vector associated to the point e in G is transported to the target point g which is equivalently obtained by the application of the derivative of L_g. Figure was taken from [3].

vector field X_v^L on G such that $X_v^L(e) = v$. Explicitly, we have

$$X_v^L(g) := D_e L_g(v) \;\; \forall g \in G$$

We call X_v^L the **left invariant extension** of v on G.

Proof. We start with a useful identity. by direct computation, we obtain that for any $g, h \in G$,

$$L_g \circ L_h = L_{gh}$$

Since both sides are differentiable by proposition 4.6, differentiating on both sides and applying the chain rule, we obtain

$$D_e(L_g \circ L_h) = D_h L_g \circ D_e L_h = D_e L_{gh}$$

Thus, for any $v \in T_e G$, defining

$$X_v(g) := D_e L_g(v) \;\; \forall g \in G$$

We compute that for any $h \in G$,

$$D_h L_g[X_v(h)] = D_h L_g[D_e L_h(v)] = D_e L_{gh}(v) = X_v(gh)$$

Which, by definition 4.7 means X_v is left invariant, concluding existence.

For uniqueness, we let $\bar{X}_v$ be another such left invariant vector field, then, we must have by its left invariance that

$$\bar{X}_v(g) = \bar{X}_v(g \cdot e) = D_e L_g(\bar{X}_v(e)) = D_e L_g(v) =: X_v(g)$$

Which concludes uniqueness. □

Corollary 4.9 Let G be a Lie group, then the map $\Phi : T_eG \to \mathfrak{X}_L(G)$ defined by

$$\Phi(v) = X_v^L \quad \forall v \in T_eG$$

is a canonical vector space isomorphism. Thus

$$T_eG \cong \mathfrak{X}_L(G)$$

Proof. Let $\alpha, \beta \in \mathbb{R}$ and $u, v \in T_eG$ and $g \in G$, then

$$\begin{aligned}
\Phi(\alpha u + \beta v)(g) &= X^L_{\alpha u + \beta v}(g) \\
&= D_e L_g(\alpha u + \beta v) \\
&= \alpha \cdot D_e L_g(u) + \beta \cdot D_e L_g(v) \\
&= \alpha X_u^L(g) + \beta X_v^L(g) \\
&= \alpha \cdot \Phi(u)(g) + \beta \cdot \Phi(v)(g)
\end{aligned}$$

Hence Φ is a linear map. Furthermore, by proposition 4.8, we know that Φ is bijective since X_v exists and is unique. This concludes the proof. □

Proposition 4.10 Let G be a Lie group and $X, Y \in \mathfrak{X}_L(G)$. Then, the Jacobi-Lie bracket $[X\,, Y]$ (defined in 2.28) of X and Y is also left invariant.

Remark This shows that the Jacobi-Lie bracket is closed on $\mathfrak{X}_L(G)$ since

$$[X\,, Y] \in \mathfrak{X}_L(G) \quad \forall X, Y \in \mathfrak{X}_L(G)$$

Proof. We first prove a useful identity. Let $f \in \mathcal{F}(G)$, $g, h \in G$ and $X \in \mathfrak{X}_L(G)$. Then,

$$\begin{aligned}
X[f \circ L_g](h) &= d(f \circ L_g)_h[X(h)] \\
(\text{chain rule}) &= df_{gh}(D_h L_g[X(h)]) \\
(X \text{ is left invariant}) &= df_{gh}(X(gh)) \\
&= X[f](gh) \\
&= X[f] \circ L_g(h)
\end{aligned}$$

And so $X[f \circ L_g] = X[f] \circ L_g$ which allows the following computation

$$\begin{aligned}
[X\,,Y]\,(f \circ L_g) &= X[Y[f \circ L_g]] - Y[X[f \circ L_g]] \\
&= X[Y[f] \circ L_g] - Y[X[f] \circ L_g] \\
&= X[Y[f]] \circ L_g - Y[X[f]] \circ L_g \\
&= [X\,,Y] \circ L_g
\end{aligned}$$

Evaluating at $h \in G$ gives

$$[X\,,Y]\,[f](gh) = [X\,,Y]\,[f \circ L_g](h) = D_h L_g([X\,,Y])[f](h)$$

Since they act the same as derivations, by theorem 2.16,

$$[X\,,Y]\,(gh) = D_h L_g([X\,,Y]\,(h))$$

as vectors, proving left invariance. □

Although we already defined the concept of *a* Lie algebra in definition 4.3, when a particular Lie group G is considered, we define ***the* Lie algebra of G** as follows.

Definition 4.11 Let G be a Lie group, then **its Lie algebra** $\mathfrak{g}$ is

$$\mathfrak{g} := T_e G$$

Its Lie bracket is defined by

$$[u\,,v] := \left[X_u^L\,, X_v^L\right](e) \quad \forall u, v \in \mathfrak{g}$$

Where X_u^L and X_u^L are the left invariant vector fields defined as in proposition 4.8 and the bracket on the right denotes the Jacobi-Lie bracket on vector fields.

Remark This indeed forms a Lie algebra since $\mathfrak{X}_L(G) \subset \mathfrak{X}(G)$ while $\mathfrak{X}(G)$ forms a Lie algebra with the Jacobi-Lie bracket and, by proposition 4.10, this operation is closed on $\mathfrak{X}_L(G)$. In this case, we say that $(\mathfrak{X}_L(G), [.\,,.])$ is a **subalgebra** of $(\mathfrak{X}(G), [.\,,.])$. Finally, the Lie bracket is well defined since, by corollary 4.9, X_v^L and X_u^L exist and are unique.

4.2 Matrix Lie Groups

Example 4.12 We will now try to characterize the case where G is a matrix Lie group. We first remark that letting $\gamma(t) : I \to G$ be a smooth curve with on a matrix lie group $G < \mathrm{GL}_n(\mathbb{R})$, we can express it by

$$\gamma(t) = [\gamma_{ij}(t)]$$

And if $\gamma(0) = g \in G$, then

$$A := \gamma'(0) = [(\gamma_{ij})'(t)] \in T_g G$$

Which allows us to view tangent vectors of G as elements of $\mathcal{M}_n(\mathbb{R})$. Therefore, since the identity element of G is $e = I_n$, we obtain that the Lie algebra of G satisfies $\mathfrak{g} = T_{I_n} G \subset \mathcal{M}_n(\mathbb{R})$.

We now want to compute the left invariant extension of $A \in \mathfrak{g}$. By proposition 4.8, we know it is defined by

$$X_A^L(g) = D_e L_g(A) \ \ \forall g \in G$$

To compute this directly, we let γ_A be smooth curve on G such that $\gamma(0) = I_n$ and $\gamma'(0) = A \in \mathfrak{g}$. Note that, by definition 2.12, γ_A must exist. Then,

$$D_e L_g(A) := (L_g \circ \gamma)'(0) = (g \cdot \gamma)'(0) = g \cdot \gamma'(0) = g \cdot A$$

Which gives us the expression

$$X_A^L(g) = g \cdot A \ \ \forall g \in G, A \in \mathfrak{g}$$

This allows us to view the left invariant vector field as a map $X_A^L : G \to \mathcal{M}_n(\mathbb{R})$ so we may compute $D_g X_A^L : T_g G \to T_{gA} \mathcal{M}_n(\mathbb{R}) \cong \mathcal{M}_n(\mathbb{R})$ which, by linearity of g in X_A^L gives

$$D_g X_A^L(v) = vA \ \ \forall v \in T_g G$$

Then, we can compute the Jacobi-Lie bracket of X_A^L and X_B^L for $A, B \in T_{I_n} G = \mathfrak{g}$ by

$$\begin{aligned} \left[X_A^L, X_B^L\right](g) &= X_A^L[X_B^L(g)] - X_B^L[X_A^L(g)] \\ &= D_g X_B^L[X_A^L(g)] - D_g X_A^L[X_B^L(g)] \\ &= D_g X_B^L[gA] - D_g X_A^L[gB] \\ &= gAB - gBA \end{aligned}$$

This now allows us to compute the Lie Bracket on $\mathfrak{g}$. By definition 4.11, for any $A, B \in \mathfrak{g}$,

$$[A, B] := \left[X_A^L, X_B^L\right](I_n) = I_n AB - I_n BA = AB - BA$$

Which is the matrix commutator.

This motivates the following definition

Definition 4.13 Let $(G, .)$ be a matrix Lie group, then **its Lie algebra** $\mathfrak{g}$ is

$$\mathfrak{g} := T_{I_n} G \subset \mathcal{M}_n(\mathbb{R})$$

Its Lie bracket is the **matrix commutator** $[.,.]$ which acts as

$$[A,\, B] = AB - BA$$

for all $A, B \in \mathfrak{g}$

Definition 4.14 Let $A \in \mathcal{M}_n(\mathbb{R})$, we generalize the notion of the exponential to

$$\exp A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

with the usual convention that $A^0 = I_n$, where I_n denotes the $n \times n$ identity matrix

Remark It can be shown that $\exp A$ converges for all $A \in \mathcal{M}_n(\mathbb{R})$ using the fact that $(\mathcal{M}_n(\mathbb{R}), ||\cdot||)$ is Banach for any matrix norm $||\cdot||$ and using the absolute convergence of the exponential on $\mathbb{R}$.

With this generalization, we recover basic properties of the exponential.

Proposition 4.15 Let $A \in \mathcal{M}_n$, the following hold :

1. $\exp(0 \cdot A) = \exp(\mathbf{0}_n) = I_n$
2. $\exp(-A) = (\exp(A))^{-1}$
3. $\dfrac{d}{dt}\exp(tA) = A\exp(tA)\ \forall t \in \mathbb{R}$

Remark We note that the third property can be used inductively to show that $f(t) = \exp(tA)$ is smooth.

Proof. First, for any square matrix A, $0 \cdot A = \mathbf{0}_n$, the $n \times n$ null matrix. Then, using the definition,

$$\exp(\mathbf{0}_n) = \sum_{k=0}^{\infty} \frac{\mathbf{0}_n^k}{k!} = \mathbf{0}_n^0 = I_n$$

by our convention.

Second, we have from the absolute convergence of both exponentials that

$$\begin{aligned}
\exp(-A)\exp(A) &= \left(\sum_{k=0}^{\infty} \frac{(-A)^k}{k!}\right)\left(\sum_{m=0}^{\infty} \frac{A^m}{m!}\right) \\
\text{(A commutes with itself)} &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^k}{k!\,m!} A^{k+m} \\
\text{(Letting } p = m + k\text{)} &= \sum_{p=0}^{\infty}\sum_{m=0}^{p} \frac{(-1)^{p-m}}{m!\,(p-m)!} A^p \\
\text{(Rearranging the terms)} &= \sum_{p=0}^{\infty} A^p \sum_{m=0}^{p} \frac{1}{p!}\binom{p}{m}(-1)^{p-m}
\end{aligned}$$

$$= \sum_{p=0}^{\infty} \frac{(-A)^p}{p!} \sum_{m=0}^{p} \binom{p}{m} (-1)^m$$

$$\text{(Binomial formula)} = \sum_{p=0}^{\infty} \frac{(-A)^p}{p!} (1-1)^p$$

$$\text{(Only non-zero term is } p = 0) = I_n$$

And so we conclude that they are inverses of each other. Finally, we compute.

$$\begin{aligned}
\frac{d}{dt} \exp(tA) &= \frac{d}{dt} \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \\
\text{(By absolute convergence)} &= \sum_{k=0}^{\infty} \frac{d}{dt} \frac{t^k A^k}{k!} \\
&= \sum_{k=1}^{\infty} \frac{k \cdot t^{k-1} A^k}{k!} \\
&= \sum_{k=1}^{\infty} \frac{t^{k-1} A^k}{(k-1)!} \\
&= \sum_{n=0}^{\infty} \frac{t^n A^{n+1}}{n!} \\
&= A \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \\
&= A \exp(tA)
\end{aligned}$$

□

The following theorem gives an important equivalent definition of the Lie algebra in the case a matrix Lie groups.

Theorem 4.16 Let G be a matrix Lie group, then its Lie algebra is given by

$$\mathfrak{g} = \{A \in \mathcal{M}_n \mid \exp(tA) \in G \; \forall t \in \mathbb{R}\}$$

Again, the matrix commutator is its Lie bracket.

Proof. First, assume that for all $t \in \mathbb{R}$, $\exp(tA) \in G$ for some $A \in \mathcal{M}_n$ then by definition 2.11 $\gamma : \mathbb{R} \to G$ defined by $\gamma(t) = \exp(tA)$ forms a smooth curve on G. In particular, using proposition 4.15, we know $\gamma(0) = I_n$ and so by definitions 2.12 and 2.13, we have that

$$\left.\frac{d\gamma}{dt}\right|_{t=0} \in T_{I_n} G = \mathfrak{g}$$

Furthermore, by proposition 4.15, we compute

$$\left.\frac{d\gamma}{dt}\right|_{t=0} = A \exp(0 \cdot A) = A \in \mathfrak{g}$$

Hence, the following inclusion holds

$$\{A \in \mathcal{M}_n \mid \exp(tA) \in G \; \forall t \in \mathbb{R}\} \subset \mathfrak{g}$$

Conversely, assume that $A \in \mathfrak{g} = T_{I_n}G$. We know from proposition 4.8 that there exists a unique left invariant vector field X_A^L on G such that $X_A^L(I_n) = A$. Furthermore, in example 4.12, we showed that

$$X_A^L(g) = gA \;\; \forall g \in G$$

We know the curve $\gamma(t) = \exp(tA)$ has the property that $\gamma(0) = I_n \in G$. Additionally, for any $t \in \mathbb{R}$,

$$\gamma'(t) = A\exp(tA) = \exp(tA)A = \gamma(t)A = X_A^L(\gamma(t))$$

And so for all $t \in \mathbb{R}$, γ is an integral curve of X_A^L and by the theorem of existence and uniqueness 2.21, it follows that $\gamma(\mathbb{R}) \subset G$ which leads to the inclusion

$$\mathfrak{g} = T_{I_n}G \subset \{A \in \mathcal{M}_n \mid \exp(tA) \in G \; \forall t \in \mathbb{R}\}$$

By mutual inclusion, we reach equality of the sets. □

Example 4.17 To demonstrate the usefulness of this theorem, we note that by proposition 4.15, the exponential of any matrix $A \in \mathcal{M}_n$ is invertible and thus $\exp(tA) \in GL_n(\mathbb{R})$ at all times. Therefore the Lie algebra of $GL_n(\mathbb{R})$ is $\mathfrak{gl}_n(\mathbb{R}) = \mathcal{M}_n(\mathbb{R})$ with the matrix commutator as its bracket.

4.3 Lie Group Actions

We now continue to develop the abstract Lie theory

Definition 4.18 Let G be a Lie group and $g \in G$. We define **conjugation** by g the operation $C_g : G \to G$ that acts as

$$C_g(h) := ghg^{-1} \;\; \forall h \in G$$

Definition 4.19 Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $g \in G$. The **adjoint action** of g on $\mathfrak{g}$ is the map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ which acts as

$$\mathrm{Ad}_g(h) := D_eC_g(h) \;\; \forall h \in \mathfrak{g}$$

Remark By smoothness of the binary operation and of inversion on G, C_g is differentiable so D_eC_g exists. We also note that the adjoint action defines a group action of G on $\mathfrak{g}$ which justifies its name. It describes how conjugation by g acts infinitesimally on the Lie algebra.

Proposition 4.20 Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $g \in G$. Then Ad_g defines an **automorphism** on $\mathfrak{g}$. That is, it defines a vector space isomorphism from $\mathfrak{g}$ to itself.

Remark Since C_g defines a group automorphism, its differential at the identity Ad_g actually defines a **Lie algebra automorphism**. That is a vector space isomorphism that preserves the Lie bracket by acting as

$$\mathrm{Ad}_g([u\,,v]) = [\mathrm{Ad}_g u\,, \mathrm{Ad}_g v] \quad \forall u, v \in \mathfrak{g}$$

For this reason, we may say that the adjoint action defines a homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ by $\mathrm{Ad}(g) := \mathrm{Ad}_g$ where $\mathrm{Aut}(\mathfrak{g})$ is the group of Lie algebra Automorphisms on $\mathfrak{g}$. This holds since a direct computation shows that $\mathrm{Ad}_{gh} = \mathrm{Ad}_g \circ \mathrm{Ad}_h \quad \forall g, h \in G$. This interesting result will however not be of use in the rest of this work.

Proof. We first note that, indeed, $D_e C_g$ is a map from $\mathfrak{g} = T_e G$ to $T_{C_g(e)}G = T_{geg^{-1}}G = T_e G = \mathfrak{g}$. Then, by linearity of derivatives, Ad_g is a linear map. finally, we claim that $\mathrm{Ad}_{g^{-1}}$ is its inverse. A direct computation shows that

$$\mathrm{Ad}_{g^{-1}}(\mathrm{Ad}_g(v)) = D_e C_{g^{-1}}(D_e C_g(v))$$

But since

$$C_{g^{-1}}(C_g(h)) = g^{-1}ghg^{-1}g = h \quad \forall h \in G$$

We get that $C_{g^{-1}} \circ C_g = \mathrm{Id}_G$. Taking the derivative at e on both sides gives

$$D_e(C_{g^{-1}} \circ C_g) = D_{geg^{-1}}C_{g^{-1}} \circ D_e C_g = D_e C_{g^{-1}} \circ D_e C_g = D_e \mathrm{Id}_G = \mathrm{Id}_{\mathfrak{g}}$$

And so

$$\mathrm{Ad}_{g^{-1}}(\mathrm{Ad}_g(v)) = \mathrm{Id}_{\mathfrak{g}}(v) = v$$

Showing that Ad_g is an invertible linear transformation. Hence it is a vector space isomorphism. □

Example 4.21 We investigate the case where G is a matrix Lie group. If we let $g \in G$ and $\gamma(t) = [\gamma_{ij}(t)]$ is a curve on G with $\gamma(0) = I_n$ and $\gamma'(0) = v \in \mathfrak{g}$. Then we compute the adjoint action of g In the following way.

$$\begin{aligned}
\mathrm{Ad}_g(v) &= D_{I_n} C_g(v) \\
&= \frac{d}{dt}(C_g \circ \gamma)(t)\Big|_{t=0} \\
&= \frac{d}{dt}(g\gamma(t)g^{-1})\Big|_{t=0} \\
&= g\gamma'(0)g^{-1} \\
&= gvg^{-1}
\end{aligned}$$

Thus, we see that in the case of matrix Lie groups, the adjoint action of g is simply matrix conjugation by g of elements of $\mathfrak{g}$ which are also viewed as matrices.

Definition 4.22 Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra and $\xi \in \mathfrak{g}$. The **adjoint representation** is the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ where $\mathrm{End}(\mathfrak{g})$ is the vector space of **endomorphism** on $\mathfrak{g}$ which consists of all linear transformations from $\mathfrak{g}$ to itself. The map is defined by

$$\mathrm{ad}(\xi) = \mathrm{ad}_\xi := [\xi\,, .]$$

We call ad_ξ the **adjoint operator** associated to ξ.

We now give a heuristic reasoning for this definition.

Example 4.23 Let G be a matrix Lie group and $\mathfrak{g}$ be its Lie algebra. We let $A \in \mathfrak{g}$ and consider the curve $\gamma(t) = \exp(tA)$ on G. As computed in example 4.21, we know that for any $B \in \mathfrak{g}$, we have

$$\mathrm{Ad}_{\gamma(t)}(B) = \gamma(t)B(\gamma(t))^{-1} = \gamma(t)B\,\gamma(-t)$$

And so differentiating at $t = 0$ yields

$$\frac{d}{dt}\mathrm{Ad}_{\gamma(t)}(B)\bigg|_{t=0} = \gamma'(0)B\gamma(0) - \gamma(0)B\gamma'(0) = AB - BA = [A\,, B] = \mathrm{ad}_A(B) = \mathrm{ad}_{\gamma'(0)}(B)$$

Which tells us that the adjoint representation is the derivative of the adjoint action along the exponential function. This is why the adjoint representation is often regarded as the infinitesimal effect of the adjoint action.

Remark One can generalize the exponential as a map $\exp : \mathfrak{g} \to G$ to arbitrary Lie Groups and define the adjoint representation in a similar way as was done in this example.

Proposition 4.24 Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra and $\xi, \eta \in \mathfrak{g}$. Then the following holds.

$$[\mathrm{ad}_\xi\,, \mathrm{ad}_\eta]_f := \mathrm{ad}_\xi \circ \mathrm{ad}_\eta - \mathrm{ad}_\eta \circ \mathrm{ad}_\xi = \mathrm{ad}_{[\xi\,,\eta]}$$

Remark This shows that ad forms a **Lie algebra homomorphism** between $(\mathfrak{g}, [.\,,.])$ and $(\mathrm{ad}(\mathfrak{g}), [.\,,.]_f)$ since it preserves the Lie bracket.

Proof. Let $\eta, \xi, \nu \in \mathfrak{g}$, then we compute

$$\begin{aligned}
[\mathrm{ad}_\xi\,, \mathrm{ad}_\eta]_f\,(\nu) &= \mathrm{ad}_\xi \circ \mathrm{ad}_\eta(\nu) - \mathrm{ad}_\eta \circ \mathrm{ad}_\xi(\nu) \\
&= [\xi\,, [\eta\,, \nu]] - [\eta\,, [\xi\,, \nu]] \\
\text{(Skew symmetry)} &= [\xi\,, [\eta\,, \nu]] + [\eta\,, [\nu\,, \xi]] \\
\text{(Jacobi identity)} &= -\,[\nu\,, [\xi\,, \eta]] \\
\text{(Skew symmetry)} &= [[\xi\,, \eta]\ , \nu] \\
&= \mathrm{ad}_{[\xi\,, \eta]}(\nu)
\end{aligned}$$

Since they agree at all $\nu \in \mathfrak{g}$, they are equal as endomorphisms. □

We now give similar definitions that apply un the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$.

Definition 4.25 Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mu \in G$. The **Coadjoint action** of g on $\mathfrak{g}^*$ is the map $\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ which acts as

$$\left\langle \mathrm{Ad}^*_g(\mu), \xi \right\rangle := \left\langle \mu, \mathrm{Ad}_{g^{-1}}(\xi) \right\rangle = \left\langle \mu, D_e C_{g^{-1}}(\xi) \right\rangle \quad \forall \mu \in \mathfrak{g}^*, \xi \in \mathfrak{g}$$

Remark Equivalently, it is the dual map of $\mathrm{Ad}_{g^{-1}}$. The inverse is taken to preserve the automorphism property of Ad.

Definition 4.26 Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra and $\xi \in \mathfrak{g}$. The **coadjoint representation** is the map $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$. The map is defined by

$$\left\langle \mathrm{ad}^*_\xi(\mu), \eta \right\rangle := \langle \mu, \mathrm{ad}_{-\xi}(\eta) \rangle = \langle \mu, [\eta\,, \xi] \rangle \quad \forall \mu \in \mathfrak{g}^*, \eta \in \mathfrak{g}$$

We call ad_ξ the **coadjoint operator** associated to ξ.

Remark Similarly, we use $\mathrm{ad}_{-\xi}$ so that the property detailed in the heuristic 4.23 holds true.

5 The Lie-Poisson Bracket

With the prerequisites in Lie theory covered, we now investigate the results of section 3 regarding the cotangent bundle in the case where the configuration space $Q = G$ is a Lie group.

In this section, due to the prominence of these expressions, we introduce the use of Einstein summation notation. That is, unless otherwise stated, if two elements with the same index are multiplied, the index is summed over its range. For example, suppose i runs from 1 to n, then we interpret

$$\alpha_i(\xi^i)$$

as

$$\sum_{i=1}^{n} \alpha_i(\xi^i)$$

We now introduce a useful tool in the setting of Lie groups.

5.1 Cartan Frames

Definition 5.1 Let G be a Lie group of dimension n. A list of n left-invariant vector fields $E_1, \dots, E_n \in \mathfrak{X}_L(G)$ is called a **Cartan frame** on G if

$$\mathfrak{X}(G) = \mathcal{F}(G)\langle E_1, \dots, E_n \rangle$$

Meaning, the $\mathcal{F}(G)$ span of $\{E_i\}$ achieves all vector fields over G. Analogous to vectors, the **Cartan coframe** dual to $\{E_i\}$ is the collection of one forms $E^1, \dots, E^n \in \Omega^1(G)$ such that

$$E^i_g(E_j(g)) = \delta^i_j \;\; \forall g \in G \quad \text{and} \quad \Omega^1(G) = \mathcal{F}(G)\langle E^1, \dots, E^n \rangle$$

.

Proposition 5.2 Every Lie group admits a Cartan frame and coframe.

Proof.

Let $e_1, \dots e_n$ be a basis of $\mathfrak{g}$ and $e^1, \dots, e^n$ be its dual basis. By proposition 4.8, the basis vectors have unique left invariant extensions $E_i(g) := D_e L_g(e_i)\ \forall g \in G$. To show that $E_1, \dots, E_n$ forms a Cartan frame, it suffices to show that $E_1(g), \dots, E_n(g)$ forms a basis of T_gG for any $g \in G$. Indeed, if this is the case, by definition 2.19, the span condition follows. Thus, we let $a^1, \dots, a^n \in \mathbb{R}$ and $g \in G$. Furthermore, suppose that

$$a^i E_i(g) = 0 \in T_gG$$

By our definition, this implies that

$$a^i D_e L_g(e_i) = 0$$

And, by linearity of the differential,

$$D_e L_g \left(a^i e_i\right) = 0$$

Now, by proposition 4.6, we know that $D_e L_g$ is an isomorphism, hence this is only possible if $a^i e_i = 0 \in \mathfrak{g}$. However, by assumption, $\{e_i\}$ forms a basis of $\mathfrak{g}$ thus this is only true if $a^1, \ldots, a^n = 0$. This proves that for all $g \in G$, $E_1(g), \ldots, E_n(g)$ are linearly independent. Since the dimension of $T_g G$ is also n, we conclude that $\{E_i\}$ is indeed a Cartan frame.

Onto the coframe, define the left extensions E^i of the dual basis vectors by $E^i_g := ((D_e L_g)^{-1})^*(e^i) = e^i \circ D_g L_{g^{-1}}$. We then compute

$$E^i_g(E_j(g)) = e^i \circ D_g L_{g^{-1}} \circ D_e L_g(e_j) = e^i(e_j) = \delta^i_j$$

This concludes the first condition. The span condition follows from that of the Cartan frame. □

Definition 5.3 Let G be a Lie group of dimension n and $\{E_i\}$ be a Cartan frame on it. We define the fiber coordinates ζ_i relative to the frame as the functions

$$\begin{aligned} \zeta_i : T^*G &\to \mathbb{R} \\ (g, \alpha_g) &\mapsto \alpha_g(E_i(g)) \end{aligned}$$

We then define the vector fields $\frac{\partial}{\partial \zeta_i} \in \mathfrak{X}(T^*G)$ such that $E_i[\zeta_j] = 0$, $\frac{\partial}{\partial \zeta_i}(\zeta_j) = \delta^i_j$, and

$$\mathfrak{X}(T^*G) = \mathcal{F}(T^*G) \left\langle \partial_{\zeta_1}, \ldots, , \partial_{\zeta_n} E_1, \ldots, E_n \right\rangle$$

where, in the last condition, $E_i \in \mathfrak{X}(T^*G)$ denotes the lift of $E_i \in \mathfrak{X}(G)$ such that $E_j[\zeta_i] = 0$.

Remark As E^i forms a coframe, we may decompose any α_g as $\alpha_i E^i_g$ which yielding $\zeta_i(g, \alpha) = \alpha_j E^j_g(E_i(g)) = \alpha_i$. This justifies the name of fiber coordinate for ζ as it gives the coordinates in the $T^*_g G$ portion of T^*G. Furthermore, the conditions on the vector field can be satisfied as E^i is tangent to the Lie group whereas ∂_{ζ_i} is tangent to the fiber hence they do not intersect. Finally, this gives $2n$ linearly independent vectors hence a basis at every point.

5.2 The Canonical 1-Form and Symplectic Form of Lie Groups

We now progress towards an expression for the canonical poisson bracket by giving expressions for its components. Using the isomorphism in proposition 2.18, there is an isomorphism

$$T_{(g,\alpha)}(T^*G) \cong T_g G \times T_\alpha(T_g G)$$

. Recalling the definition of the canonical one form θ in 3.4, we obtain

$$\begin{aligned} \theta_{(g,\alpha)} : T_gG \times T_\alpha(T_gG) &\to \mathbb{R} \\ (\xi, \eta) &\mapsto \alpha(\xi) \end{aligned} \tag{3}$$

Proposition 5.4 Given a Cartan frame $\{E_i\}$ on a Lie group G, we may write the canonical one form θ on its cotangent bundle as

$$\theta = \zeta_i E^i$$

Remark Note that this expression is global due to the existence of the Cartan frame.

Proof. Let $((g, \alpha), (\xi, \eta)) \in T(T^*G)$, by definition 5.1, we may decompose α and ξ as

$$\alpha = \alpha_i E^i_g \quad \text{and} \quad \xi = \xi^i E_i(g)$$

Then,

$$\zeta_i E^i((g, \alpha), (\xi, \eta)) = \zeta_i(\alpha) E^i(\xi) = \alpha(E_i) E^i(\xi) = \alpha_i \xi^i = \alpha(\xi) = \theta_{(g,\alpha)}(\eta, \xi)$$

□

Recalling definition 3.5, the symplectic form on T^*G is given by $\omega := -d\theta$. Under the identification of proposition 5.4, we obtain the following.

Proposition 5.5 Let G be a Lie group with Cartan frame $\{E_i\}$ and c^i_{jk} be the corresponding structure constants. The symplectic form on T^*G is then given by

$$\omega = -d\zeta_i \wedge E^i + \frac{1}{2} \zeta_i c^i_{jk} E^j \wedge E^k$$

Remark Note that the use of structure constants is well defined since the frame is left invariant and the Lie bracket is closed over left invariant vector fields. Hence the Lie bracket evaluated at e and the Jacobi-Lie bracket are equivalent in this setting.

Proof. Using proposition 2.30, we notice that

$$\begin{aligned} dE^i(E_j, E_k) &:= E_j(E^i(E_k)) - E_k(E^i(E_j) - E^i([E_j\,, E_k]) \\ &= E_j(\delta^i_k) - E_k(\delta^i_j) - c^l_{jk} E^i(E_l) \\ &= -c^l_{jk} \delta^i_l \\ &= -c^i_{jk} \end{aligned}$$

Where the third inequality comes from the fact that E_j, E_k are derivations and δ^i_j is constant when i, j, k are fixed. Then, since $dE^i \in \Omega^2(G) = \mathcal{F}(G)\langle E^i \wedge E^j \rangle$, we may decompose it into its wedged components

of the Cartan coframe as

$$\begin{aligned} dE^i &= dE^i(E_j, E_k) \cdot (E^j \wedge E^k) \\ &= -\frac{1}{2} c^i_{jk} E^j \wedge E^k \end{aligned}$$

Where the division by two comes from the double counting of the structure constant. Then, using proposition 2.34, we obtain

$$\begin{aligned} \omega &:= -d\theta \\ &= -d(\zeta_i E^i) \\ &= -d\zeta_i \wedge E^i - \zeta_i \wedge dE^i \\ &= -d\zeta_i \wedge E^i + \frac{1}{2} \zeta_i c^i_{jk} E^j \wedge E^k \end{aligned}$$

□

5.3 The Canonical Poisson Bracket on Lie Groups

We may now investigate the symplectic vector fields of functions on T^*G which will lead to an expression for the canonical Poisson bracket on the phase space of G.

Proposition 5.6 Let $f \in \mathcal{F}(T^*G)$, then its symplectic vector field $\tilde{X}_f$ is given by

$$\tilde{X}_f = \frac{\partial f}{\partial \zeta_k} E_k + \left(\zeta_i c^i_{jk} \frac{\partial f}{\partial \zeta_j} - E_k[f] \right) \partial_{\zeta_k}$$

Proof. By definition 5.3, since $\tilde{X}_f \in \mathfrak{X}(T^*G)$, we may write it as

$$\tilde{X}_f = a^i E_i + b_i \partial_{\zeta_i}$$

for some $a^i, b_i \in \mathcal{F}(T^*G)$. Recalling the defining property of symplectic vector fields in corollary 3.7, we must have

$$\iota_{\tilde{X}_f} \omega = df$$

Expanding the left expression, letting $V \in \mathfrak{X}(T^*G)$ and using proposition 5.5,

$$\begin{aligned} \iota_{\tilde{X}_f} \omega(V) &= \omega\left(\tilde{X}_f, V\right) \\ &= -d\zeta_i \wedge E^i\left(\tilde{X}_f, V\right) + \frac{1}{2} \zeta_i c^i_{jk} E^j \wedge E^k\left(\tilde{X}_f, V\right) \\ &= -d\zeta_i(\tilde{X}_f) E^i(V) + d\zeta_i(V) E^i(\tilde{X}_f) + \frac{1}{2} \zeta_i c^i_{jk} \left(E^j(\tilde{X}_f) E^k(V) - E^k(\tilde{X}_f) E^j(V) \right) \end{aligned}$$

Then, using our decomposition,

$$d\zeta_i(\tilde{X}_f) = d\zeta_i(a^j E_j + b_j \partial_{\zeta_j}) = a^j \mathcal{L}_{E_j}(\zeta_i) + b_j d\zeta_i(\partial_{\zeta_j}) = b_j \delta_i^j = b_i$$
$$E^i(\tilde{X}_f) = E^i(a^j E_j + b_j \partial_{\zeta_j}) = a^j E^i(E_j) = a^i$$

Making the substitution in the expression above, we get

$$\iota_{\tilde{X}_f}\omega(V) = -b_i E^i(V) + d\zeta_i(V)a^i + \frac{1}{2}\zeta_i c^i_{jk}\left(a^j E^k(V) - a^k E^j(V)\right)$$

Thus, as one forms, we reach the equality

$$\iota_{\tilde{X}_f}\omega = -b_i E^i + a^i d\zeta_i + \frac{1}{2}\zeta_i c^i_{jk}\left(a^j E^k - a^k E^j\right)$$

Furthermore, by definition 4.4, since the Jacobi-Lie bracket is antisymmetric, we must have $c^i_{jk} = -c^i_{kj}$ hence

$$\iota_{\tilde{X}_f}\omega = a^i d\zeta_i + \left(\zeta_i c^i_{jk} a^j - b_k\right) E^k$$

Whereas on the right we have

$$df = \frac{\partial f}{\partial \zeta_i} d\zeta_i + E_i[f]E^i$$

Matching the two equivalent expressions, we must have

$$a^k = \frac{\partial f}{\partial \zeta_k}, \quad b_k = \zeta_i c^i_{jk}\frac{\partial f}{\partial \zeta_j} - E_k[f]$$

□

We now have every computation needed to obtain the canonical Poisson bracket on T^*G with respect to a Cartan frame.

Proposition 5.7 Let G be a Lie group with Cartan frame $\{E_i\}$. Using the decomposition of $\mathfrak{X}(T^*G)$ in definition 5.3, the canonical Poisson bracket $\{.,.\} : \mathcal{F}(T^*G) \times \mathcal{F}(T^*G) \to \mathcal{F}(T^*G)$ on the Cotangent bundle T^*G of G may be expressed as

$$\{f\,, g\} = -\zeta_i c^i_{jk}\frac{\partial f}{\partial \zeta_j}\frac{\partial h}{\partial \zeta_k} - \frac{\partial f}{\partial \zeta_i}E_i[h] + \frac{\partial h}{\partial \zeta_i}E_i[f]$$

Proof. Recall from theorem 3.8, that the canonical Poisson bracket on T^*G is given by

$$\begin{aligned}\{f\,, g\} &= \omega\left(\tilde{X}_f, \tilde{X}_g\right)\\ &= \iota_{\tilde{X}_f}\omega(\tilde{X}_g)\\ &= df\left(\tilde{X}_g\right)\end{aligned}$$

and thus, by proposition 5.6, we get

$$\begin{aligned}\{f\,,g\} &= \frac{\partial f}{\partial \zeta_k} d\zeta_k(\tilde{X}_h) + E_k[f]E^k(\tilde{X}_h) \\ &= \frac{\partial f}{\partial \zeta_k}\left(\zeta_i c^i_{jk}\frac{\partial h}{\partial \zeta_j} - E_k[h]\right) + E_k[f]\frac{\partial h}{\partial \zeta_k} \\ &= \zeta_i c^i_{jk}\frac{\partial f}{\partial \zeta_k}\frac{\partial h}{\partial \zeta_j} - \frac{\partial f}{\partial \zeta_k}E_k[h] + E_k[f]\frac{\partial h}{\partial \zeta_k}\end{aligned}$$

Finally, performing the switch of indices $j \leftrightarrow k$ in the first term and using the antisymmetry of c^i_{jk} in the lower indices, we reach the statement. □

5.4 Lie-Poisson Reduction and the Lie-Poisson Bracket

In this section, we investigate what happens to the Poisson bracket when the functions we are interested are independent of the configuration space variable $h \in G$.

Definition 5.8 Let G be a Lie group. The **cotangent lift of the left translation action** is the group action $\rho_L : G \times T^*G \to T^*G$ of G on its cotangent bundle T^*G defined by

$$\rho_L(g,(h,\alpha_h)) = (L_g(h),(L_{g^{-1}})^*(\alpha_h)) = (gh, \alpha_h \circ D_{gh}L_{g^{-1}})$$

Indeed, this is well defined since $\alpha_h \circ D_{gh}L_{g^{-1}}$ is a map $T_{gh}G \to T_hG \to \mathbb{R}$ thus $(L_{g^{-1}})^*(\alpha_h) \in T^*_{gh}G$. Furthermore properties shown in section 4.1 imply that this defines a group action.

Remark The name of this group action comes from the fact that we include both the action of left translation on G itself and on the covectors which "lifts" the effect to the cotangent bundle.

The next proposition gives a useful isomorphism.

Proposition 5.9 Let G be a Lie group and $\mathfrak{g}$ its Lie algebra , then

$$TG \cong G \times \mathfrak{g} \quad \text{and} \quad T^*G \cong G \times \mathfrak{g}^*$$

Proof. We recall the definition

$$TG := \{(g,v) \mid g \in G,\, v \in T_gG\}$$

From proposition 4.6, we know that, for any $g \in G$, L_g is a diffeomorphism and its differential at the identity e of G, namely $D_eL_g : T_eG = \mathfrak{g} \to T_gG$, is an isomorphism since it is invertible. So

$$T_eG \cong T_gG \;\; \forall g \in G$$

by associating $\xi \in \mathfrak{g}$ with $D_e L_g(\xi) \in T_g G$. Hence the map $\Lambda : G \times \mathfrak{g} \to TG$ which acts as

$$\Lambda(g, \xi) := (g, D_e L_g(\xi)) \ \ \forall (g, \xi) \in G \times \mathfrak{g}$$

is an isomorphism. Thus $TG \cong G \times \mathfrak{g}$.

We can then define the dual map $(D_e L_g)^* : T_g^* G \to T_e^* G = \mathfrak{g}^*$ such that for $\mu_g \in T_g^* G$,

$$(D_e L_g)^*(\mu_g) := \mu_g \circ D_e L_g$$

Since $D_e L_g$ is an isomorphism, so is the dual map. If we now define $\Lambda^* : T^*G \to G \times \mathfrak{g}^*$ by

$$\Lambda^*(g, \mu_g) := (g, (D_e L_g)^*(\mu_g)) \ \ \forall g \in G, \mu_g \in T_g^* G$$

Since it has an inverse which is explicitly

$$(\Lambda^*)^{-1}(g, \mu) = (g, [(D_e L_g)^*]^{-1}(\mu)) \ \ \forall g \in G, \mu \in \mathfrak{g}^*$$

We conclude that

$$T^*G \cong G \times \mathfrak{g}^*$$

□

This motivates the following definition.

Definition 5.10 Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then the **left trivialization** of its cotangent bundle is the map $\Lambda^* : T^*G \to G \times \mathfrak{g}^*$ defined by

$$\Lambda^*(g, \mu_g) := (g, (D_e L_g)^*(\mu_g)) \ \ \forall (g, \mu_g) \in T^*G$$

The previous proposition showed that it defines an isomorphism.

We now investigate the cotangent lift of the left translation action under this trivialization.

Proposition 5.11 Let G be a Lie group, and let T^*G/G denote the quotient of T^*G by the action ρ_L of G, then

$$T^*G/G \cong \mathfrak{g}^*$$

Proof. Let $g \in G$ and $(h, \mu_h) \in T^*G$. We recall from definition 5.10 that it has a corresponding element $\Lambda^*(h, \mu_h) = (h, (D_e L_h)^*(\mu_h)) \in G \times \mathfrak{g}^*$. Defining $\mu := (D_e L_h)^*(\mu_h)$, we have $(h, \mu_h) \sim_{\Lambda^*} (h, mu)$ If we act on the element by g through the cotangent lift, we get

$$\rho_L(g, (h, \mu_h)) = (gh, \mu_h \circ D_{gh} L_{g^{-1}}) \in T^*G$$

If we now express the result in the left trivialized space, we get

$$\begin{aligned}
\Lambda^* \circ \rho_L(g,(h,\mu_h)) &= (gh, (D_eL_{gh})^*[\mu_h \circ D_{gh}L_{g^{-1}}]) \\
&= (gh, \mu_h \circ D_{gh}L_{g^{-1}} \circ D_eL_{gh}) \\
\text{(see the proof of proposition 4.8)} &= (gh, \mu_h \circ D_eL_{g^{-1}gh}) \\
&= (gh, \mu_h \circ D_eL_h) \\
&= (gh, \mu)
\end{aligned}$$

Thus in trivialized coordinates, the cotangent lift of the left multiplication action only acts on the G component. Thus modding out by it results only in $(G \times \mathfrak{g}^*)/G \cong \{(e,\mu) \mid \mu \in \mathfrak{g}^*\} \cong \mathfrak{g}^*$. Therefore, $T^*G/G \cong \mathfrak{g}^*$. □

Definition 5.12 Let G be a Lie Group and $f \in \mathcal{F}(T^*G)$. We say that f is **left invariant** if, for any $g \in G$ and $(h,\alpha_h) \in T^*G$,

$$f \circ \rho_L(g,(h,\alpha_h)) = f(h,\alpha_h)$$

We denote the set of left invariant functions on T^*G by $\mathcal{F}_L(T^*G)$.

Proposition 5.13 Let G be a Lie Group, then any left invariant function of T^*G can be represented by a function on $\mathfrak{g}^*$. Thus $\mathcal{F}_L(T^*G) \cong \mathcal{F}(\mathfrak{g}^*)$

Proof. Let $(h,\alpha_h) \in T^*G$ and $f \in \mathcal{F}_L(T^*G)$. By the definition of left invariant functions, we have

$$\begin{aligned}
f(h,\alpha_h) &= f \circ \rho_L(h^{-1},(h,\alpha_h)) \\
&= f(e, \alpha_h \circ D_{h^{-1}h}L_h) \\
&= f(e, \alpha_h \circ D_eL_h)
\end{aligned}$$

And thus f depends only on $\alpha_h \circ D_eL_h \in \mathfrak{g}^*$ which is exactly the left trivialization of α_h. Conversely, taking $f \in \mathcal{F}(\mathfrak{g}^*)$, we may define a function $\tilde{f}$ over T^*G such that for any $(h,\mu_h) \in T^*G$, we have

$$\tilde{f}(h,\mu_h) := f(\mu_h \circ D_eL_h)$$

Then, we take $g \in G$ and compute

$$\begin{aligned}
\tilde{f} \circ \rho_L(g,(h,\alpha_h)) &= \tilde{f}(gh, \alpha_h \circ D_{gh}L_{g^{-1}}) \\
&= f(\alpha_h \circ D_{gh}L_{g^{-1}} \circ D_eL_{gh}) \\
&= f(\alpha_h \circ D_eL_h) \\
&= \tilde{f}(h,\alpha_h)
\end{aligned}$$

Thus, $\tilde{f} \in \mathcal{F}_L(T^*G)$. Since the identification is one to one, we get

$$\mathcal{F}_L(T^*G) \cong \mathcal{F}(\mathfrak{g}^*)$$

□

Functions that are left invariant with respect to a lie group are exactly what we mean by a function having a continuous symmetry described by G. For example, a free particle in three dimensional space has a configuration space of $G = \mathbb{R}^3$. Its cotangent bundle has global coordinates $(q_i, p_i)_{i=1}^3$. However, its total energy is given by

$$H(q_i, p_i) = \sum_{i=1}^{3} \frac{p_i^2}{2m}$$

We see that H is therefore left invariant with respect to the lie group of translations $(\mathbb{R}^3, +)$. We may therefore suppose that the dynamics of this system may entirely be described by its momentum.

This motivates finding a Poisson bracket on left invariant functions of T^*G. As remarked above, the set of such functions actually consists of $\mathcal{F}(\mathfrak{g}^*)$.

Theorem 5.14 Let G be a Lie group and $\{E_i\}$ be a Cartan frame on it. Then, the canonical bracket on T^*G is closed over left invariant functions. That is $\forall f, g \in \mathcal{F}_L(T^*G)$ we have that $\{f\,, g\} \in \mathcal{F}_L(T^*G)$. Explicitly, letting ζ_i be the fiber coordinates defined in 5.3 we obtain that

$$\{f\,, g\} = -c^i_{jk}\zeta_i \frac{\partial f}{\partial \zeta_j}\frac{\partial g}{\partial \zeta_k}, \quad \forall f, g \in \mathcal{F}_L(T^*G)$$

Furthermore, under the identification $T^*G/G \cong \mathfrak{g}^*$ and $\mathcal{F}_L(T^*G) \cong \mathcal{F}(\mathfrak{g}^*)$ shown in propositions 5.11 and 5.13, we obtain a Poisson bracket $\{.\,,.\}_{LP} : \mathcal{F}(\mathfrak{g}^*) \times \mathcal{F}(\mathfrak{g}^*) \to \mathcal{F}(\mathfrak{g}^*)$ defined intrinsically by

$$\{f\,, g\}_{LP}(\mu) = -\langle \mu, [df_\mu\,, dg_\mu]\rangle$$

where $[.\,,.]$ denotes the Lie bracket on $\mathfrak{g}$. This Poisson bracket on the dual of a Lie algebra is called **the Lie-Poisson bracket** of $\mathfrak{g}^*$.

Remark Here, using df_μ, dg_μ is an abuse of notation. In reality, $df_\mu \in T^*_\mu\mathfrak{g}^*$ but since $\mathfrak{g}^*$ is a vector space, $T^*_\mu\mathfrak{g}^* = (T_\mu\mathfrak{g}^*)^* \cong (\mathfrak{g}^*)^* \cong \mathfrak{g}$ by proposition 2.5. Thus treating them as elements of $\mathfrak{g}$ is justified.

Proof. We first start by showing that the fiber coordinates ζ_i are themselves left invariant. Indeed, letting $g \in G$ and $(h, \alpha_h) \in T^*G$, then, by definition 5.3

$$\begin{aligned}\zeta_i \circ \rho_L(g, (h, \alpha_h)) &= \zeta_i(gh, \alpha_h \circ D_{gh}L_{g^{-1}}) \\ &= \alpha_h \circ D_{gh}L_{g^{-1}}(E_i(gh))\end{aligned}$$

Since E_i is part of a Cartan frame, it is a left invariant vector field. Refering to definition 4.7, we have

that

$$D_{gh}L_{g^{-1}}(E_i(gh)) = E_i(g^{-1}gh) = E_i(h)$$

Thus,

$$\begin{aligned}\zeta_i \circ \rho_L(g,(h,\alpha_h)) &= \alpha_h(E_i(h)) \\ &= \zeta_i(h,\alpha_h)\end{aligned}$$

proving the claim. Now, taking $f \in \mathcal{F}_L(T^*G)$, by proposition 5.11, we know that $f(h,\alpha_h)$ depends only on α_h and thus only on its fiber coordinates ζ_i. Given that vector fields decompose into distinct E_i and ζ_i coordinates, it follows that $E_k[f] = 0$. Applying this to proposition 5.7, we obtain

$$\{f\,,g\} = -c^i_{jk}\zeta_i \frac{\partial f}{\partial \zeta_j}\frac{\partial g}{\partial \zeta_k}, \quad \forall f,g \in \mathcal{F}_L(T^*G)$$

We then notice that since f,g depend only on ζ_i, so do $\frac{\partial f}{\partial \zeta_j}, \frac{\partial g}{\partial \zeta_k}$. Hence $\{f\,,g\}$ depends entirely on the fiber coordinates. Since they are left invariant, so is $\{f\,,g\}$. Moving to the Lie-poisson bracket, by proposition 5.13, we know f and g descend to well-defined functions on $\mathfrak{g}^*$. In that setting, using ζ_i as coordinates on $\mathfrak{g}^*$ (which is possible by left invariance), we obtain their exterior derivatives using proposition 2.26 as

$$df_\mu = \frac{\partial f}{\partial \zeta_i} d\zeta_i$$

As remarked above, we want the corresponding elements in $\mathfrak{g}$ which is simply

$$df_\mu = \frac{\partial f}{\partial \zeta_i} e_i$$

Where $e_i = E_i(e)$ to respect the original Cartan frame. Then, by the definition of structure constants, we get that

$$[df_\mu\,, dg_\mu] = \left[\frac{\partial f}{\partial \zeta_j}e_j\,, \frac{\partial g}{\partial \zeta_k}e_k\right] = \frac{\partial f}{\partial \zeta_j}\frac{\partial g}{\partial \zeta_k}[e_j\,,e_k] = c^i_{jk}\frac{\partial f}{\partial \zeta_j}\frac{\partial g}{\partial \zeta_k}e_i$$

Finally, for an element $v^i e_i \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$ we compute

$$\langle \mu, v^i e_i\rangle = v^i \mu(e_i) = v^i \zeta_i$$

Putting all of this together, we have

$$\begin{aligned}
\{f\,,g\}_{LP} &= -\langle \mu, [df_\mu\,, dg_\mu]\rangle \\
&= -\left\langle \mu, c^i_{jk}\frac{\partial f}{\partial \zeta_j}\frac{\partial g}{\partial \zeta_k}e_i \right\rangle \\
&= -c^i_{jk}\frac{\partial f}{\partial \zeta_j}\frac{\partial g}{\partial \zeta_k}\langle \mu, e_i\rangle \\
&= -c^i_{jk}\zeta_i\frac{\partial f}{\partial \zeta_j}\frac{\partial g}{\partial \zeta_k} \\
&= \{f\,,g\}
\end{aligned}$$

Proving the equivalence of the canonical Poisson bracket and the Lie-Poisson bracket when dealing with left-invariant functions. Since the operation is closed, it follows that the Lie-Poisson bracket is a Poisson Bracket. □

Now that we have derived a Poisson bracket when the system has a symmetry associated to a Lie group G, the next section will give concrete examples from mechanics.

6 Rigid Motion

6.1 The Lie Algebra of Rotations

Multiple physical system display rotational symmetries. In other words, the dynamics are left unchanged when the system has been rotated. For example, a spherically symmetric rigid body in the absence of external forces has this symmetry. Mathematically, rotations in 3 dimensional Euclidean space are represented by the group $SO(3)$ of **special orthogonal** 3×3 matrices with real entries. Formally, it is defined as

$$SO(3) = \left\{R \in \mathcal{M}_3(\mathbb{R}) \mid \det(R) = 1,\, R^{-1} = R^T\right\}$$

. We see by its definition that every element of this group is invertible since they have determinant equal to one and thus, $SO(3) \subset GL_3(\mathbb{R})$. We also may remark that SO(3) does indeed form a group, since for $A, B \in SO(3)$ we have

1. $\det(AB) = \det(A)\det(B) = 1$
2. $(AB)^{-1} = B^{-1}A^{-1} = B^T A^T = (AB)^T$

so it is closed under matrix multiplication. Furthermore, since $A^{-1} = A^T$, we have $\det(A^{-1}) = 1$ and $(A^{-1})^{-1} = (A^T)^{-1} = (A^{-1})^T$ thus $A^{-1} \in SO(3)$. Finally, the identity is in $SO(3)$ since $det(I_n) = 1$ and $I_3^{-1} = I_3 = I_3^T$. Thus $SO(3)$ is a matrix Lie group. We can thus use the framework of the last two sections to find its Lie algebra.

From theorem 4.16, the Lie algebra $\mathfrak{so}(3)$ of the matrix Lie group $SO(3)$ is

$$\mathfrak{so}(3) = \{A \in \mathcal{M}_3 \mid \exp(tA) \in SO(3)\ \forall t \in \mathbb{R}\}$$

So if A is an element of $\mathfrak{so}(3)$ then, as $\exp(tA) \in SO(3)$, we have

$$\exp(-tA) = \exp(tA)^{-1} = \exp(tA)^T = \left(\sum_{k=0}^{\infty} \frac{(tA)^k}{k!}\right)^T = \sum_{k=0}^{\infty} \frac{(tA^T)^k}{k!} = \exp\left(tA^T\right)$$

Differentiating the first and last terms and evaluating at $t = 0$, we obtain

$$-A\exp(-0 \cdot A) = A^T \exp\left(0 \cdot A^T\right) \iff -A = A^T$$

So we see that A must be skew symmetric. Conversely, if A is skew symmetric then $\exp(-tA) = \exp(tA)^T$ satisfying the first property of $SO(3)$. Furthermore, the following formula taken from [4] (section 7, property 4)

$$\det(\exp(A)) = \exp(\mathrm{Tr}(A))$$

gives that, since the diagonals of a skew symmetric matrix are 0, $\det(\exp(tA)) = 1$ and so $A \in \mathfrak{so}(3)$. So we conclude that

$$\mathfrak{so}(3) = \{A \in M_3(\mathbb{R}) \mid A^T = -A\}$$

which has a three dimensional basis consisting of

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \; E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

A straightforward computation of the matrix commutator forming the lie bracket gives:

$$[E_1\,, E_2] = E_3,\; [E_2\,, E_3] = E_1,\; [E_3\,, E_1] = E_2$$

We notice that if we identify the matrix E_i with the i^{th} standard basis vector e_i of $\mathbb{R}^3$, then $[e_i\,, e_j] = e_i \times e_j$. And so we conclude that

$$\mathfrak{so}(3) \cong \left(\mathbb{R}^3, [.\,,.] = \cdot \times \cdot\right)$$

With this in hand, we can give an expression for the Lie-Poisson bracket on $\mathfrak{so}(3)^*$. We recall from theorem 5.14, that the Lie-Poisson bracket on the dual of a Lie algebra is defined as

$$\{f\,, g\}(\mu) := -\langle \mu, [df_\mu\,, dg_\mu](\mu)\,\rangle$$

where $f, g \in \mathcal{F}(\mathfrak{g}^*)$. We derived above that $(\mathfrak{so}(3), [.\,,.]) \cong (\mathbb{R}^3, \times)$ and so $\mathfrak{so}(3)^*$ can be identified with $(\mathbb{R}^3)^*$. Since the standard basis of $\mathbb{R}^3$ is orthonormal, we may observe that

$$e_i \cdot e_j = \delta^i_j$$

and so the action of e^i is the same as that of "$e_i\cdot$" so we may treat any $\mu = \sum_{i=1}^3 \mu_i e^i \in \mathfrak{so}(3)^*$ as $\mu = \sum_{i=1}^3 \mu_i e_i \in \mathfrak{so}(3)$ keeping in mind that $\langle \mu, v\rangle = \mu \cdot v$ for any $v \in \mathbb{R}^3$. As in the remark earlier, we identify de^i with e_i. For a function $f \in \mathcal{F}(\mathfrak{so}(3)^*)$, which we identified with $\mathcal{F}(\mathbb{R}^3)$, this yields

$$df = \sum_{i=1}^3 \frac{\partial f}{\partial x^i} de^i \mapsto \sum_{i=1}^3 \frac{\partial f}{\partial x_i} e_i = \nabla f$$

We then get an equivalent expression for the bracket by

$$\{f\,, g\}(\mu) = -\langle \mu, [df_\mu\,, dg_\mu]\,\rangle = -\mu \cdot (\nabla f \times \nabla g)|_\mu$$

Remark Consider a free rigid body. Let $\mu = I\Omega \in \mathfrak{so}(3)^*$ be the angular momentum vector, where I is

the body's moment of inertia tensor and Ω is its angular velocity. Also, let H be its Hamiltonian (total energy of the system). Using the Poisson bracket derived just above let us evaluate the following:

$$\begin{aligned}\dot{\mu}_i &= \{\mu_i\,, H\}\,(\mu) \\ &= -\mu \cdot (\nabla\mu_i \times \nabla H) \\ &= \mu \cdot (\Omega \times \nabla\mu_i) \\ &= \nabla\mu_i \cdot (\mu \times \Omega) \\ &= e_i \cdot (\mu \times \Omega) = (\mu \times \Omega)_i\end{aligned}$$

where we used $\nabla H = \Omega$, $\nabla\mu_i = e_i$, and we flipped things around in the triple product. We can write this in vector form as

$$\dot{\mu} = \mu \times \Omega$$

which are the famous Euler equations for a rigid body.

6.2 Incorporating Translations

In the previous part, we derived the Lie-Poisson bracket associated with rotations. This was a first step in describing how a rigid body moves since in the center of mass frame, this is the only way of changing its state. However, to fully describe trajectories of such objects, we need to allow the center of mass to move in euclidean space. The motion of the center of mass is completely described by translations in $\mathbb{R}^3$. That is functions $T_a : \mathbb{R}^3 \to \mathbb{R}^3$, where $a \in \mathbb{R}^3$ acting as

$$T_a(v) = v + a \;\; \forall v \in \mathbb{R}^3$$

And thus the most general rigid motion is described by rotations and translations which are the elements of the **special euclidean** group, namely

$$SE(3) = \{T_{R,a} : \mathbb{R}^3 \to \mathbb{R}^3 \mid T_{R,a}(v) = Rv + a \;\; \forall v \in \mathbb{R}^3,\; R \in SO(3),\; a \in \mathbb{R}^3\}$$

They are called euclidean transformations since they preserve euclidean distance and angles between points in $\mathbb{R}^3$. Importantly, these are affine transformations rather than linear ones since the origin may be shifted, so the connection with matrix Lie groups is not obvious. Nevertheless, it is possible to use the previous machinery by homogenizing our coordinates. Analogous to how it is often useful to view $\mathbb{R}^2$ as the x, y plane

in $\mathbb{R}^3$, we will consider $\mathbb{R}^3$ to be a subspace of $\mathbb{R}^4$. We choose the transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longleftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix}$$

If we consider a block matrix defined with $R \in \mathcal{M}_3(\mathbb{R})$ and $\mathbf{0}, a \in \mathbb{R}^3$ by

$$\begin{pmatrix} R & a \\ \mathbf{0}^T & 1 \end{pmatrix}$$

Then we may compute for $v \in \mathbb{R}^3$

$$\begin{pmatrix} R & a \\ \mathbf{0}^T & 1 \end{pmatrix} \cdot \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} Rv + a \\ 1 \end{pmatrix}$$

And so, if $R \in SO(3)$, this is exactly a transformation in $SE(3)$ if we revert back to the original $\mathbb{R}^3$ space. This motivates an equivalent definition for $SE(3)$.

Proposition 6.1 The group $SE(3)$ of orientation-preserving euclidean transformations of three dimensional space when viewed in $\mathbb{R}^4$ as

$$SE(3) = \left\{ \begin{pmatrix} R & a \\ \mathbf{0}^T & 1 \end{pmatrix} \in M_4(\mathbb{R}) \;\middle|\; R \in SO(3),\, a \in \mathbb{R}^3 \right\}$$

forms a matrix Lie group.

Proof. Let $R, \tilde{R} \in SO(3)$ and $a, \tilde{a} \in \mathbb{R}^3$. Then,

$$\begin{pmatrix} R & a \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \tilde{R} & \tilde{a} \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} R\tilde{R} & R\tilde{a} + a \\ \mathbf{0}^T & 1 \end{pmatrix}$$

Evidently, $R\tilde{a} + a \in \mathbb{R}^3$ and, by the closedness of $SO(3)$ shown in section 6.1, $R\tilde{R} \in SO(3)$. Therefore $SE(3)$ is closed under matrix multiplication. Furthermore, since $SO(3)$ is a group, $R^{-1} = R^T \in SO(3)$ so

$$\begin{pmatrix} R^T & -R^T a \\ \mathbf{0}^T & 1 \end{pmatrix} \in SE(3)$$

And we have

$$\begin{pmatrix} R^T & -R^T a \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} R & a \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} R^T R & R^T a - R^T a \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} I_3 & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} = I_4$$

And thus every $A \in SE(3)$ is invertible and its inverse is in $SE(3)$. Hence $SE(3)$ is a subgroup of $GL_4(\mathbb{R})$

and a matrix Lie Group. □

With this in hand, we compute its lie algebra.

Proposition 6.2 The lie algebra $\mathfrak{se}(3)$ of $SE(3)$ can be represented by $\mathbb{R}^3 \times \mathbb{R}^3$ that is (η, v) with $\eta, v \in \mathbb{R}^3$ and lie bracket

$$[(\eta, v), (\xi, u)] = (\eta \times \xi, \eta \times u - \xi \times v)$$

Where η represents the rotation part of the motion i.e. an element of $\mathfrak{so}(3)$ and v represents the translation motion.

Remark We notice that taking only the first components, this simplifies to the lie algebra of $\mathfrak{so}(3)$ while the translation component gets mixed with rotation. This comes from the fact that rotations act on translation. That is, a translation done before or after a rotation will not give the same vector. This mixing between operations is evidence of a semidirect product $\mathfrak{se}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3$ where $\mathbb{R}^3$ has the trivial bracket $[u, v] = 0$ since translations commute with other translations. Similarly, as groups, we have $SE(3) \cong SO(3) \ltimes \mathbb{R}^3$ where $\mathbb{R}^3$ is the additive abelian group.

Proof. Let $\gamma(t)$ be a smooth curve on $SE(3)$, that is,

$$\gamma(t) = \begin{pmatrix} R(t) & a(t) \\ \mathbf{0}^T & 1 \end{pmatrix} \text{ with } R(t) \in SO(3),\ a(t) \in \mathbb{R}^3 \ \ \forall t$$

with the condition that

$$\gamma(0) = I_4 = \begin{pmatrix} I_3 & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

Then, we must have that $R(t)$ is a smooth curve in $SO(3)$ with $R(0) = I_3$ and so

$$R'(0) \in T_{I_3} SO(3) = \mathfrak{so}(3)$$

From our computations in section 6.1, we know that $R'(0) \in \mathfrak{so}(3) \iff R'(0)$ is skew symmetric. Additionally, $a(t)$ is a smooth curve in $\mathbb{R}^3$ with $a(0) = 0$ so $a'(0) \in T_0\mathbb{R}^3$ but since $\mathbb{R}^3$ is a vector space, its tangent space at any point is isomorphic to itself and so $a'(0)$ is any element of $\mathbb{R}^3$. Returning to the original matrix, we have

$$\gamma'(0) = \begin{pmatrix} R'(0) & a'(0) \\ \mathbf{0}^T & 0 \end{pmatrix} \in T_{I_4} SE(3) = \mathfrak{se}(3)$$

And so we reach

$$\mathfrak{se}(3) = \left\{ \begin{pmatrix} A & v \\ \mathbf{0}^T & 0 \end{pmatrix} \in \mathcal{M}_4 \;\middle|\; A \in \mathfrak{so}(3),\ v \in \mathbb{R}^3 \right\}$$

as usual, the lie bracket is the matrix commutator which we now compute. Let $A, B \in \mathfrak{so}(3)$ and $u, v \in \mathbb{R}^3$,

then let

$$E(A,u) := \begin{pmatrix} A & u \\ \mathbf{0}^T & 0 \end{pmatrix}$$

We then obtain

$$\begin{aligned}
[E(A,v), E(B,u)] &= E(A,v)\cdot E(B,u) - E(B,u)\cdot E(A,v) \\
&= \begin{pmatrix} A & v \\ \mathbf{0}^T & 0 \end{pmatrix}\begin{pmatrix} B & u \\ \mathbf{0}^T & 0 \end{pmatrix} - \begin{pmatrix} B & u \\ \mathbf{0}^T & 0 \end{pmatrix}\begin{pmatrix} A & v \\ \mathbf{0}^T & 0 \end{pmatrix} \\
&= \begin{pmatrix} AB & Au \\ \mathbf{0}^T & 0 \end{pmatrix} - \begin{pmatrix} BA & Bv \\ \mathbf{0}^T & 0 \end{pmatrix} \\
&= \begin{pmatrix} [A,B]_{\mathfrak{so}(3)} & Au - Bv \\ \mathbf{0}^T & 0 \end{pmatrix}
\end{aligned}$$

Now, we may recall from section 6.1 that $\mathfrak{so}(3)$ was represented by the standard basis on $\mathbb{R}^3$ with the cross product as its bracket$[.,.]_{\mathfrak{so}(3)}$. That is, suppose

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

Then we represent A with the vector

$$\eta := \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We may then notice that

$$Au = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} bu_3 - cu_2 \\ cu_1 - au_3 \\ au_2 - bu_1 \end{pmatrix} = \eta \times u$$

Thus, if $\eta, \xi \in \mathbb{R}^3$ are the vector representing A and B respectively, then $Au = \eta \times u$ and, if we identify $E(A,v) \in \mathfrak{se}(3)$ with $(\eta, v) \in \mathbb{R}^3 \times \mathbb{R}^3$, we get

$$[(\eta, v), (\xi, u)] = (\eta \times \xi, \eta \times u - \xi \times v)$$

□

We now proceed to obtain the Lie-Poisson bracket on $\mathfrak{se}(3)^*$. As per the previous treatment, an element

(η, u) of $\mathfrak{se}(3)^* \cong (\mathbb{R}^3)^* \times (\mathbb{R}^3)^*$ may be viewed as an element of $\mathbb{R}^3 \times \mathbb{R}^3$ where we define

$$\langle (\eta, u),\, (\xi, v) \rangle := \eta \cdot \xi + \, u \cdot v \;\; \forall (\xi,\, v) \in \mathfrak{se}(3)$$

Then, any function $f \in \mathcal{F}(\mathfrak{se}(3)^*)$ may be viewed as a function in $\mathcal{F}(\mathbb{R}^3 \times \mathbb{R}^3)$ with

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x_R^i} de_R^i + \sum_{i=1}^{3} \frac{\partial f}{\partial x_T^i} de_T^i$$

where the subscripts R, T indicate the rotation and translation part respectively, that is first and second entries. If we identify de_i with e_i, we then get

$$df = (\nabla_R f,\, \nabla_T f)$$

where ∇_i represents the three dimensional gradient in the respective Cartesian product entry. Now applying the definition in theorem 5.14 as well as the results of proposition 6.2, we get the following Lie-Poisson bracket

$$\begin{aligned} \{f\,, g\}\,(\Pi, \Gamma) &= -(\Pi, \Gamma) \cdot (\nabla_R(f) \times \nabla_R(g),\, \nabla_R(f) \times \nabla_T(g) - \nabla_R(g) \times \nabla_T(f))|_{(\Pi,\Gamma)} \\ &= -\Pi \cdot (\nabla_R(f) \times \nabla_R(g))|_{(\Pi,\Gamma)} + \Gamma \cdot (\nabla_R(f) \times \nabla_T(g) - \nabla_R(g) \times \nabla_T(f))|_{(\Pi,\Gamma)} \end{aligned} \tag{4}$$

for any $(\Pi, \Gamma) \in \mathfrak{se}(3)$ and $f, g \in \mathcal{F}(\mathfrak{se}(3)^*)$.

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