# Coxeter Groups and Root Systems

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**Abstract**

This paper provides an introduction to Coxeter groups. We study how one can build a faithful representation of Coxeter groups using a root system and, through this representation, highlight several remarkable results about the structure of such groups. We then explore the action of Coxeter groups on their Cayley graphs by introducing the concepts of walls and half-spaces, emphasizing their connection to the underlying root system. Finally, we discuss the Parallel Wall Theorem and its applications.

## 1 Introduction

The theory of Coxeter groups was introduced through the work of H.S.M. Coxeter, and has since become a fundamental object in numerous areas of mathematics. These groups can be understood in several different ways. This includes algebraic, geometric, and combinatorial approaches, each of which highlight different aspects of these groups and contribute to the richness and depth of the subject. In this paper, we focus primarily on the representation of Coxeter groups via root systems. Through this representation, we are able to build a concrete visualization of these groups allowing us to uncover many of its remarkable properties such as the Strong Exchange Condition.

**Organization**: We start with some preliminary material; In section 2, we define the free group, group presentations, Cayley graphs, and Coxeter groups. We emphasize how a Coxeter group $W$ can be thought of as a group generated by reflections which motivates the construction of a representation provided in section 3. After introducing an appropriate bilinear form, we were able to formally identify the generators of our group with linear transformations on a vector space $V$ resembling reflections. We then consider how this can be extended into a faithful representation $\sigma$ of $W$ by introducing a root system given by a set $\Phi \subset V$. In section 4, we shift our focus towards the action of $W$ on its Cayley graph and examine the relationship between this action with its action on $V$ given by $\sigma$. We study the associated geometry of walls and half-spaces in the Cayley graph which leads to the discussion and interpretation of the Parallel Wall Theorem and its consequences. In particular, we note that this theorem is used to show that Coxeter groups are automatic.

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## 2 Preliminaries

### 2.1 Free Groups and Presentations

Let $S$ be a set, called an **alphabet**, whose elements are called **letters**. We consider the set $S^{-1} = \{s^{-1} : s \in S\}$ consisting of the formal inverses of the letters in $S$. By convention, we say $(s^{-1})^{-1} = s$ for any $s \in S$.

A **word** over $S \cup S^{-1}$ is a finite string of letters in $S \cup S^{-1}$. We may denote the empty word by $e$. Such a word is **reduced** if it does not contain any occurrences of $ss^{-1}$ or $s^{-1}s$ for $s \in S$.

Let $F(S)$ be the set of reduced words over $S \cup S^{-1}$. We now define a binary operation on $F(S)$ as follows: for any two words $w, w' \in F(S)$, their product $ww'$ is the word obtained by concatenating $w$ and $w'$, and then deleting any occurrences of $ss^{-1}$ or $s^{-1}s$ that appear in the resulting word. The latter process is called **reduction**.

**Proposition 2.1.** *The binary operation on $F(S)$ of concatenation and reduction is associative. More precisely, if for a given word $w$ over $S \cup S^{-1}$, there exists two sequences of reductions yielding reduced words $w_1, w_2 \in F(S)$, then $w_1 = w_2$.*

*Proof.* We proceed by induction of the length of $w$ (this being the number of letters in $w$). If $w$ has length 0 or 1, $w$ is either empty or equal to some $s \in S$. In both cases, $w$ is already reduced. Now suppose we have two sequences of reductions $w \to w_1$ and $w \to w_2$ as in the statement of the proposition. We may write $w = s_1 \cdots s_i s_{i+1} \cdots s_n$ with $s_1, \ldots, s_n \in S \cup S^{-1}$ where $s_i = s_{i+1}^{-1}$, and we can assume the first sequence of reductions starts with the removal of $s_i s_{i+1}$. Let $w_1'$ denote the result of removing $s_i s_{i+1}$ from $w$. Similarly, we have $w = s_1 \cdots s_j s_{j+1} \cdots s_n$ with $s_j = s_{j+1}^{-1}$ where the second sequence of reductions starts with the removal of $s_j s_{j+1}$ giving $w_2'$. If the subwords $s_i s_{i+1}$, $s_j s_{j+1}$ of $w$ do not overlap, then $w_1'$ contains the subword $s_j s_{j+1}$, and $w_2'$ contains the subword $s_i s_{i+1}$. Then, removing $s_j s_{j+1}$ from $w_1'$ and $s_i s_{i+1}$ from $w_2'$ will result in the same word $w_3'$. If $w_3' \to w_3$ is a sequence of reductions with $w_3 \in F(S)$, we apply the induction hypothesis to the pair of sequences $w_1' \to w_1$ and $w_1' \to w_3' \to w_3$ to obtain $w_1 = w_3$. Similarly, $w_3 = w_2$ which gives $w_1 = w_2$. If the subwords $s_i s_{i+1}$ and $s_j s_{j+1}$ overlap, it follows that $s_i = s_j^{-1}$ and $w_1' = w_2'$. By induction, we obtain $w_1 = w_2$. ☐

With this, it's easy to see that $F(S)$ forms a group under this operation where $e$ is the identity element and $(s_1 \cdots s_n)^{-1} = s_n^{-1} \cdots s_1^{-1}$. We call $F(S)$ the **free group** on the alphabet $S$ [5].

Note that there is an alternative definition of the free group on the set $S$ which is equivalent to our previous definition. It is called the universal property of the free group and it is stated as follows.

**Definition 2.1.** *Let $F$ be a group and $S \subset F$. We say that $F$ is a free group on $S$ if for all groups $G$ and functions $f : S \to G$ there exists a unique homomorphism $\varphi : F \to G$ such that $\varphi(s) = f(s)$ for all $s \in S$. In this case we write $F = F(S)$.*

This definition implies that every group is a quotient of a free group. Indeed, if $G$ is a group with generating set $S$, we can see that there is a surjective homomorphism $\varphi : F(S) \to G$ where $\varphi(s) = s$. By the first isomorphism theorem, we obtain

$$F(S)/Ker\varphi \cong G.$$

Let $S$ be an alphabet and let $R \subset F(S)$. We will denote $\langle\langle R \rangle\rangle$ to be the smallest normal subgroup of $F(S)$ containing $R$. Formally,

$$\langle\langle R \rangle\rangle = \bigcap_{R \subset H \trianglelefteq F(S)} H.$$

Then, the group **presented** by $S$ and $R$ is given by

$$\langle S | R \rangle := F(S)/\langle\langle R \rangle\rangle.$$

The elements of $S$ are called the **generators** and the elements of $R$ are called the **relations** of $\langle S | R \rangle$.

**Example 2.1.** Let $a = (1, 0), b = (0, 1)$. Then, $\{a, b\}$ forms a generating set for the group $\mathbb{Z}^2$ under addition. In fact, we have $\mathbb{Z}^2 \cong \langle a, b \,|\, aba^{-1}b^{-1} \rangle$.

## 2.2 Cayley Graphs

If $G$ is a group and $S \subset G$ a generating set for $G$, one can construct a graph called the **Cayley Graph** of $G$ with respect to $S$, denoted $\mathrm{Cay}(G, S)$. In this graph, $G$ corresponds to the vertex set, and for each

$g \in G$, $s \in S$ we draw the directed edge with initial point $g$ and terminal point $gs$ and we label the edge by $s$ (or color the edge appropriately). Hence, all of the edges in $\mathrm{Cay}(G, S)$ will be of the form $(g, gs)$.

**Example 2.2.** Using the same generators for $\mathbb{Z}^2$ as in the previous example, we can construct part of $\mathrm{Cay}(\mathbb{Z}^2, \{a, b\})$ as illustrated below. The red edges correspond to $a$ and the blue edges to $b$.

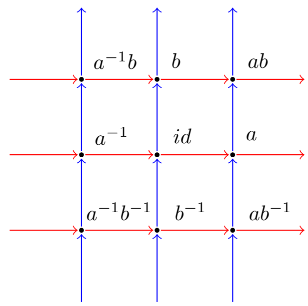



Figure 1: $\mathrm{Cay}(\mathbb{Z}^2, \{a, b\})$

With this, we can discuss some special properties of $\mathrm{Cay}(G, S)$. Let $d : G \times G \to \mathbb{R}$ be the function sending the pair $(g, h)$ to the length of the shortest path in $\mathrm{Cay}(G, S)$ from $g$ to $h$ (where each edge has length 1). Then it is easy to verify that $d$ is a metric on $G$ [5].

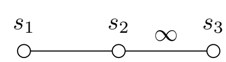
Another key observation is that there is an action of $G$ on $\mathrm{Cay}(G, S)$ by left multiplication. More precisely, each $x \in G$ will send the vertex $g$ to the vertex $xg$. Consequently, $x$ sends the edge $(g, gs)$ to the edge $(xg, xgs)$ which preserves the structure of $\mathrm{Cay}(G, S)$.

## 2.3 Coxeter Groups

Let $S$ be a set. A **Coxeter matrix** is a symmetric matrix $m : S \times S \to \{1, 2, \ldots \infty\}$ such that for all $s, t \in S$, $m(s, t) = 1$ if and only if $s = t$. Alternatively, $m$ can be represented by a **Coxeter diagram** in which $S$ is the set of vertices and for each unordered pair $\{s, t\} \in S$ with $m(s, t) \geq 3$, there is an edge connecting $s, t$. Moreover, such as egde is labelled by $m(s, t)$ whenever $m(s, t) \geq 4$.

**Example 2.3.**

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & \infty \\ 2 & \infty & 1 \end{bmatrix} \longleftrightarrow$$




Given a Coxeter matrix $m$ or a Coxeter diagram, define the group $W := \langle S | R \rangle$ where

$$R = \{(st)^{m(s,t)} : s, t \in S,\ m(s, t) < \infty\}.$$

The relations in $R$ are called **braid relations**, and this presentation is called a **Coxeter presentation**. Any group $W$ that admits such a presentation is called a **Coxeter group**. In this case, we call $S$ the set of **Coxeter generators**, and $|S|$ the **rank** of $W$ [2].

In such a group, we notice that $(ss)^{m(s,s)} = id$ for each $s \in S$. But since $m(s, s) = 1$, we obtain $s^2 = id$. Intuitively, one may think of the group $W$ as *generated by reflections*. From this perspective, each generator $s \in S$ corresponds to a reflection, and the product of two generators may correspond to a composition of two reflections. If $m(s, t) = \infty$, $st$ can be regarded as a translation which has infinite order. On the other hand, if $m(s, t) < \infty$, we would like to interpret $st$ as a rotation of finite order.

Note that that the braid relation $(st)^{m(s,t)} \in R$ tells us that the order of $st$ in $W$ is *at most* $m(s, t)$. In the next section, we will construct a representation of $W$ that associates $st$ with a rotation of order exactly

$m(s,t)$, allowing us to conclude that the order of $st$ in $W$ is also $m(s,t)$.

As an example, let's consider the Coxeter group $W$ corresponding to the matrix and diagram from our last example. We have

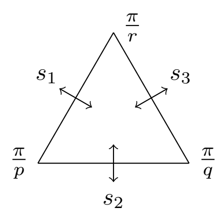
$$W = \langle\, s_1,\, s_2,\, s_3 \,|\, s_1^2,\, s_2^2,\, s_3^2,\, (s_1s_3)^2,\, (s_1s_2)^3 \,\rangle.$$

In this case, $W$ has rank 3 where $s_1s_3$, and $s_1s_2$ will be rotations of order 2 and 3 respectively, and $s_2s_3$ will be a translation.

One that note, there is actually a full classification of all the rank 3 Coxeter groups; If $W$ takes the general form of a rank 3 Coxeter group, namely

$$W = \langle s_1, s_2, s_3 \,|\, s_1^2, s_2^2, s_3^2, (s_1s_2)^p, (s_2s_3)^q, (s_1s_3)^r \rangle$$

for $p, q, r \geq 2$, then we can consider the three reflection lines of $s_1, s_2, s_3$. For the corresponding rotations to have the desired order, we find that each pair of these lines must meet at the appropriate acute angle forming a triangle as in the figure.




In the Euclidean plane, we must have $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} = \pi$ or equivalently, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. This leaves finitely many possibilities, each of which associated $W$ with a symmetry group of the tiling of the Euclidean plane generated by the reflections in the triangle. Alternatively, we could consider the Hyperbolic plane or the sphere whereby the main condition becomes $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$ and $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} > \pi$ respectively. $W$ is then associated with the symmetry group of tilings of the Hyperbolic plane and of the sphere.

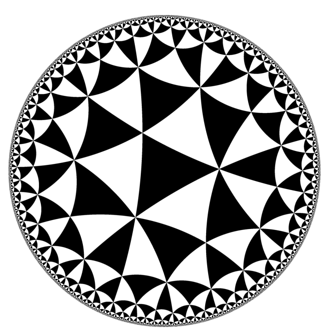
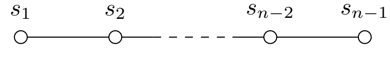


Figure 2: The tiling of the hyperbolic plane for $(p, q, r) = (3, 3, 4)$

Another example to consider is the symmetric group $S_n$. This group is generated by simple transpositions $s_i := (i\ i+1)$ for $i = 1, \ldots, n-1$ and therefore has rank $n-1$. Its diagram will take the following form.




As for the rank 3 symmetric group, $S_4$, its Cayley graph is actually given by the *permutahedron* which is the truncation of the regular octahedron.

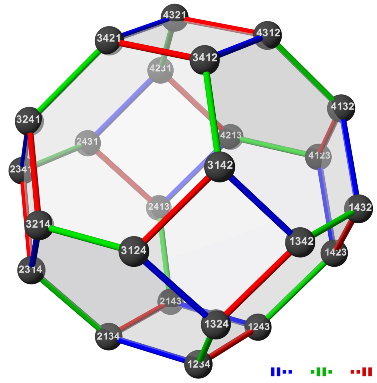



Figure 3: Cay($S_4$, {(12), (23), (34)})

Before moving forward, we highlight a special remark about Coxeter groups that will be useful in what follows [2]. It will provide us with sufficient conditions for constructing a homomorphism from $W$ to a group $G$.

**Remark 2.1.** (Universal Property) *Let $G$ be a group and $f : S \to G$ a function satisfying*

$$(f(s)f(t))^{m(s,t)} = id$$

*for all $s, t \in S$ such that $m(s,t) < \infty$. Then, there exists a unique group homomorphism $\varphi : W \to G$ such that $\varphi(s) = f(s)$ for all $s \in S$.*

We will illustrate this property with an important example which allows us to define the *sign* of an element $w \in W$, much like we would for permutations groups $S_n$. We let $G = \{\pm 1\}$ be the multiplicative group of order 2. Define $f : S \to \{\pm 1\}$ by letting $f(s) = -1$ for all $s \in S$. Then, for $m(s,t) < \infty$, we have

$$(f(s)f(t))^{m(s,t)} = (-1)^{2m(s,t)} = 1,$$

as desired. So, $f$ can be extended into a unique homomorphism $\varepsilon : W \to \{\pm 1\}$ called the **sign homomorphism** of $W$. Furthermore, since $S$ generates $W$, any $w \in W$ can be written $w = s_1 \cdots s_n$ where $s_i \in S$ are not necessarily distinct. Then it is easy to check that $\varepsilon(w) = (-1)^n$.

In fact, it will be particularly useful to consider the *smallest* such $n$; Let $\ell : W \to \mathbb{R}$ be the function assigning to $w$ the smallest $n$ such that $w = s_1 \cdots s_n$ for $s_i \in S$. We call $\ell(w)$ the **length** of $w$. We observe that $\ell(s) = 1$ if and only if $s \in S$, and we say $\ell(id) = 0$ as a convention. As it turns out, $\ell(w) = d(id, w)$ where $d$ is the metric on Cay($W, S$) introduced earlier.

We end this section with a list of elementary results about the length function given by [4].

L1. $\ell(w) = \ell(w^{-1})$

L2. $\ell(ww') \leq \ell(w) + \ell(w')$

L3. $\ell(w) - \ell(w') \leq \ell(ww')$

L4. $\ell(w) - 1 \leq \ell(ws) \leq \ell(w) + 1$

*Proof.* (L1) If $\ell(w) = n$, then we may write $w = s_1 \cdots s_n$ where $s_i \in S$. Then, $w^{-1} = s_n^{-1} \cdots s_1^{-1} \implies \ell(w^{-1}) \leq \ell(w)$. Similarly, $\ell(w) \leq \ell(w^{-1})$. (L2) If $w = s_1 \cdots s_n$ and $w' = s'_1 \cdots s'_k$, then $ww' = s_1 \cdots s_n s'_1 \cdots s'_k$ has length at most $n + k$. (L3) follows from L1 and L2. (L4) follows from L2 and L3. In fact, one can use our sign homomorphism $\varepsilon$ to obtain a stronger statement, namely that $\ell(ws) \in \{\ell(w) - 1, \ell(w) + 1\}$. $\square$

# 3 A Geometric representation of $W$

In this section, we will spend some time building a concrete representation of $W$ as a group generated by reflections as in [4]. To do this, we will allow a more general definition of a reflection. That is, a linear transformation that will send a vector $v$ to $-v$ and preserve all the points belonging to some hyperplane.

Let $V$ be a vector space over $\mathbb{R}$ such that $\dim(V) = |S|$. We can pick a basis $\{\alpha_s : s \in S\}$ indexed by the elements in $S$. We would like to endow $V$ with a geometry that allows us to interpret the "angle" between basis vectors $\alpha_s, \alpha_t$ in terms of $m(s,t)$. For this, we introduce a symmetric bilinear form $B$ on $V$ by specifying its values on the basis vectors and extending by linearity to a map $B : V \times V \to \mathbb{R}$. For all $s, t \in S$ with $m(s,t) < \infty$, we let

$$B(\alpha_s, \alpha_t) := -cos\frac{\pi}{m(s,t)}.$$

When $m(s,t) = \infty$, we say $B(\alpha_s, \alpha_t) = -1$. Some immediate remarks about $B$ are that $B(\alpha_s, \alpha_s) = 1$, and when $s \neq t$ we have $B(\alpha_s, \alpha_t) \leq 0$ since $0 \leq \frac{\pi}{m(s,t)} \leq \frac{\pi}{2}$. In particular, when $m(s,t) = 2$ we obtain $B(\alpha_s, \alpha_t) = 0$.

Next, we will use this bilinear form to define a reflection in $V$. Given some $s \in S$, we let $\sigma_s : V \to V$ be defined by

$$\sigma_s(v) = v - 2B(v, \alpha_s)\alpha_s$$

for all $v \in V$. Since $B$ is bilinear, it follows that $\sigma_s$ is linear. From our definition of $\sigma_s$, we can see that $\sigma_s(\alpha_s) = -\alpha_s$. Moreover, if we consider the orthogonal complement with respect to $B$ of the subspace spanned by $\alpha_s$, namely $H_s := \{v \in V : B(v, \alpha_s) = 0\}$, we notice that $\sigma_s$ will fix this space pointwise. Some crucial properties of $\sigma_s$ are stated as propositions in what follows. The proofs are straightforward and computational, yet we provide them for clarity.

**Proposition 3.1.** *$\sigma_s \in GL(V)$ has order 2.*

*Proof.* Clearly, $\sigma_s \neq id$ since it sends a non zero vector $\alpha_s$ to its negative. For any $v \in V$, we have

$$\begin{aligned}
\sigma_s(\sigma_s(v)) &= \sigma_s(v) - 2B(\sigma_s(v), \alpha_s)\alpha_s \\
&= v - 2B(v, \alpha_s)\alpha_s - 2B(v - 2B(v, \alpha_s)\alpha_s, \alpha_s)\alpha_s \\
&= v - 2B(v, \alpha_s)\alpha_s - 2B(v, \alpha_s)\alpha_s + 2B(B(v, \alpha_s)\alpha_s, \alpha_s)\alpha_s \\
&= v - 4B(v, \alpha_s)\alpha_s + 4B(v, \alpha_s)B(\alpha_s, \alpha_s)\alpha_s \\
&= v - 4B(v, \alpha_s)\alpha_s + 4B(v, \alpha_s)\alpha_s \\
&= v.
\end{aligned}$$

□

**Proposition 3.2.** *$\sigma_s$ preserves the bilinear form $B$ on $V$. More precisely, for $v, u \in V$, we have $B(\sigma_s(v), \sigma_s(u)) = B(v, u)$.*

*Proof.*

$$\begin{aligned}
B(\sigma_s(v), \sigma_s(u)) &= B(v - 2B(v, \alpha_s)\alpha_s, u - 2B(u, \alpha_s)\alpha_s) \\
&= B(v, u) - B(v, 2B(u, \alpha_s)\alpha_s) - B(2B(v, \alpha_s)\alpha_s, u) + B(2B(v, \alpha_s)\alpha_s, 2B(u, \alpha_s)\alpha_s) \\
&= B(v, u) - 2B(v, \alpha_s)B(u, \alpha_s) - 2B(v, \alpha_s)B(\alpha_s, u) + 4B(v, \alpha_s)B(u, \alpha_s)B(\alpha_s, \alpha_s) \\
&= B(v, u) - 4B(v, \alpha_s)B(u, \alpha_s) + 4B(v, \alpha_s)B(u, \alpha_s) \\
&= B(v, u).
\end{aligned}$$

□

Our next focus is to argue that $s \mapsto \sigma_s$ can be extended into a unique homomorphism from $W$ to $GL(V)$. Naturally, we will appeal to the Universal Property stated in the previous section. It would then remain to show that $(\sigma_s\sigma_t)^{m(s,t)} = id$ whenever $m(s,t) < \infty$. However, if we want this homomorphism to be injective, we should want to show an even stronger result, namely

1. If $m(s,t) < \infty$, then $\sigma_s\sigma_t$ has order $m(s,t)$.
2. If $m(s,t) = \infty$, then for all $k$ $(\sigma_s\sigma_t)^k \neq id$.

This way, we are preventing some non trivial elements in $W$ from lying in the kernel. By Proposition 3.2, we can assume for the moment $s \neq t$. For simplicity, let $m = m(s,t)$. We will start by restricting our view to a two dimensional subspace of $V$, namely $U := \text{Span}(\alpha_s, \alpha_t)$.

We would like to find the order of $\sigma_s\sigma_t$ in $U$. Such a task is possible, since $U$ is invariant under both $\sigma_s$ and $\sigma_t$ (this should be clear by looking at the definition). Hence, the restrictions of these reflections to $U$ are well defined.

Examining the behavior of $B$ on this space shows that for all $u \in U$, $B(u,u) \geq 0$. Indeed, if $u \in U$, we can write $u = a\alpha_s + b\alpha_t$ for $a, b \in \mathbb{R}$ and we have

$$\begin{aligned} B(u,u) &= B(a\alpha_s + b\alpha_t, a\alpha_s + b\alpha_t) \\ &= a^2 B(\alpha_s,\alpha_s) + 2abB(\alpha_s,\alpha_t) + b^2 B(\alpha_t,\alpha_t) \\ &= a^2 - 2ab\, cos\tfrac{\pi}{m} + b^2. \end{aligned}$$

If $m < \infty$, this is precisely the law of cosines; we have $c^2 = a^2 - 2ab\,cos\theta + b^2$ where $\theta = \frac{\pi}{m}$ is the acute angle in a triangle and, $c$ is the length of the opposite side. Of course, we have $c^2 \geq 0$. In fact, $c^2 > 0$ since $\frac{\pi}{m} \neq 0$. On the other hand, if $m = \infty$, $B(u,u) = a^2 - 2ab + b^2 = (a-b)^2 \geq 0$ with equality possibly when $u \neq 0$. In short, the bilinear form $B : U \times U \to \mathbb{R}$ is *positive definite* when $m < \infty$ and *semidefinite* when $m = \infty$. We will treat both cases separately.

The simplest and maybe most familiar case is when $m < \infty$, as $B$ becomes an inner product on $U \cong \mathbb{R}^2$. With respect to some orthonormal basis for $U$, $\alpha_s, \alpha_t$ can be interpreted as unit vectors, since $B(\alpha_s,\alpha_s) = B(\alpha_t,\alpha_t) = 1$. It then follows that $B(\alpha_s,\alpha_t)$ is the cosine of the angle between $\alpha_s, \alpha_t$. Since $-cos\frac{\pi}{m} = cos(\pi - \frac{\pi}{m})$, $\alpha_s, \alpha_t$ meet at an angle of $\pi - \frac{\pi}{m}$. Furthermore, the reflection lines for $\sigma_s$, $\sigma_t$ are orthogonal to $\alpha_s, \alpha_t$ respectively. A short geometric argument shows that these reflection lines must meet at an angle of $\frac{\pi}{m}$. In turn, this means that $\sigma_s\sigma_t$ is a rotation by an angle of $\frac{2\pi}{m}$ which then implies that the order of $\sigma_s\sigma_t$ in $U$ is exactly $m$.

Does this imply that the order of $\sigma_s\sigma_t$ on $V$ is also $m$? Since $B$ is an inner product on $U$, we have that $V = U \oplus U^\perp$, where $U^\perp = \{v \in V : B(v,\alpha_s) = B(v,\alpha_t) = 0\}$. Then, it is easy to see that both $\sigma_s$ and $\sigma_t$ fix $U^\perp$ pointwise. Consequently, $\sigma_s\sigma_t$ also fixes $U^\perp$ pointwise. Since $\sigma_s\sigma_t$ has order $m$ in $U$ and acts trivially on its orthogonal complement, it follows that its order is $m$ on all of $V$.

Now, we consider the case when $m = \infty$. Remember that we want to show that $(\sigma_s\sigma_t)^k \neq id$ for all $k$. It suffices to show that for all $k$ there exists some $v \in V$ such that $(\sigma_s\sigma_t)^k(v) \neq v$. Let $v := \alpha_s + \alpha_t$. Since $B(\alpha_s,\alpha_t) = -1$, we notice that $B(v,\alpha_s) = B(v,\alpha_t) = 0$. We will leave it for you to verify that $(\sigma_s\sigma_t)^k(v) = 2kv + \alpha_s$ for all $k$, and that $2kv + \alpha_s \neq v$ when $k \geq 1$. Hence, $\sigma_s\sigma_t$ has infinite order on $V$.

We can now apply the universal property to obtain the following result.

**Proposition 3.3.** *There is a unique homomorphism $\sigma : W \to GL(V)$ such that $\sigma(s) = \sigma_s$. Moreover, the subgroup $\sigma(W)$ of $GL(V)$ preserves the bilinear form $B$ on $V$.*

As of yet, we do not know if $\sigma$ is an injective homomorphism. Proving this will require some more work in the section that follows.

## 3.1 Root systems

Here, we return to the length function $\ell$ introduced earlier. We want to look further into the fourth property listed about $\ell$. Namely, that $\ell(ws) \in \{\ell(w)-1, \ell(w)+1\}$. Using the action of $W$ on $V$ that we've built, we will develop precise conditions for when $\ell(ws)$ is greater or less than $\ell(w)$.

To avoid clutter, let us write $w(v)$ instead of $\sigma(w)(v)$ for $w \in W$ and $v \in V$. We define the set $\Phi := \{w(\alpha_s) : w \in W, s \in S\}$ called the **root system** of $W$ and we call its elements **roots**. Note that, since all $\alpha_s$ are unit vectors and since $W$ preserves $B$, all vectors in $\Phi$ are unit vectors. Given some $\alpha \in \Phi$, we can write $\alpha$ as a linear combination of our basis vectors

$$\alpha = \sum_{s \in S} c_s \alpha_s$$

for uniquely determined coefficients $c_s \in \mathbb{R}$. If $c_s \geq 0$ for all $s$, we say that $\alpha$ is **positive** and we write $\alpha > 0$. Similarly, of $c_s \leq 0$ for all $s$, we say that $\alpha$ is **negative** and we write $\alpha < 0$. Let $\Phi^+ := \{\alpha \in \Phi : \alpha > 0\}$ and $\Phi^- := \{\alpha \in \Phi : \alpha < 0\}$. We will soon be able to see how these two sets are related to $\Phi$. The following theorem and proof is based on that of [4].

**Theorem 3.1. (Tits)** *Let $s \in S$ and $w \in W$. Then,*

$$\ell(ws) > \ell(w) \iff w(\alpha_s) > 0.$$

**Remark 3.1.** *It suffices to prove the 'only if' direction. Assuming we've done so, we can switch the roles of $w$ and $ws$, we get $\ell(ws) < \ell(w)$ implies $ws(\alpha_s) > 0$, where $ws(\alpha_s) = w(-\alpha_s) = -w(\alpha_s)$ which means $w(\alpha_s) < 0$. We will have $\ell(ws) < \ell(w) \implies w(\alpha_s) < 0$ which is the contrapositive of the 'if' direction.*

*Proof.* We will proceed by induction on $\ell(w)$. If $\ell(w) = 0$, then $w = id$ which means for all $s \in S$, $\ell(ws) = 1 > \ell(w)$. Since $\sigma(w) = id$ and $\alpha_s$ is a basis vector, we have $w(\alpha_s) = \alpha_s > 0$, so the result holds.

Now we assume $\ell(ws) > \ell(w) > 0$. We can find some $t \in S$ such that $\ell(wt) = \ell(w) - 1$ (this is always possible, since we can simply take $t$ to be the rightmost letter appearing in $w$ as a reduced word). It follows that $s \neq t$. We let $I := \{s, t\}$ and consider the subgroup of $W$ generated by $I$

$$W_I := \langle s, t \,|\, s^2, t^2, (st)^{m(s,t)} \rangle \leq W.$$

We note that $W_I$ is a Coxeter group in its own right. Accordingly, this Coxeter group will have its *own* length function, $\ell_I$. In particular, we can see that when $w \in W_I$, we have $\ell(w) \leq \ell_I(w)$ since $I \subset S$. Consider the following set

$$A := \{h \in W \,|\, h^{-1}w \in W_I,\ \ell(h) + \ell_I(h^{-1}w) = \ell(w)\} \subset wW_I.$$

Let $h \in A$ be such that $\ell(h)$ is minimal, and set $h_I := h^{-1}w \in W_I$. So, we have $\ell(h) + \ell_I(h_I) = \ell(w)$. Since $(wt)^{-1}w = t \in W_I$ and $\ell(wt) + \ell_I(t) = \ell(wt) - 1 + 1 = \ell(w)$, we obtain $wt \in A$. By the minimality of $\ell(h)$, we get $\ell(wt) \geq \ell(h)$. But since $\ell(wt) = \ell(w) - 1$, we can apply the induction hypothesis on $\ell(h)$ using the same $s$ as for $w$. For this, we need that $\ell(hs) > \ell(h)$. Or equivalently, $\ell(hs) = \ell(h) + 1$.

Suppose for contradiction that $\ell(hs) = \ell(h) - 1$. Then,

$$\begin{aligned}
\ell(w) &\leq \ell(hs) + \ell((hs)^{-1}w) && \text{By L2 from section 2.3} \\
&\leq \ell(hs) + \ell_I((hs)^{-1}w) && \text{Since } (hs)^{-1}w = sh_I \in W_I \text{ and } \ell \leq \ell_I \\
&= \ell(h) - 1 + \ell_I(sh_I) && \text{By assumption} \\
&\leq \ell(h) - 1 + 1 + \ell(h_I) && \text{Since } \ell_I(sh_I) \leq 1 + \ell_I(h_I) \\
&= \ell(h) + \ell_I(h_I) \\
&= \ell(w)
\end{aligned}$$

So, each inequality is actually an equality. This implies that $\ell(w) = \ell(hs) + \ell_I((hs)^{-1}w)$ which means $hs \in A$, contradicting the minimality of $\ell(h)$.

Therefore, we have $\ell(hs) > \ell(h)$ as desired. By induction, $h(\alpha_s) > 0$. Analogously, $h(\alpha_t) > 0$. Remember the goal is to show that $w$ maps $\alpha_s$ to a positive root. But since $w = hh_I$, it suffices to show that $h_I$ maps $\alpha_s$ to a positive root (or more precisely, a linear combination of $\alpha_s, \alpha_t$ with nonnegative coefficients).

Recall that the reduced word $h_I \in W_I$ consists of only $s, t$. We claim that the rightmost letter of $h_I$ is $t$. Otherwise, $h_I$ would end with $s$ which implies that $\ell_I(h_I s) \leq \ell_I(h_I)$. This gives

$$\begin{aligned}
\ell(ws) &= \ell(hh^{-1}ws) \\
&\leq \ell(h) + \ell(h^{-1}ws) && \text{By L2.} \\
&= \ell(h) + \ell(h_I s) \\
&\leq \ell(h) + \ell_I(h_I s) && \text{Since } h_I s \in W_I \text{ and } \ell \leq \ell_I \\
&\leq \ell(h) + \ell_I(h_I) && \text{By assumption} \\
&= \ell(w).
\end{aligned}$$

This contradicts $\ell(ws) > \ell(w)$. So, our claim is true. Consequently, the reduced word $h_I$ simply corresponds to a string starting with either $s$ or $t$ and goes back and forth between $s, t$ and ending with $t$. With this in mind, we have two cases to consider. Again we let $m = m(s, t)$.

1. If $m = \infty$, we can use the fact that $B(\alpha_s, \alpha_t) = -1$ to compute

$$\begin{aligned}
t(\alpha_s) &= \alpha_s + 2\alpha_t, \\
st(\alpha_s) &= 2\alpha_t + 3\alpha_s, \\
tst(\alpha_s) &= 3\alpha_s + 4\alpha_t, \text{ and so on...}
\end{aligned}$$

   It then follows that $h_I(\alpha_s) > 0$.

2. If $m < \infty$, then $\ell_I(h_I) \leq 2m - 1$. This is because a string of $s, t$ with length greater or equal to $2m$ can always be simplified to something smaller since $(st)^m = id$ where $(st)^m$ is a string of length $2m$. In fact, one can use the relation $(st)^m = 1$ to reduce any such string to one of length $\leq m$. And, the assumption that $\ell_I(h_I s) \geq \ell_I(h_I)$ actually gives a tighter upper bound of $m - 1$.
   Let's consider an explicit example and run through the necessary computations. Let $m = 3$. Using the relation $ststst = id$, every word in $s, t$ can be reduced to one of the following forms;

$$\begin{aligned}
t &= ststs, \\
st &= tsts, \\
tst &= sts, \\
stst &= ts, \\
tstst &= s.
\end{aligned}$$

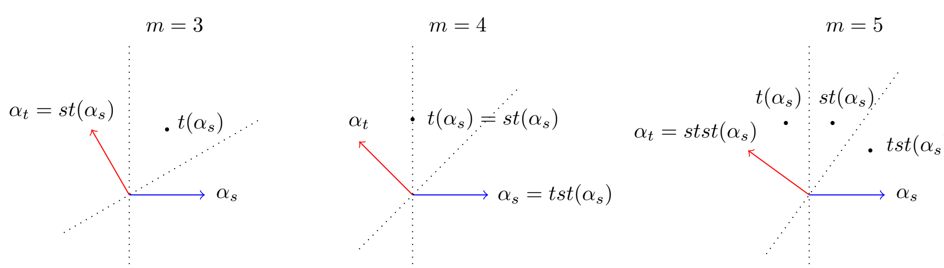
   Since $h_I$ must end with $t$, we consider the five words on the left. Notice that if $h_I = tst$ then $\ell_I(h_I s) \geq \ell_I(h_I)$ will not hold. This is because we can use the corresponding equality above to get $\ell_I(h_I s) = \ell_I((tst)s) = \ell_I((sts)s) = \ell_I(st) = 2 < \ell_I(h_I)$. Similarly, we cannot have $h_I = stst$ or

$h_I = tstst$. Hence, $h_I \in \{t, st\}$ where

$$t(\alpha_s) = \alpha_s - 2B(\alpha_s, \alpha_t)\alpha_t$$
$$= \alpha_s + 2cos\tfrac{\pi}{3}\alpha_t$$
$$= \alpha_s + \alpha_t,$$

$$st(\alpha_s) = s(\alpha_s + \alpha_t)$$
$$= s(\alpha_s) + s(\alpha_t)$$
$$= -\alpha_s + \alpha_t - 2B(\alpha_s, \alpha_t)\alpha_s$$
$$= (2cos\tfrac{\pi}{3} - 1)\alpha_s + \alpha_t$$
$$= \alpha_t,$$

are both positive.
For general $m$, we can make a geometric argument. Recall we are working in the Euclidean plane where $\alpha_s, \alpha_t$ are unit vectors meeting at an angle of $\pi - \frac{\pi}{m}$ and $st$ is a rotation by an angle of $\frac{2\pi}{m}$ starting from $\alpha_s$ going towards $\alpha_t$. You will find that the maximum number of rotations occurs when $h_I = (st)^k$ where $k = \lfloor \frac{m-1}{2} \rfloor$. In this case, $\alpha_s$ is rotated by a total angle $\leq \frac{2\pi}{m}\frac{m-1}{2} = \pi - \frac{\pi}{m}$ and hence, $h_I(\alpha_s)$ will remain in the cone bounded by the positive rays determined by $\alpha_s, \alpha_t$. Otherwise, $\alpha_s$ is first reflected by $t$ where $t(\alpha_s)$ will make an angle of $\frac{\pi}{m}$ with $\alpha_t$. Then there will be at most $\lfloor \frac{m-2}{2} \rfloor$ rotations by $\frac{2\pi}{m}$, leaving its angle with $\alpha_s$ at most $\frac{\pi}{m} + \frac{2\pi}{m}\frac{m-2}{2} = \pi - \frac{\pi}{m}$, again, leaving $h_I(\alpha_s)$ in the appropriate cone, as desired.




∎

**Corollary 3.1.** *The representation $\sigma : W \to GL(V)$ is faithful i.e. $ker\sigma = \{id\}$.*

*Proof.* Otherwise, we would have some non trivial $w \in ker\sigma$. By taking the rightmost $s$ is the reduced expression for $w$, we have $\ell(ws) < \ell(w)$. By Remark 3.1, $w(\alpha_s) < 0$ which contradicts $\sigma(w) = id$. ∎

**Corollary 3.2.** $\Phi = \Phi^+ \sqcup \Phi^-$

*Proof.* Let $\alpha \in \Phi$. Then $\alpha = w(\alpha_s)$ for some $w \in W, s \in S$. If $w(\alpha_s)$ is not positive, then we can't have $\ell(ws) > \ell(w)$ by Theorem 3.1. Since $\ell(ws) = \ell(w) \pm 1$, we must have $\ell(ws) < \ell(w)$ which implies $w(\alpha_s) < 0$, again by Remark 3.1. ∎

## 3.2 Interpreting $\ell$ in terms of $\Phi$

We would like to take a closer look at the action of $W$ on $\Phi$.

**Lemma 3.1.** *For all $s \in S$, $s$ acts as a permutation on $\Phi^+ \smallsetminus \{\alpha_s\}$.*

*Proof.* Let $\alpha \in \Phi^+ \smallsetminus \{\alpha_s\}$. Note that $\alpha$ cannot be a multiple of $\alpha_s$ since both $\alpha, \alpha_s$ are unit vectors and

$\alpha \neq \alpha_s$. Hence, for some $c_s, c_t \geq 0$ with at least one $c_t > 0$, we have

$$\begin{aligned}
&\alpha = c_s\alpha_s + \sum_{t \in S \smallsetminus \{s\}} c_t\alpha_t \\
\implies &s(\alpha) = c_s s(\alpha_s) + \sum_{t \in S \smallsetminus \{s\}} c_t s(\alpha_t) \\
&\quad = -c_s\alpha_s + \sum_{t \in S \smallsetminus \{s\}} c_t\alpha_t + \sum_{t \in S \smallsetminus \{s\}} -2B(\alpha_t, \alpha_s)\alpha_s \\
&\quad = \Big( -c_s - \sum_{t \in S \smallsetminus \{s\}} 2B(\alpha_t, \alpha_s)\Big)\alpha_s + \sum_{t \in S \smallsetminus \{s\}} c_t\alpha_t.
\end{aligned}$$

By Corollary 3.2, $s(\alpha)$ is either positive or negative. Since the coefficients for $\alpha_t$ in $s(\alpha)$ are exactly those in $\alpha$ with at least one strictly positive, we must have $s(\alpha) > 0$. Hence, $s(\Phi^+ \smallsetminus \{\alpha_s\}) \subset \Phi^+ \smallsetminus \{\alpha_s\}$, and we have the other inclusion by applying $s$ to both sides. □

Now we will use Lemma 3.1 to prove a much more general statement from [4].

**Proposition 3.4.** *For all $w \in W$, $\ell(w) = |w(\Phi^+) \cap \Phi^-|$. In words, $\ell(w)$ is equal to the number of positive roots that get sent to negative roots by $w$.*

*Proof.* For all $w \in W$, we let $n(w) := |w(\Phi^+) \cap \Phi^-|$. Given $w \in W$, we will show that $n(w) = \ell(w)$ by induction on $\ell(w)$. If $\ell(w) = 0$, then $w = id$ acts trivially on $\Phi$, so the result is obvious. It is also clear by the Lemma that the result also holds for $\ell(w) = 1$. Now suppose $\ell(w) > 1$ and the result holds for all $w' \in W$ with $\ell(w') \leq \ell(w) - 1$. Then, we can find $s \in S$ such that $\ell(ws) = \ell(w) - 1$. By induction, $\ell(ws) = n(ws)$. Since $\ell(wss) = \ell(w) > \ell(ws)$, we have $ws(\alpha_s) > 0$ by Theorem 3.1. So, $\alpha_s \notin ws(\Phi^+) \cap \Phi^-$. Since $s(\alpha_s) = -\alpha_s$ and $s$ permutes the remaining positive roots, it follows that $wss(\Phi^+) \cap \Phi^- = (ws(\Phi^+) \cap \Phi^-) \sqcup \{\alpha_s\}$ which gives

$$\begin{aligned}
n(w) &= n(wss) \\
&= |wss(\Phi^+) \cap \Phi^-| \\
&= |(ws(\Phi^+) \cap \Phi^-) \sqcup \{\alpha_s\}| \\
&= |ws(\Phi^+) \cap \Phi^-| + 1 \\
&= n(ws) + 1 \\
&= \ell(ws) + 1 \\
&= \ell(w) - 1 + 1 \\
&= \ell(w).
\end{aligned}$$

□

For our next result, we will first observe how the *conjugates* of $s$ act on $V$. Given some $s \in S$ and $w \in W$, we let $\alpha := w(\alpha_s) \in \Phi$. For any $v \in V$, we obtain

$$\begin{aligned}
wsw^{-1}(v) &= w(s(w^{-1}(v))) \\
&= w[w^{-1}(v) - 2B(w^{-1}(v), \alpha_s)\alpha_s] \\
&= w(w^{-1}(v)) - 2B(w^{-1}(v), \alpha_s)w(\alpha_s) \\
&= v - 2B(w^{-1}(v), \alpha_s)w(\alpha_s) \\
&= v - 2B(v, w(\alpha_s))w(\alpha_s) && \text{Since } w \text{ preserves } B \\
&= v - 2B(v, \alpha)\alpha.
\end{aligned}$$

Notice that the function $v \mapsto v - 2B(v,\alpha)\alpha$ depends only on $\alpha \in \Phi$. It resembles the reflection with respect to $\alpha$ in the same way that $s$ is the reflection with respect to $\alpha_s$. That is, $\alpha \mapsto -\alpha$ and all the points on the hyperplane orthogonal to $\alpha$ are fixed. We will denote this reflection by $s_\alpha$. In what follows, we may use our representation $\sigma$ to identify $s_\alpha$ as both an element of $W$ and a map in $GL(V)$.

**Remark 3.2.** *Let $\alpha, \beta \in \Phi$. If for some $w \in W$ $w(\alpha) = \beta$, then $ws_\alpha w^{-1} = s_\beta$.*

Naturally, we have the following generalization of Theorem 3.1.

**Theorem 3.2.** *For all $w \in W, \alpha \in \Phi^+$,*

$$\ell(ws_\alpha) > \ell(w) \iff w(\alpha) > 0.$$

*Proof.* As in Remark 3.1, it suffices to prove the 'only if' direction. Again, we proceed by induction on $\ell(w)$. If $\ell(w) = 0$, then $w = id$ and $w(\alpha) = \alpha > 0$. If $\ell(w) > 0$, then we can find $s \in S$ such that $\ell(sw) < \ell(w)$. If $\ell(ws_\alpha) > \ell(w)$, then

$$\begin{aligned}\ell((sw)s_\alpha) &= \ell(s(ws_\alpha)) \\ &\geq \ell(ws_\alpha) - 1 \\ &> \ell(w) - 1 \\ &= \ell(sw).\end{aligned}$$

Applying the induction hypothesis on $sw$, we get $sw(\alpha) > 0$. By Lemma 3.1, it suffices to show that $sw(\alpha) \neq \alpha_s$. Otherwise, we could apply Remark 3.2 to obtain $(sw)s_\alpha(sw)^{-1} = s_{sw(\alpha)} = s_{\alpha_s} = s$. We have $(sw)s_\alpha(sw)^{-1} = s$ which implies $ws_\alpha = sw$ and hence $\ell(ws_\alpha) = \ell(sw) < \ell(w)$ contradicting our assumption. Therefore, $sw(\alpha) \neq \alpha_s$ and $w(\alpha) = ssw(\alpha) > 0$ as desired. $\square$

## 3.3 The Strong Exchange Condition and Further Results

We introduce some new notation; we let $s_1 \cdots \widehat{s_i} \cdots s_r = s_1 \cdots s_{i-1}s_{i+1} \cdots s_r$. The following theorem is called the **Strong Exchange Condition**. If instead of $s_\alpha$ we had $s \in S$ it would be called the **Exchange Condition**.

**Theorem 3.3.** *Let $w \in W$ and suppose we have an expression for $w$ that is not necessarily reduced, $w = s_1 \cdots s_r$ for $s_i \in S$. Suppose further that there exists $\alpha \in \Phi$ such that $\ell(ws_\alpha) < \ell(w)$. Then, for some $i$, $ws_\alpha = s_1 \cdots \widehat{s_i} \cdots s_r$. Moreover, if this expression is reduced, then $i$ is unique.*

*Proof.* Without loss of generality, we can assume $\alpha > 0$ (since $s_\alpha = s_{-\alpha}$). Assuming $\ell(ws_\alpha) < \ell(w)$, we obtain $w(\alpha) < 0$ by Theorem 3.2. Since $s \in S$ will only change the sign of the root $\alpha_s$, there must exist at least one index $1 \leq i \leq r$ for which $s_{i+1} \cdots s_r(\alpha) > 0$ but $s_i s_{i+1} \cdots s_r(\alpha) < 0$. Hence, we must have $s_{i+1} \cdots s_r(\alpha) = \alpha_{s_i}$. By Remark 3.2, $(s_{i+1} \cdots s_r)s_\alpha(s_{i+1} \cdots s_r)^{-1} = s_i$, and if we multiply both sides by $s_1 \cdots s_i$ on the left, we obtain $ws_\alpha(s_{i+1} \cdots s_r)^{-1} = s_1 \cdots s_{i-1}$. Finally, isolating for $ws_\alpha$ gives $ws_\alpha = s_1 \cdots \widehat{s_i} \cdots s_r$ as desired.
Moreover, if $s_1 \cdots s_r$ is reduced, or equivalently if $\ell(w) = r$, then we can assume for contradiction that there are distinct $1 \leq i < j \leq r$ such that $ws_\alpha = s_1 \cdots \widehat{s_i} \cdots s_r = s_1 \cdots \widehat{s_j} \cdots s_r$. Then we can cancel the appropriate number of terms on both the left and right sides of the equation to get $s_{i+1} \cdots s_j = s_i \cdots s_{j-1}$.

But if this is the case, then we can substitute and simplify as follows

$$\begin{aligned}
w &= s_1 \cdots s_i (s_{i+1} \cdots s_j) s_{j+1} \cdots s_r \\
&= s_1 \cdots s_i (s_i \cdots s_{j-1}) s_{j+1} \cdots s_r \\
&= s_1 \cdots (s_i^2) \cdots s_j s_{j+1} \cdots s_r \\
&= s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_r \\
&= s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_r,
\end{aligned}$$

contradicting the fact that $\ell(w) = r$. □

The following corollary is called the **Deletion Criterion**.

**Corollary 3.3.** *Assume $w = s_1 \cdots s_r$ with $s_i \in S$ and $\ell(w) < r$ i.e. the word $s_1 \cdots s_r$ is not reduced. Then, there are distinct $1 \leq i < j \leq r$ such that $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_r$.*

*Proof.* Let $w \in W$ satisfy the hypothesis. If for all $i$, $\ell(s_1 \cdots s_i) = i$, we would not have $\ell(w) < r$. There must exist some $j$ such that $\ell(s_1 \cdots s_{j-1}) = j - 1$ and $\ell(s_1 \cdots s_j) < \ell(s_1 \cdots s_{j-1})$. Applying the Strong Exchange Condition to $w' := s_1 \cdots s_{j-1}$ and $s_j$, there exists $i$ such that $w' s_j = s_1 \cdots \widehat{s_i} \cdots s_{j-1}$. Multiplying both sides by $s_{j+1} \cdots s_r$ gives $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_r$ as desired. □

**Remark 3.3.** *Notice that we can apply induction to the previous proof to obtain an even stronger statement, namely that any $w = s_1 \cdots s_r \in W$ can be reduced by omitting an even number of $s_i$.*

We end this section with a theorem that provides us with alternative ways to identify Coxeter groups [1].

**Theorem 3.4.** *Let $(W, S)$ be an involution system i.e. $W$ is a group generated by a set $S$ of involutions. Then the following are equivalent.*

(i) *$(W, S)$ is a Coxeter System.*

(ii) *$(W, S)$ satisfies the Exchange Condition or the Deletion Condition.*

(iii) *$(W, S)$ satisfies the Reflection Cocycle Condition.*

(iv) *$(W, S)$ satisfies the Braid Condition and $(W, S)$ has a sign character.*

Let $T := \bigcup_{w \in W} w S w^{-1}$. Given an involution system $(W, S)$, we say that $(W, S)$ satisfies the **Reflection Cocycle Condition** if there is an action of $W$ on the set $T \times \{\pm 1\}$ such that

$$s \cdot (t, \varepsilon) = \begin{cases} (sts, \varepsilon) & s \neq t \\ (s, -\varepsilon) & s = t \end{cases}$$

for all $s, t \in S$, $\varepsilon \in \{\pm 1\}$. If $W$ is a Coxeter group under our representation $\sigma$, then each element of $T$ determines some reflection in $V$. Since each such reflection is in some hyperplane, we can think of $T$ as a set of hyperplanes. Then, $T \times \{\pm 1\}$ can be thought of as the *sides* of the hyperplanes. From this perspective, an element $(s_\alpha, 1)$ can represent the side of the hyperplane orthogonal to $\alpha$ containing $\alpha$, and $(s_\alpha, -1)$ would then represent the side containing $-\alpha$.

On the other hand, we say that the involution system $(W, S)$ satisfies the **Braid Condition** if for all monoids $M$ and functions $f : S \to M$ such that

$$[f(s), f(t)]_{m(s,t)} = [f(t), f(s)]_{m(s,t)}, \text{ whenever } m(s,t) < \infty,$$

then the map $\varphi : W \to M$ given by $s_1 \cdots s_n \mapsto f(s_1) \cdots f(s_n)$ is well defined.

# 4 Properties of Cay($W, S$) and the Parallel Wall Theorem

In this section, we will consider Coxeter groups from a slightly different perspective. We want to revisit the action of $W$ on its Cayley graph, and explore how it relates to our representation $\sigma$ of $W$. This will provide us with the knowledge we need to then discuss the Parallel Wall Theorem from two different points of view; one based on our root system, and one based on Cay($W, S$).

Recall that every root $\alpha \in \Phi$ is associated with a reflection $s_\alpha \in GL(V)$, namely the reflection sending $v \mapsto v - 2B(v, \alpha)\alpha$. In particular, we notice that $s_\alpha = s_{-\alpha}$ since $B(v, \alpha)\alpha = B(v, -\alpha)(-\alpha)$, so we can assume $\alpha \in \Phi^+$. Moreover, we can write $\alpha = w(\alpha_s)$ for some $w \in W$, $s \in S$, which gives $s_\alpha = \sigma(wsw^{-1})$ as shown in the previous section. Since $\sigma$ is injective, it follows that there is a well defined map from $\Phi^+$ to our set $T$ given by $\alpha \mapsto \sigma^{-1}(s_\alpha)$. As before, we can identify $\sigma^{-1}(s_\alpha)$ with $s_\alpha \in W$.

At the expense of a group structure for $\Phi^+$, we can then think of $\Phi^+$ as acting on Cay($W, S$). From this perspective, $\alpha$ would act on the graph in the exact same way as the group element $s_\alpha$ acts on the graph (by left multiplication). We will revisit this idea soon.

**Definition 4.1.** *Let $r \in T$. The **wall** $M_r$ is the fixed point set of the action of $r$ on* Cay($W, S$).

Assuming $id \notin S$, we have $id \notin T$. Thus if $r \in T$, $M_r$ does not contain any vertex of Cay($W, S$) since $rw \neq w$ for all $w \in W$. It follows that $M_r$ consists of the midpoints of certain edges in the graph. More precisely, $M_r$ contains the midpoint of the edge $(w, ws)$ if and only if $r = wsw^{-1}$. Hence, a single point in a wall will determine the entire wall and distinct walls are disjoint.

**Definition 4.2.** *Let $M_r$ be a wall. A **half-space** of $M_r$ is a connected component of* Cay($W, S$) $\setminus M_r$.

As the names *wall* and *half-space* suggest, it turns out that a wall $M_r$ admits exactly two half-spaces. Moreover, one can show that if $M_r$ contains the midpoint of the edge $(w, w')$, then the vertices $w$ and $w'$ must lie on distinct half-spaces of $M_r$. It then follows that the action of $r$ on the graph will exchange the two half-spaces and hence resembles a *reflection* in $M_r$. For this reason, we can think of each $\alpha \in \Phi^+$ as a reflection on Cay($W, S$) in a wall which we can denote $M_\alpha$ (formally, $M_\alpha := M_{s_\alpha}$).

**Proposition 4.1.** *Let $w, w' \in W$. Then the distance between $w$ and $w'$ in* Cay($W, S$) *is equal to the number of distinct walls $M_r$ separating $w, w'$ i.e. such that $w, w'$ lie in distinct half-spaces of $M_r$.*

With that said, let us consider the following two definitions stated in terms of our root system given by [3] and [6].

**Definition 4.3.** *Let $\alpha, \beta \in \Phi^+$. We say that $\alpha$ **dominates** $\beta$ if for all $w \in W$ we have*

$$w(\alpha) < 0 \implies w(\beta) < 0.$$

**Definition 4.4.** *A root $\alpha \in \Phi^+$ is called **elementary** if the only root that it dominates is itself.*

If $\alpha$ dominates $\beta$ and if $w \in W$ is any word, we have

$$w^{-1}(\alpha) < 0 \implies w^{-1}(\beta) < 0$$

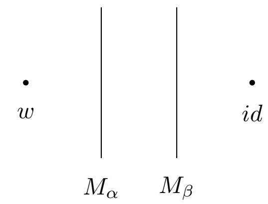
since the implication holds for all words in $W$. By Theorem 3.2, this is equivalent to

$$\ell(w^{-1}s_\alpha) < \ell(w^{-1}) \implies \ell(w^{-1}s_\beta) < \ell(w^{-1}).$$

We recall that $\ell$ is invariant under taking inverses. In our case we have $(w^{-1}s_\alpha)^{-1} = s_\alpha w$ (since $s_\alpha$ is a conjugate of an involution), and similarly $(w^{-1}s_\beta)^{-1} = s_\beta w$. This gives us

$$\ell(s_\alpha w) < \ell(w) \implies \ell(s_\beta w) < \ell(w).$$

Recall that $\ell(w)$ can be interpreted as the length of the shortest path from $id$ to $w$ in $\text{Cay}(W,S)$. Accordingly, the previous statement says that if the reflection in $M_\alpha$ brings $w$ closer to $id$, then so will the reflection in $M_\beta$.




In this case, we can see that wall $M_\beta$ separates $M_\alpha$ and $id$. Considering definition 4.4, $\alpha$ is elementary when there is no such wall $M_\beta$. In this case, we will say that the wall $M_\alpha$ is elementary. Let us consider the following example.

**Example 4.1.** Let $W = \langle s,t,r \,|\, s^2, t^2, r^2, (st)^4, (tr)^4, (sr)^2 \rangle$. The Cayley graph of $W$ with respect to $S = \{s,t,r\}$ is given below. One that show that it is dual to the tiling of the Euclidean plane by right angled isosceles triangles. Since $W$ is infinite, there are infinitely many roots. However, there are finitely many elementary roots. More precisely, there are eight elementary roots each of which is associated with a wall that is "closest" to $id$ as indicated in the figure. Note that the lines themselves are not the walls; rather, the walls are their intersections with the graph.

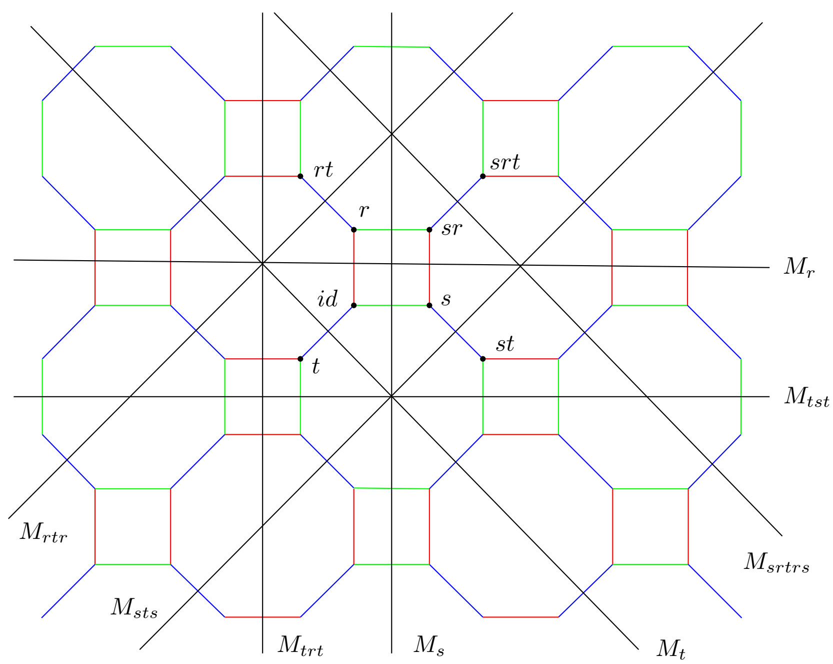



Figure 4: The elementary walls in $\text{Cay}(W,S)$

**Definition 4.5.** *The* ***depth*** *of* $\alpha \in \Phi^+$ *is given by*

$$dp(\alpha) := min\{n : \exists w \in W,\, \ell(w) = n,\, w(\alpha) < 0\}.$$

As a consequence of Proposition 4.1, $dp(\alpha)$ can be thought of as the minimal number of steps it would take to reach the other side of $M_\alpha$ from $id$. If we return to Example 4.1, the following table lists each elementary root, beside its associated elementary wall, and its depth which can be computed using our figure.

| Elementary walls | Elementary roots | Depth |
|---|---|---|
| $M_s$ | $\alpha_s$ | 1 |
| $M_r$ | $\alpha_r$ | 1 |
| $M_t$ | $\alpha_t$ | 1 |
| $M_{tst}$ | $t(\alpha_s) = \alpha_s + \sqrt{2}\,\alpha_t$ | 2 |
| $M_{sts}$ | $s(\alpha_t) = \alpha_t + \sqrt{2}\,\alpha_s$ | 2 |
| $M_{rtr}$ | $r(\alpha_t) = \alpha_t + \sqrt{2}\,\alpha_r$ | 2 |
| $M_{trt}$ | $t(\alpha_r) = \alpha_r + \sqrt{2}\,\alpha_t$ | 2 |
| $M_{srtrs}$ | $sr(\alpha_t) = \alpha_t + \sqrt{2}\,\alpha_s + \sqrt{2}\,\alpha_r$ | 3 |

Figure 5: The corresponding elementary roots and depth

**Example 4.2.** Let $W = \langle s, t, r \mid s^2, t^2, r^2, (st)^3, (sr)^3, (tr)^3 \rangle$. The Cayley graph of $W$ with respect to $S = \{s, t, r\}$ is given by a tiling of the Euclidean plane by regular hexagons. As in the previous example, we find that $W$ is infinite but with finitely many elementary roots and elementary walls.

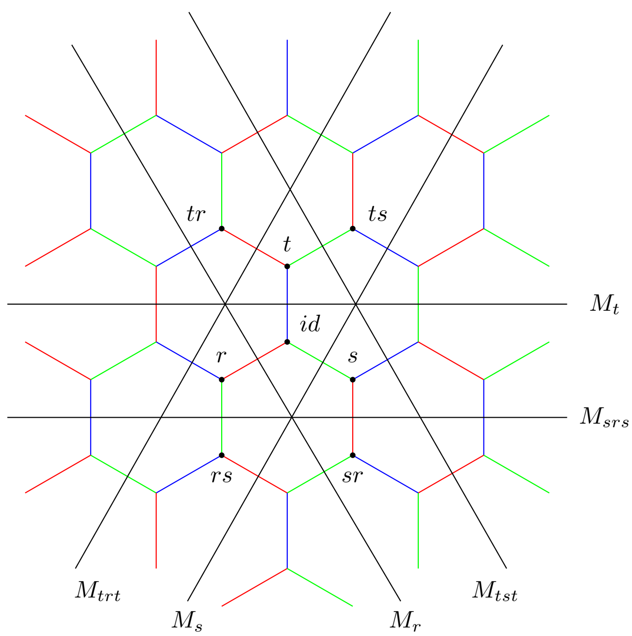



Figure 6: The elementary walls in $\mathrm{Cay}(W, \{s, t, r\})$

| Elementary walls | Elementary roots | Depth |
|---|---|---|
| $M_s$ | $\alpha_s$ | 1 |
| $M_r$ | $\alpha_r$ | 1 |
| $M_t$ | $\alpha_t$ | 1 |
| $M_{tst}$ | $t(\alpha_s) = \alpha_s + \alpha_t$ | 2 |
| $M_{trt}$ | $t(\alpha_r) = \alpha_r + \alpha_t$ | 2 |
| $M_{srs}$ | $s(\alpha_r) = \alpha_r + \alpha_s$ | 2 |

Figure 7: The corresponding elementary roots and depth

Notice that in both examples $W$ is infinite but the set of elementary roots, which we may denote by $\mathcal{E}$, is finite. In turns out that this will always be the case.

**Theorem 4.1. (Parallel Wall Theorem)** *Let $(W, S)$ be a Coxeter system. Then the associated set of elementary roots $\mathcal{E}$ is finite.*

**Remark 4.1.** *This theorem is equivalent to the following statement; For a given Coxeter system $(W, S)$, there is a constant $C$ such that for all $w \in W$ and walls $M \subset \mathrm{Cay}(W, S)$, if $w$ and $M$ are at distance greater than $C$ in $\mathrm{Cay}(W, S)$, then there exists a wall $M'$ separating $w$ and $M$ in $\mathrm{Cay}(W, S)$.*

The proof of this theorem is omitted due to its technical complexity. It utilizes the partial order defined on $\Phi^+$ given below as well as the Lemma that follows.

**Definition 4.6.** *For two roots $\alpha, \beta \in \Phi^+$, we write $\alpha \prec \beta$ if there exists $s_1, \ldots, s_n \in S$ such that*

(i) *$s_n s_{n-1} \cdots s_1(\beta) = \alpha$, and*

(ii) *$dp(s_i \cdots s_1(\beta)) = dp(\beta) + i$ for each $1 \leq i \leq n$.*

**Lemma 4.1.** *Let $(W, S)$ be a Coxeter system with associated set of positive roots $\Phi^+$. Then, $\mathcal{E}$ can be computed inductively. More precisely, $\mathcal{E} = \bigcup_{i \geq 1} \mathcal{E}_i$ where $\mathcal{E}_0 := \{\alpha_s : s \in S\}$ and for each $i \geq 1$, $\mathcal{E}_{i+1} := \{s(\alpha) : 1 < B(\alpha_s, \alpha) < 0\}$.*

In short, the proof starts with an elementary root $\alpha$ with $dp(\alpha) = n$ and an associated chain of elementary roots with respect to our partial order $\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_n := \alpha$ for $\alpha_i \in \mathcal{E}_i$. It then uses several properties of our bilinear form $B$ as well as the partial order $\prec$ to conclude that $n$ must be bounded. Since the number of elementary roots of a fixed depth $n$ is bounded above by $|S|^{n+1}$, it then follows that $\mathcal{E}$ is finite.

Rather than focusing on the details of the proof, the emphasis here is placed more on the interpretations and implications of the theorem. This result was proved in 1993 by Brink and Howlett in [3] and was then used to prove that Coxeter groups are automatic. Loosely speaking, a Coxeter group $W = \langle S | R \rangle$ is **automatic** if there is a language $L$ over $S$ representing each element in $W$, such that such that for each $s \in S$, one can decide whether two words $w, w' \in L$ differ by multiplication by $s$ in $W$. Alternatively, one can replace the second condition with something called the **fellow traveler property**. This property states that there exists a constant $k$ such that for all words $w, w' \in L$ and all $s \in S$, if $ws = w'$ in $W$ then the paths labeled by $w, w'$ in $\mathrm{Cay}(W, S)$ are at distance at most $k$. Here, the distance between two paths refers to the distance between *corresponding points* in each path. That is, if we let $w(i)$ and $w'(i)$ denote the points along the paths $w, w'$ at distance $i$ from $id$, then for each such $i$ we have $d(w(i), w'(i)) \leq k$.

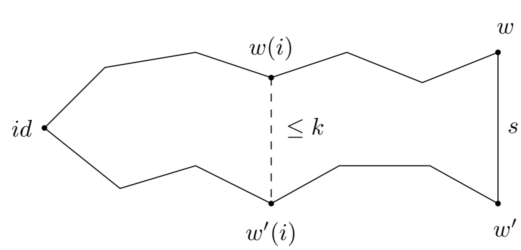



Figure 8: Fellow Traveler Property

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