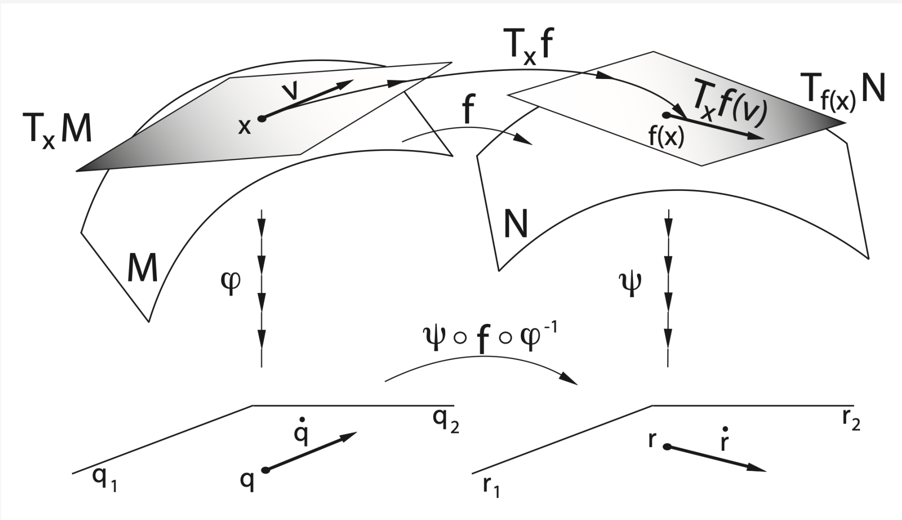
Calculus on Manifolds and Riemannian Geometry

The Bridge to Modern Mechanics



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May 2026

Abstract

We make sense of the world and its mechanical processes through a very familiar notion of space and geometry—the one we see with our eyes and measure with rulers. Indeed, Euclid laid the groundwork over two thousand years ago for what we know as Euclidean geometry: an establishment of numbers, measurements, angles, and "space", with which nearly all of undergraduate mechanics is built and understood upon.

It is in fact very convenient to view physical space in this way; we live in a three-dimensional world, and the notion of what is "flat" is the most simple for us to understand (try finding the surface area of a piece of paper versus an apple with a ruler). Although "curved" objects are harder to measure, we have developed a beautiful theory of multivariable calculus to help us find the volume of a wine glass, or to find the electric flux of a charge inside of a spherical ball.

Despite this, your average toddler would likely have a much easier time multiplying numbers together to find the volume of a box ($volume = length \times width \times height$) over computing a surface integral over an apple to find its surface area ($area = \oiint_{\text{apple}} dS$). Even most adults could not perform such a computation. It is thus worth asking, if the majority of objects we encounter are curved and cannot be measured by the average CEO, why do we centralize Euclidean space and geometry in our perception of the world?

Indeed, this prioritization has previously led us to believe, for example, that the Earth is flat. As such, it is worth establishing a substantial generalization of our familiar notion of "space", as well as how we make measurements and determine what it actually means to be "straight" or "flat". This motivates the abstract theory of differential geometry and Riemannian geometry, which provides a broad framework for the notion of space endowed with geometric structure.

We begin this paper with an abstract definition of smooth manifolds. We next use tangent spaces, differentials, vector fields, and differential forms to naturally establish notions of calculus on these abstract spaces. Finally, we introduce Riemannian metrics to endow manifolds with geometric structure—namely lengths, angles, and geodesic motion.

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1 Introduction

1.1 Introduction

First, I'd like to establish that this paper is written for McGill's Directed Reading Program, which partners undergraduate students with graduate mentors to do a guided independent study. I'd like to thank my mentor, William Holman-Bissegger, for his guidance and for his encouragement in the face of such an abstract topic. As such, the target audience of this paper is undergraduate students, such as me, who value abstraction and revel in realizing the world we know as a specific instance of deeper mathematical rules.

This paper uses the generalization of calculus on manifolds to establish a natural segue into mechanics, which is the primary application of the theory. Often, while discussing math with my friends and relatives, I get the question:

"How is this useful?"

This turns my enthusiastic explanation of a theory into a resume of applied fields. It is reasonable to be curious about the applied sciences—fields like physics, computer science, and engineering exist for this reason. Questioning math's uses, though, often detracts value from its construction. Those I know who study math do so for the beautiful puzzle it provides, and for the struggle that comes with new theory. Just as we value the contributions of artists, literature, and cinema, it is important to value math for the beauty of its problem solving just as much as for its practical applications.

As such, the motivations of differential geometry can be explained from two points of view.

Practically, many real-world problems benefit from abandoning the notions of Euclidean space. Here are a few:

- As we shall see at the end of this paper, much of modern physics relies on mechanical systems that can be generalized to any notion of space. Most notably, Albert Einstein's general theory of relativity uses the language of differential geometry to label the universe as a smooth manifold equipped with a pseudo-Riemannian metric. This sets the groundwork for how we position satellites, and study black holes.
- Differential geometry is used in the study of econometrics—a heavy quantitative branch of economics that applies statistical methods and mathematical models.
- In engineering, differential geometry is applied to solve problems digital signal processing.
- In biology and chemistry, differential geometry is used to study the effects of pressure on cell membrane structure.
- Neuroimaging, structural geology and computer design are among other fields for which differential geometry is used for modeling.

From a theoretical standpoint, differential geometry fuses topological spaces with calculus, allowing us to lift the complex math we have developed for Euclidean space (i.e. vector calculus) into abstract topological ones. Mathematical spaces are often better understood in the language of differential geometry.

Vector calculus is a gleaming example of some of the most high-tech math we have in Euclidean space. It alludes to the fact that integration in $\mathbb{R}^n$ is not enough—in order to find the total quantity of something on a curved surface we require a surface integral, which does not make sense alone in two-dimensional or three-dimensional Euclidean space. A surface, as we study in multivariable calculus, is a two dimensional object that exists in three-dimensional space. Think, for example, of a string lying flat on a table:

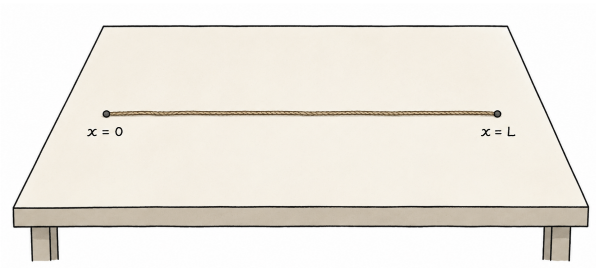

Fig. 1: String in One Dimension

This string is one-dimensional. Now, perturb it so that it is makes a curved shape:

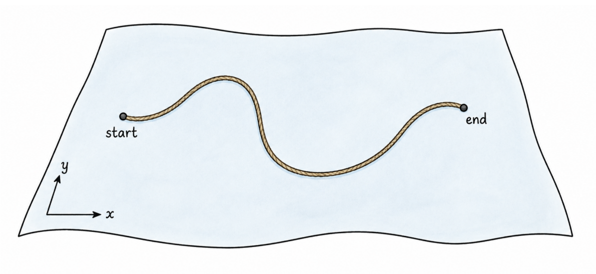

Fig. 2: String in Two Dimensions

The string, a one-dimensional object, now exists in two-dimensional space. We need both x and y-coordinates to describe the points on the string. Similarly, we can lift the string off of the flat surface:

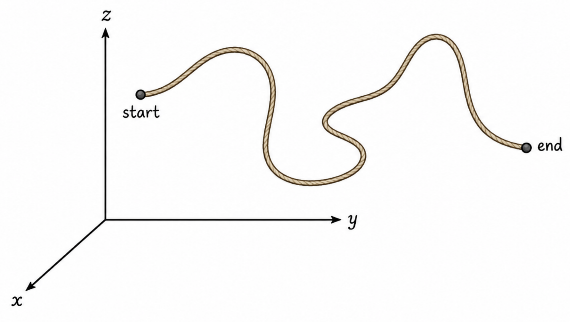

Fig. 3: String in Three Dimensions

Now, the one-dimensional object occupies coordinates in three dimensions! This establishes the idea of an *embedding* into higher-dimensional space. This string example could be carried into four dimensions (or as many as we want), we just cannot visualize it.

It thus makes sense to think of the curved objects we study and use heavily in vector calculus in this way—n-dimensional objects embedded in $n+k$-dimensional space.

Stokes' theorem is one of the main theorems we use in vector calculus. It uses the concept of curl, which only makes sense in three dimensions:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

Differential geometry generalizes the theorem to arbitrary-dimensional manifolds:

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

It thus makes sense to view differential geometry as a very natural generalization of calculus that succeeds as it does not cling to principles of Euclidean space.

1.2 Mathematical Background

This paper is aimed at curious undergraduate students (as I am one of them myself!). As such, the recommended mathematical background is familiarity with the following topics that a curious undergraduate likely has heard about:

- **Multivariable Calculus/Differential Equations.** The basics of coordinate geometry. Partial derivatives and the Riemann integral in multiple dimensions. Vector calculus is definitely nice to know, but not required. Definition of an ODE/PDE.
- **Real Analysis.** Basics of real analysis and logic. Inverse and implicit function theorems. Metric spaces. Topological spaces; bases, separation axioms, homeomorphisms.
- **Abstract Algebra.** Quotient spaces, familiarity with groups and fields. Linear algebra; vector spaces, isomorphisms, dual spaces. Inner product spaces and normed vector spaces.

Also, for chapter eight, very elementary physics background (i.e. a first-year mechanics course) is recommended but not required.

1.3 Paper Overview

Here is a brief overview of the contents of the paper:

We start by discussing manifolds immersed in $\mathbb{R}^n$, as motivated in section 1.1. Then we generalize—chapter two establishes the broad topological definition of a smooth manifold. While a beautiful definition, it can be a lot to take in. Hence lots of the chapter will be examples and non-examples illustrating the intricacies of the definition.

To begin thinking about calculus, we must establish the general notion of tangency and linear approximations. In chapter four we introduce the broad definition of a tangent space as a vector space associated with a point on a manifold. We consider the various maps to and from manifolds and tangent spaces, as well as the cotangent space.

After discussing vector fields and flows, we will be ready in chapter six to define differential forms and integration—the fundamental notions of calculus on manifolds. This is a chapter packed with rich definitions and generalizations of the calculus operators on Euclidean space. Similar to chapter two, we will go over examples to illustrate the functionality of these new objects.

Similar to how we establish geometric properties on vector spaces with norms and inner products, we next define Riemannian metrics which accomplish the same task on our abstract manifolds. There are many similarities between Riemannian manifolds and topological normed vector spaces, which we will explore. At last we will define geodesics, which generalize the notion of straight lines and shortest paths on our manifolds.

We conclude by demonstrating differential geometry's natural role in Newtonian and Lagrangian mechanics, providing insight into the physical applications of the field.

2 Submanifolds of $\mathbb{R}^n$

One of the main ideas of vector calculus is integrating over a curve or a surface. These objects, as presented in the introduction, are special. A curve, for example

$$\phi(t) = (t, t^2),$$

is an object whose domain ($t \in \mathbb{R}$) is fundamentally one-dimensional, despite occupying two-dimensional space ($\mathrm{im}(\phi) \subset \mathbb{R}^2$). We'd like to provide a formal definition of these objects that exist in $\mathbb{R}^n$ but are not themselves $\mathbb{R}^n$.

2.1 Submersions

Recall the definition of the *graph* of a function $f : X \to Y$, given by:

$$\mathrm{graph}(f) = \{(x, f(x)) : x \in X\}$$

Consider $X = \mathbb{R}^n$. An m-**dimensional embedded submanifold** of $\mathbb{R}^n$ is a subset M of $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$, there exists a neighborhood U of x such that $M \cap U$ is the graph of some smooth function, expressing $n-m$ coordinates in terms of the other m.

Example 1. The circle $x^2+y^2 = 1$ is a 2-dimensional embedded submanifold of $\mathbb{R}^2$. While the points $(-1, 0)$ and $(1, 0)$ have vertical tangent lines (i.e. $\frac{dy}{dx} = \infty$), this does not contradict the definition. The definition does not specify which coordinates are functions of the other, and thus at these points x can be represented as a perfectly smooth function of y (i.e. $x = \pm\sqrt{1-y^2}$).

Example 2. Any subset of an m-dimensional linear subspace of $\mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$ iff it has empty boundary. Note that as per our definition, if the manifold has nonempty boundary, we cannot take a neighborhood on which the submanifold is entirely well-defined.

Example 3. Let $f(x, y) = xy$ and take the level sets $M_c := f^{-1}(c)$, for $c \in \mathbb{R}$. Fixing a c, we get a family of hyperbolae $xy = c$. For every $c \neq 0$ these level sets are embedded submanifolds of $\mathbb{R}^2$. However, if $c = 0$, the origin is the troublesome point at which we cannot intersect a smooth graph for any open neighborhood. In general, a level set $f(\mathbf{x}) = c$ is an embedded submanifold if $\nabla f(\mathbf{x}) \neq 0$ for every x in the level set. This means that if there exists one point at which the gradient equals zero, the level set is not an embedded submanifold.

A differentiable function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a **submersion of** $x \in \Omega$ if the Jacobian $Df(\mathbf{x})$ is surjective, i.e. has rank m. A function that satisfies this condition at all points in Ω is called a **submersion**.

Consequences:

- in order for f to be a submersion it is necessary that we have $n \geq m$.
- A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is a submersion iff $\nabla f \neq 0$.

- A *diffeomorphism* is a differentiable bijective map across topological spaces with differentiable inverse. All diffeomorphisms are submersions. The converse is false, for example $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x$. We see $\nabla f = (1, 0) \neq \mathbf{0}$, but f is not a diffeomorphism since f is not injective.

A **regular value** of $f : \Omega \to \mathbb{R}^m$ is a $\mathbf{c} \in \mathbb{R}^m$ such that $Df(\mathbf{x})$ is surjective for all $\mathbf{x} \in f^{-1}(\mathbf{c})$. It follows that if $f : \Omega \to \mathbb{R}^m$ is smooth and $\mathbf{c}$ is a regular value of f, then $f^{-1}(\mathbf{c})$ is a submanifold of $\mathbb{R}^n$ of codimension m. For example, the unit sphere $\mathbb{S}^2$ is $g^{-1}(1)$, where $g(x, y, z) = x^2 + y^2 + z^2$, and is thus a submanifold of $\mathbb{R}^3$ of dimension 2 and codimension 1.

2.2 Immersions and Embeddings

Similar to submersions, an **immersion** is a differentiable function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ such that Df is injective everywhere.

Consequences:

- in order for f to be an immersion it is necessary that we have $n \leq m$.
- All diffeomorphisms are immersions. The converse is again false, for example $f : \mathbb{R} \to \mathbb{R}^2, x \mapsto (x, 0)$. We can see that Df is injective, but f not a diffeomorphism since f itself is not injective.

Note that an immersion brings some number of variables into a higher-dimensional space, while a submersion collapses the coordinates into a smaller dimensional space.

Example 1. Every C^1 surface parametrization is an immersion from two to three dimensional space.

Example 2. Every scalar function with non-vanishing gradient is a submersion from higher-dimensional space to one dimension. Also, projections are submersions.

The image of an immersion is called an **immersed submanifold** whose dimensional is equal to the codomain of the immersion. Parametrized curves and surfaces are examples of immersed submanifolds.

An **embedding** is an immersion $\psi : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ such that $\psi^{-1} : \psi(U) \to U$ is continuous. As a result, $\psi : U \subseteq \mathbb{R}^n \to \psi(U)$ is an embedding iff it is homeomorphic onto its image, and this image is a submanifold of $\mathbb{R}^m$.

Note that an immersion need not be an embedding. For example, define

$$f : \mathbb{S}^1 \to \mathbb{R}^2, \qquad f(\theta) = \Big(\sin\theta,\ \sin(2\theta)\Big).$$

Then

$$f'(\theta) = \Big(\cos\theta,\ 2\cos(2\theta)\Big).$$

If $f'(\theta) = (0,0)$, then $\cos\theta = 0$ and $\cos(2\theta) = 0$. But $\cos\theta = 0$ implies $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, and then

$$\cos(2\theta) = \cos(\pi) = -1 \neq 0.$$

Thus $f'(\theta) \neq (0,0)$ for all $\theta \in (0, 2\pi)$, so $\operatorname{rank} Df_\theta = 1$ everywhere and f is an immersion.

Moreover,

$$f(0) = (0,0) = f(\pi),$$

so the image (a *lemniscate*) crosses itself at the origin. In particular, f is not injective, so it is not an embedding.

Even if one restricts attention to defining an "inverse" on the image away from this issue, the inverse cannot be continuous at $(0,0)$. For example, let $\theta_n = \frac{1}{n}$. Then

$$f(\theta_n) = \left(\sin\frac{1}{n},\ \sin\frac{2}{n}\right) \longrightarrow (0,0) = f(\pi),$$

but

$$\theta_n \longrightarrow 0 \neq \pi.$$

Thus any specific choice of inverse $f^{-1} : L \to (0, 2\pi)$ fails to be continuous at $(0,0)$, and so f is an immersion whose image is not a submanifold of $\mathbb{R}^2$, and f is not an embedding.

2.3 Equivalent Definitions of an Immersed Submanifold of $\mathbb{R}^n$

With our knowledge of submersions, immersions, and embeddings, we can now establish and prove an important theorem which gives equivalent definitions of an immersed submanifold of $\mathbb{R}^n$ in terms of these maps:

Theorem. Let $M \subset \mathbb{R}^n$, and let $0 < m < n$ and $k = n - m$. Then *the following are equivalent*:

1. For every $a \in M$ there exists a neighbourhood U of a, a smooth submersion

$$f : U \to \mathbb{R}^k$$

   and a $c \in \mathbb{R}^k$ such that

$$M \cap U = f^{-1}(c).$$

2. For every $a \in M$ there exists a neighborhood U of a such that $M \cap U$ is the graph of a smooth function expressing k of the standard coordinates in terms of the other m.

3. For every $a \in M$ there exists a neighborhood U of a and a smooth embedding

$$\psi : V \subset \mathbb{R}^m \to \mathbb{R}^n$$

   such that

$$\psi(V) = M \cap U.$$

Proof. (1) $\Leftrightarrow$ (2):

" $\implies$ " Suppose $\mathbf{a} \in M$, and $M \cap U = f^{-1}(\mathbf{c})$, where f is a smooth submersion. Since f is a submersion, it is a submersion at $\mathbf{a}$, hence the Jacobian matrix

$$Df(\mathbf{a})$$

is surjective.

" $\impliedby$ " Now assume that $M \cap U$ is the graph of some smooth function g.

Assume without loss of generality that $U = U_1 \times U_2$, such that $M \cap U = \{(\mathbf{x}, g(\mathbf{x})) : \mathbf{x} \in U_1\}$. Define $f : U = U_1 \times U_2 \to \mathbb{R}, (\mathbf{x}, \mathbf{z}) \mapsto \mathbf{z} - g(\mathbf{x})$. Then g is a smooth submersion with $f^{-1}(0) = M \cap U$.

(2) $\Leftrightarrow$ (3):

" $\implies$ " Define $V := U_1$ and $\psi : V \to \mathbb{R}^n, \mathbf{x} \mapsto (\mathbf{x}, g(\mathbf{x}))$, for U_1 and g as in the previous part of the proof. Then ψ is a smooth immersion. Furthermore, ψ^{-1} is continuous as it is a projection onto the first m coordinates, and such projections are continuous maps.

" $\impliedby$ " Take the same map ψ, and denote $\mathbf{b} := \psi^{-1}(\mathbf{a})$. Since ψ is an immersion, we have $\operatorname{rank}(D\psi(\mathbf{b})) = m$. By rank-nullity we must have $\dim(\operatorname{im}(D\psi(\mathbf{b}))) \leq m$.

Denote the map $\pi : \mathbb{R}^n \to \mathbb{R}^m$ as the projection onto the first m coordinates.

Since π is a projection, we must have $\ker(\pi) \oplus \operatorname{im}(\pi) = \mathbf{0}$. Again by rank-nullity, $\operatorname{rank}(\pi|_{\mathrm{im(D}\psi(\mathbf{b}))}) = m$, and hence $\operatorname{rank}(D(\pi \circ \psi)(\mathbf{b})) = m$.

By the inverse function theorem, we can find a neighborhood W_1 of $\mathbf{b}$ such that $(\pi \circ \psi)|_{W_1}$ is smooth with smooth inverse. Since ψ^{-1} is continuous, $\psi(W_1)$ is relatively open in the subspace topology on $M \cap U$, so we can find a $U' \subseteq U$ such that $\psi(W_1) = M \cap U'$. Now denote $W_2 := (\pi \circ \psi)(W_1) = \pi(M \cap U')$, and let $h := \psi \circ (\pi \circ \psi)^{-1} : W_2 \to \mathbb{R}^n$.

Note that $\pi \circ h$ is the identity map on W_2, while the components of $\pi \circ h$ are the first m components of h. Let $g : W_2 \to \mathbb{R}^k$ be the last k components of h. Then g is smooth, and its graph is given by

$$\{(\mathbf{x}, g(\mathbf{x}) : \mathbf{x} \in W_2\} = \{h(\mathbf{x}) : \mathbf{x} \in W_2\} = h(W_2) = \psi(W_1) = M \cap U.$$

$\square$

The unit sphere $\mathbb{S}^2$ is a good example that illustrates the equivalence of these three points:

- The sphere is the preimage $f^{-1}(1)$, where f is the smooth submersion $(x, y, z) \mapsto x^2 + y^2 + z^2$.
- At any point on the sphere, we can express two coordinates in terms of the other one, i.e. $z = \sqrt{1 - x^2 - y^2}$ for any point in the upper hemisphere, or $y = -\sqrt{1 - x^2 - z^2}$ as a separate hemisphere.

- The sphere can be locally parametrized, which is a smooth embedding.

Some more theorems and definitions follow:
Theorem. Let M_1 be an m_1-dimensional submanifold of $\mathbb{R}^{n_1}$, and let M_2 be an m_2-dimensional submanifold of $\mathbb{R}^{n_2}$. Then $M_1 \times M_2$ is an $(m_1 + m_2)$-dimensional submanifold of $\mathbb{R}^{n_1+n_2}$.

For example, the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, submanifold of $\mathbb{R}^4$.

An embedding ψ as in condition 3 of the above theorem is called a **parametrization** of the manifold M. If the domain & image of ψ are V and $M \cap V$, then $\varphi = \psi^{-1} : M \cap V \to V$ is called a **coordinate chart**. The components are called **local coordinates**. Two coordinate charts with overlapping domains induce a **transition function** $\varphi_2 \circ \varphi_1^{-1}$.

A coordinate chart inputs manifold points and outputs the coordinates that map to them.

Theorem. The transition function on overlapping domains is a diffeomorphism.

If φ_1, φ_2 are coordinate charts and $f : M \to N$, then f is **differentiable** at a point if the map $\varphi_2 \circ f \circ \varphi_1^{-1}$ is differentiable from $\mathbb{R}$ to $\mathbb{R}$.

Why is this needed?

We do not have a notion of differentiability for functions on curved domains. Instead, we take the composition of coordinate charts with f, and instead verify that the change in coordinates under f is differentiable as a real-valued function.
So we say a parametrized embedding is differentiable if the change of coordinates map is differentiable as a real map.

f is a *diffeomorphism* on manifolds if it has smooth & smooth inverse.

3 Topological and Smooth Manifolds

3.1 Definitions

Let M be a space. The goal of this section is to define the conditions under which we can label M as a smooth manifold, regardless of any ambient geometry (say, of $\mathbb{R}^3$).

A **chart** on M is a pair (U, φ) such that:

- $U \subseteq M$
- $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism, and $\varphi(U)$ is open in $\mathbb{R}^{\ltimes}$.

An **atlas** on M is a collection of charts $\mathcal{A} := \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ such that:

- $\bigcup_{\alpha \in I} \supseteq M$
- $\forall \alpha, \beta \in I : U_\alpha \cap U_\beta \neq \varnothing$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a C^∞ map.

Then we say M satisfies the definition of an **abstract manifold** if:

- It is equipped with a Hausdorff and second countable topology
- It has an maximal atlas $\mathcal{A}$.

3.2 Instances and Failures

A stereotypical failure of the abstract definition of a manifold is as follows:

$$\{t \in \mathbb{R} : t \geq 0\}.$$

3.2.1 The Stereographic Projection

Ex. Take the 2-sphere $\mathbb{S}^2$. We'll establish an atlas on $\mathbb{S}^2$ by computing charts on open neighborhoods and verifying that the transition maps between them are smooth.

$$\ell_{xyz}(t) = (xt, yt, 1 + (z-1)t)$$

We'll solve $\ell_{xyz}(t) = (X, Y, 0)$ in order to find the point on the plane that intersect the line through (x, y, z) and the north pole.

$$1 + (z-1)t = 0 \quad \Rightarrow \quad t = \frac{1}{1-z}$$

$$\Rightarrow (X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

Now we have the *stereographic projections*:

$$S_N(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right), \qquad S_S(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

We'll find their inverses in order to verify the smoothness of transition maps.

$$\ell_{xy}(t) = (xt, yt, 1-t)$$

We need

$$x^2t^2 + y^2t^2 + (1-t)^2 = 1$$

$$\Rightarrow t^2(x^2 + y^2 + 1) - 2t = 0$$

Reject $t = 0$, since this itself is the north pole. Then

$$t = \frac{2}{x^2 + y^2 + 1}$$

$$\Rightarrow S_N^{-1}(x, y) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right)$$

$$S_S^{-1}(x, y) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2x}{x^2 + y^2 + 1}, \frac{1 - x^2 - y^2}{x^2 + y^2 + 1}\right)$$

Transition maps:

$$T_N = S_N \circ S_S^{-1}, \qquad T_S = S_S \circ S_N^{-1}$$

For the north:

$$T_N(x, y) = S_N(S_S^{-1}(x, y))$$

$$= \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right)$$

$$\text{while} \quad T_S(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right),$$

and these are smooth away from 0.

3.2.2 The Torus

Ex. 2. We'll construct an atlas on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, $x \mapsto [x]$. Let $0 < \varepsilon < 1$. Define the four neighborhoods:

$$U_1 := (-\varepsilon, \tfrac{1}{2} + \varepsilon) \times (-\varepsilon, \tfrac{1}{2} + \varepsilon),$$

$$U_2 := (\tfrac{1}{2} - \varepsilon, 1 + \varepsilon) \times (-\varepsilon, \tfrac{1}{2} + \varepsilon),$$

$$U_3 := (-\varepsilon, \tfrac{1}{2} + \varepsilon) \times (\tfrac{1}{2} - \varepsilon, 1 + \varepsilon),$$

$$U_4 := (\tfrac{1}{2} - \varepsilon, 1 + \varepsilon) \times (\tfrac{1}{2} - \varepsilon, 1 + \varepsilon).$$

Let $V_i := \psi(U_i), \quad i \in \{1, 2, 3, 4\}$, and set $\mathcal{A} = \{(V_i, \varphi_i) : i \in \{1, 2, 3, 4\}\}$ Then

$$\bigcup_{i=1}^{4} V_i = \mathbb{T}^2.$$

Take

$$\varphi_i : V_i \to U_i, \quad [x] \mapsto x.$$

These is well-defined by construction and also bijective, so

$$\varphi_i^{-1} : U_i \to V_i, \quad x \mapsto [x]$$

is well-defined.

Now let $a, b \in \{1, 2, 3, 4\}$ with $V_a \cap V_b \neq \emptyset$. Then the transition maps are

$$\varphi_{ab} = \varphi_b \circ \varphi_a^{-1} : V_a \to V_b,$$

$$(x, y) \mapsto (x + k_{ab},\ y + j_{ab}),$$

which are C^∞ as maps $\mathbb{R}^2 \to \mathbb{R}^2$. Thus $\mathcal{A}$ forms an atlas of the torus $\mathbb{T}^2$.

4 Tangent Spaces and Differentials

4.1 Tangent Spaces and Tangent Bundles

For a point on a submanifold $M \subseteq \mathbb{R}^n$, the **tangent space** is the set of all possible tangent vectors through any paths. The tangent space is a vector space centered at $\vec{0}$, whereas a tangent plane, and its analogues are centered at x.

Formally, a **tangent vector** to M at x is $g'(0)$ for some smooth path g such that $g(0) = x$. x is called the **base point**. The tangent space is the set of all tangent vectors for a point x, we write:

$$T_x M = \{(x, v) : v \text{ is a tangent vector}\}$$

The **tangent bundle** of a submanifold M is defined

$$TM := \bigcup_{x \in M} T_x M$$

The **tangent bundle projection** is the map

$$\pi : TM \to M, \quad (x, v) \mapsto x$$

Let $x \in M$ be an m-dimensional submanifold of $\mathbb{R}^n$. If $M \subset V$ is the image of some parametrization ψ (i.e. local embedding) with $\psi(a) = x$, then

$$T_x M = \operatorname{im}(D\psi(a)).$$

Theorem. If M is locally the level set of a smooth submersion f then

$$T_x M = \ker(Df(x)).$$

Ex. The 2-sphere $\mathbb{S}^2$ is the level set of the submersion

$$f(x, y, z) = x^2 + y^2 + z^2.$$

And thus

$$T_{(x,y,z)}\mathbb{S}^2 = \{(u, v, w) \in \mathbb{R}^3 : ux + vy + wz = 0\}.$$

Let $\psi : W \subset \mathbb{R}^m \to U \subset \mathbb{R}^n$ be a parametrization of an m-dimensional submanifold M with $\psi(a) = x$. Note that $D\psi(a)$ is injective since ψ is a parametrization. By an above theorem, $\operatorname{im}(D\psi(a)) = T_x M$. Since $D\psi(a)$ is linear, then $D\psi(a)$ is an isomorphism from $\mathbb{R}^m$ to $T_x M$.

The **tangent lift** of a differentiable map $\psi : W \subset \mathbb{R}^m \to \mathbb{R}^n$ is given by

$$T\psi : TW \to T\mathbb{R}^n, \quad (q, \dot{q}) \mapsto (\psi(q), D\psi(q) \cdot \dot{q}),$$

where $\dot{q}$ represents a vector attached to q. Essentially, the tangent lift takes a vector and lifts it to $\operatorname{im}(D\psi(q)) = T_q M)$.

Theorem. The tangent bundle TM of any m-dimensional submanifold M of $\mathbb{R}^n$ is a 2-dimensional submanifold of $\mathbb{R}^n$.

Now consider a differentiable map $f : M \to N$ between manifolds. The **tangent map** at x is the map

$$T_x f : T_x M \to T_{f(x)} N, \quad v \mapsto T_x f(v) := \frac{d}{dt}\bigg|_{t=0} f(g(t)),$$

where g is a path in M where $g(0) = x$ and $\frac{d}{dt}|_{t=0} g(t) = v$.

The map $T_x f$ is often written as $D(f)$ and called the derivative of f at x.

All of the tangent maps $T_x f$ for $x \in M$ together define the **tangent lift** of f:

$$Tf : TM \to TN, \quad (x, v) \mapsto (f(x), T_x f(v)).$$

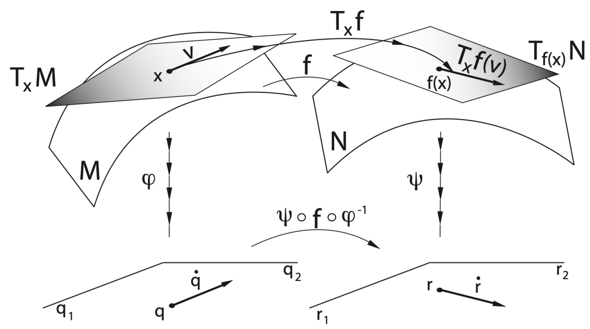

Fig. 4: The various maps associated with $f : M \to N$.

4.2 The Cotangent Space

Let $f : M \to \mathbb{R}$ be smooth. Then $T_{f(x)}\mathbb{R} = \mathbb{R}$, so

$$T_x f : T_x M \to \mathbb{R}.$$

In this case, $T_x f$ is also called the **differential** of f at x and written $df(x)$.

The **cotangent space** to M at x, denoted $T_x^* M$, is the *dual space* of $T_x M$. The **cotangent bundle** T^*M of M is given as

$$T^*M = \bigcup_{x \in M} T_x^* M.$$

Example. Take the circle $\mathbb{S}^1$, and point $(-1, 0)$. Then

$$T_{(-1,0)}\mathbb{S}^1 = \{(0, t) : t \in \mathbb{R}\}.$$

Take $p(0, t) := t$. Then $p \in T_{(-1,0)}^* M$. Note that df is also an element of the cotangent space, for any $f : M \to \mathbb{R}$.

Theorem. T^*M is a manifold with dimension twice that of M.

Also note that $\mathbb{R}^m \cong T_x M$ and $T_x^* M \cong (\mathbb{R}^m)^* \cong \mathbb{R}^m$, as well as that $T^*(f^{-1}) = (T^* f)^{-1}$ and $T(f^{-1}) = Tf^{-1}$.

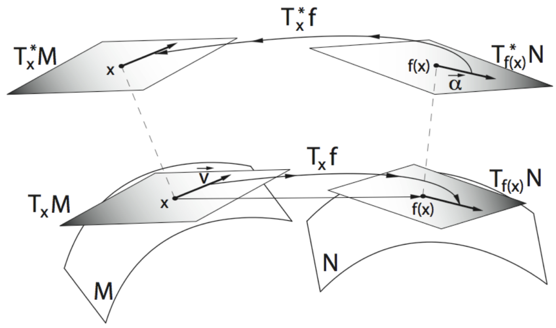

Fig. 5: The tangent and cotangent maps.

5 Vector Fields and Flows

In this chapter we'll discuss two important sets of objects pertaining to geometry on manifolds.

Let M be a manifold. A **vector field** on M is formally defined as map $X : M \to TM$, such that $X(z) \in T_z(M)$ for all $z \in M$. Essentially, it is a map that sends each point on a manifold to its tangent space.

In hindsight, this coincides with the vector fields we know in $\mathbb{R}^n$, as $T_z(\mathbb{R}^n) = \mathbb{R}^n$ for every $z \in \mathbb{R}^n$. Thus, vector fields in $\mathbb{R}^n$ are maps $X : \mathbb{R}^n \to \mathbb{R}^n$.

Denote k a smooth map $M \to \mathbb{R}$. Such maps are called **scalar fields**. The set of all scalar fields on a manifold is denoted $\mathcal{F}(M)$, which forms a commutative ring under the standard addition and multiplication operations on functions. The set of smooth vector fields on a manifold is denoted $\mathfrak{X}(M)$.

We can define the following operations on $\mathfrak{X}(M)$:

- Addition, defined by $(X_1 + X_2)(z) := X_1(z) + X_2(z)$.
- Scalar field multiplication, defined by $(kX)(z) := k(z)X(z)$.

It is worth observing that $\mathfrak{X}(M)$, equipped with the above operations, forms a module over the ring $\mathcal{F}(M)$.

We can define an **integral curve** of a vector field X as a differentiable map $c : I \to M$, where I is an open interval subset of $\mathbb{R}$, as a curve satisfying

$$c'(t) = X(c(t))$$

for every $t \in I$.

As a result, when expressed in terms of coordinates, vector fields represent solutions to first order differential equations. Under certain conditions, integral curves exist and are unique (for example, the Picard-Lindelöf theorem for ordinary differential equations in $\mathbb{R}$.)

For $z \in M$, we can denote the integral curve through z as $\Phi_z(t)$. Combining all of these maps gives us a map $\Phi : M \times \mathbb{R} \to M, (z,t) \to \Phi_z(t)$. We denote this map the **flow map** of a vector field X.

Theorem. Let X be a differentiable vector field on a manifold M, and let Φ be the induced flow map. Then Φ satisfies the following properties:

- $\Phi(z, 0) = \text{id}$, i.e. $\Phi(z, 0) = z$ for all z on the domain of definition of Φ.
- $\Phi(z, t + s) = \Phi(z, t) + \Phi(z, s)$. This means the map $t \mapsto \Phi_t(z)$ is a homomorphism on the group of diffeomorphisms of M.
- Fix t. For every t, $\Phi(z, t)$ is a diffeomorphism onto its image.

6 Differential Forms and Integration

6.1 Tensors, Wedge Products, and the Exterior Algebra

A **differential 1-form** is a map $\theta : M \to T^*M$ such that $\theta(x) \in T^*_x(M) \quad \forall x \in M$. Let ω be a 1-form. A 1-form can be viewed as a smooth assignment $\omega_p : T_pM \to \mathbb{R}$ of points to elements of the cotangent bundle.

A **covariant k-tensor** on a vector space V is a multilinear map

$$V^k \to \mathbb{R}.$$

A **contravariant k-tensor** is a multilinear map

$$(V^*)^k \to \mathbb{R}.$$

k is the **rank** of the tensor.

The **tensor product** of a covariant k-tensor S and a covariant ℓ-tensor T is the covariant $(k+\ell)$-tensor $S \otimes T$ defined by

$$(S \otimes T)(v_1, \dots, v_{k+\ell}) = S(v_1, \dots, v_k)\, T(v_{k+1}, \dots, v_{k+\ell}).$$

If T is a covariant 2-tensor, it can be described by a matrix:

$$T(v, w) = v^T M w.$$

(e.g. an inner product is a positive definite symmetric covariant 2-tensor.)

The **wedge product** of two 1-forms α, β is defined by

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha.$$

A more thorough definition is as follows:

Let $\omega_1, \cdots, \omega_n$ be a collection of k-forms. Then the wedge product $\omega_1 \wedge \cdots \wedge \omega_n$ can be written as:

$$\omega_1 \wedge \cdots \wedge \omega_n = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \bigotimes_{k=1}^{n} \omega_{\sigma(n)}$$

The proof is by noting that the wedge product is associative by associativity of the tensor product, and inductively expanding the definition on two 1-forms.

Note that this is very similar in definition to the determinant! Indeed, using this definition we can show that

$$\omega_i \wedge \omega_j = \begin{cases} 0, & \text{if } i = j \\ -\omega_j \wedge \omega_i, & \text{if } i \neq j \end{cases},$$

i.e. the wedge product establishes an alternating algebra known as the **exterior algebra**.

Let f be a scalar field with local coordinates $(x_1, \cdots, x_n)$. We can define the 1-forms $\mathrm{d}x_i$ such that $\mathrm{d}x_i((v_1, \cdots, v_n)) = v_i \in \mathbb{R}$. Then the differential $\mathrm{d}f$ is a 1-form defined by $\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i$.

It is also useful to note that a 0-form is by convention a function in coordinates, while a k-form is a smooth assignment $\omega_p : \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ times}} \to \mathbb{R}$ that is both multilinear and alternating. As a result, we can also express most k-forms as k consecutive wedge products of 1-forms.

6.2 The Exterior Derivative

We are now ready to define the exterior derivative. We will do this axiomatically, as establishing a general definition is tedious and I'm only human, after all.

To introduce some notation, the space of multilinear functionals on a vector space V is denoted $\Lambda^k(V)$. Notably, a k-form is a map $\omega_p : \Lambda^k(T_pM) \to \mathbb{R}$. Further, the space of all smooth k-forms on a manifold is denoted $\Omega^k(M)$.

The **exterior derivative** is the unique map $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ which satisfies the following four axioms:

- The operator d applied to a 0-form f is the differential $\mathrm{d}f$.
- If α, β are 1-forms, then $\mathrm{d}(a\alpha + b\beta) = a\,\mathrm{d}\alpha + b\,\mathrm{d}\beta$, where a and b are scalar elements of the ambient field.
- If α is a k-form and β is an l-form, then $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$. This is known as the *graded product rule*.
- If α is a k-form, then $\mathrm{d}(\mathrm{d}\alpha) = 0$.

The proof of uniqueness exists, but is quite long and complicated.

We'll also define the *pullback* of a form axiomatically:

Let f be a scalar field and let α, β be arbitrary 1-forms. Let $\varphi : M \to N$ be a smooth map between manifolds. The **pullback** of φ is the unique map φ^* that satisfies the following properties:

- $\varphi^*(\mathrm{d}f) = \mathrm{d}(\varphi^* f) = \mathrm{d}(f \circ \varphi)$
- $\varphi^*(f\alpha) = (\varphi^* f)(\varphi^* \alpha) = (f \circ \varphi)(\varphi^* \alpha)$
- $\varphi^*(\alpha \otimes \beta) = (\varphi^* \alpha) \otimes (\varphi^* \beta)$
- $\varphi^*(\alpha\beta) = (\varphi^* \alpha)(\varphi^* \beta)$
- $\varphi^*(\alpha \wedge \beta) = (\varphi^* \alpha) \wedge (\varphi^* \beta)$

It is easy (and correct, usually as a definition from which the axioms are proven) to view the pullback of a form as the composition of the form and the smooth map that the pullback is defined by.

6.3 Integration of Forms

We can define the notion of an integral in the broader setting of an abstract manifold as an operation that we can perform on k-forms. The idea will be to take a k-dimensional manifold and a smooth k-form, and to integrate the form over the manifold by relating forms to something we can Riemann integrate.

Let M be a smooth oriented k-dimensional manifold equipped with an atlas $\mathcal{A} := \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$. Let ω be a smooth k-form, and let $\{\rho_\alpha\}_{\alpha\in I}$ be a collection of smooth functions with the property that:

- $\mathrm{supp}(\rho_\alpha) \subseteq U_\alpha$ for every $\alpha \in I$
- For all $x \in M$, $\sum_{\alpha\in I} \rho_\alpha(x) = 1$

Such a collection is called a *partition of unity*.

We can thus decompose our form as $\omega = 1 \cdot \omega = \left(\sum_{\alpha\in I} \rho_\alpha\right)\omega = \sum_{\alpha\in I} \rho_\alpha\omega$, and we can denote each $\rho_\alpha\omega := \omega_\alpha$.

Recall the definition of the *pullback* of a form from the previous section. Suppose for every $\alpha \in I$, the pullback of ω_α under φ_α^{-1} is given by $f_\alpha \, dx_1 \wedge \cdots \wedge dx_k$ We can then define the integral:

$$\int_M \omega := \sum_{\alpha\in I} \int_{U_\alpha} \omega_\alpha := \sum_{\alpha\in I} \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^*\omega_\alpha := \sum_{\alpha\in I} \int_{\varphi_\alpha(U_\alpha)} f_\alpha \, dx_1 \cdots dx_k$$

This definition of the integral of a form expresses it as a Riemann integral in $\mathbb{R}^n$!

Ex. We'll compute the integral of the 2-form

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$

over the unit sphere $S^2 \subset \mathbb{R}^3$ using stereographic projection.

Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$, and define the stereographic charts:

$$U_N = S^2 \setminus \{N\}, \quad U_S = S^2 \setminus \{S\}.$$

The inverse stereographic projection from the south pole is:

$$\Phi_S(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{1 - u^2 - v^2}{1 + u^2 + v^2} \right).$$

We compute the pullback:

$$\Phi_S^*\omega = (\Phi_S \cdot (\partial_u \Phi_S \times \partial_v \Phi_S))\, du \wedge dv.$$

A direct computation yields:

$$\Phi_S^*\omega = \frac{4}{(1 + u^2 + v^2)^2}\, du \wedge dv.$$

The upper hemisphere $H_+ = \{z \geq 0\}$ corresponds to the disk:

$$D = \{u^2 + v^2 \leq 1\}.$$

Thus:

$$\int_{H_+} \omega = \int_D \frac{4}{(1+u^2+v^2)^2}\, du\, dv.$$

Switching to polar coordinates:

$$u = r\cos\theta, \quad v = r\sin\theta, \quad du\, dv = r\, dr\, d\theta,$$

we obtain:

$$\int_D \frac{4}{(1+u^2+v^2)^2}\, du\, dv = \int_0^{2\pi} \int_0^1 \frac{4r}{(1+r^2)^2}\, dr\, d\theta.$$

Let $t = 1 + r^2$, so $dt = 2r\, dr$. Then:

$$\int_0^1 \frac{4r}{(1+r^2)^2}\, dr = 2\int_1^2 t^{-2}\, dt = 2\left[-\frac{1}{t}\right]_1^2 = 1.$$

Hence:

$$\int_{H_+} \omega = \int_0^{2\pi} 1\, d\theta = 2\pi.$$

By symmetry (or using the north chart with corrected orientation), we also have:

$$\int_{H_-} \omega = 2\pi.$$

Therefore:

$$\int_{S^2} \omega = 4\pi.$$

An important theorem regarding integrals of forms is *Stokes' Theorem.* It states that provided you have a smooth oriented k-dimensional manifold M and a smooth k-form ω, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

This theorem relates integration over a k+1-dimensional space to integration over a k-dimensional space.

7 Riemannian Geometry

One of the key characteristics of our studies of calculus in n-dimensional Euclidean space is our ability to measure vectors and take angles between them.

The area form of any parametrized surface $\phi(u,v) = (f(u,v), g(u,v), h(u,v))$ in $\mathbb{R}^2$, for example, is given by

$$\omega = \left\| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right\| du \wedge dv.$$

Evidently we make use here of both the Euclidean norm and the cross product to measure the scale difference from the standard area form $dx \wedge dy$ in $\mathbb{R}^2$. The norm establishes length and the cross product relies heavily on an angular separation between vectors.

The ability to treat the manifold $\mathbb{R}^n$ as a vector space with an inner product is essential for nearly all the primary applications of vector calculus. However, a manifold does not need to be a vector space.

To generalize the foundational notions of geometry to abstract manifolds, we must first leverage the fact that, although our manifolds themselves are not vector spaces, each point on a manifold is assigned to a tangent space that is one.

7.1 Riemannian Metrics: Inner Products and Induced Norms

Let M be a manifold.

A **Riemannian metric** g on M is a smooth, symmetric, positive-definite covariant 2-tensor field on M. This means that every point $p \in M$ is assigned an inner product on the tangent space T_pM by the map g. We denote the map as follows:

$$g : M \to \Lambda^2(TM), \quad p \mapsto \langle \cdot, \cdot \rangle_p$$

A **Riemannian manifold** is a pair (M, g), i.e. a Manifold equipped with a Riemannian metric.

For example, $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is the dot product, is the classical example of a Riemannian manifold, in which each point is assigned to the same inner product. Indeed, any manifold that is itself an inner product space automatically has a constant Riemannian metric.

The **norm** associated to a Riemannian metric is defined as $\| \cdot \|_p := \sqrt{\langle \cdot, \cdot \rangle_p}$.

Theorem. Let M be a manifold. Let (N, g) be a Riemannian manifold and let $\varphi : M \to N$ be a smooth immersion. Then the pullback $\varphi^* g$ of g by φ is a Riemannian metric on M.

In the proof, while most properties of a Riemannian metric are achieved for free, the immersion assumption is required to guarantee the positive-definiteness property. The above theorem is significant because it allows us to easily establish Riemannian metrics on any immersed submanifold of $\mathbb{R}^n$ (for example, the sphere) via a pullback.

Any Riemannian metric on an immersed submanifold M of $\mathbb{R}^n$ induced by a pullback from $\mathbb{R}^n$ is called the **first fundamental form** on M.

7.2 Arc Lengths and Isometries

Having established notions of length and angles, we can now discuss arc length.

Let (M, g) be a Riemannian manifold. The **arc length** of a smooth path $\gamma : [a, b] \subset \mathbb{R} \to M$ is defined by

$$L = \int_\gamma \|\gamma'(t)\|_{\gamma(t)}\, dt,$$

where the norm in the integrand is the associated norm to the metric g.

Let (Q, g) and (R, h) be Riemannian manifolds. An **isometry** is defined as a diffeomorphic map $\varphi : (Q, g) \to (R, h)$ such that $\varphi^* h = g$. If such a φ exists, we say that the Riemannian manifolds (Q, g) and (R, h) are **isometric**.

Theorem. Let φ be a diffeomorphism between Riemannian manifolds. Then *the following are equivalent:*

- φ is an isometry;
- φ preserves lengths of tangent vectors;
- φ preserves arc length;
- φ preserves distances;
- φ preserves geodesics.

We will define *geodesics in the following section.*

7.3 Geodesic Curves

Geodesic curves on Riemannian manifolds provide us with standard notions of *straight lines*. In Euclidean space, the shortest path from one point to another is given by their linear interpolation, i.e. the straight line between the points. We'd like to establish this familiar notion on abstract manifolds.

Let (M, g) be a Riemannian manifold. A smooth path $\gamma : [a, b] \subset \mathbb{R} \to M$ is a **geodesic** if it has constant nonzero speed (i.e. $\|\gamma'(t)\| \equiv c \neq 0$), and furthermore it is a stationary point of the length functional given by

$$\mathcal{S}_l(q) := \int_q \|q'(t)\|_{q(t)}\, dt.$$

8 Toward Mechanics

8.1 Mechanical Systems

The establishment of geometry on manifolds allows us to work easily with general *mechanical systems.* These are sets of differential equations that govern mechanical processes in a space.

Newtonian mechanics, for example, are governed by the equation

$$\frac{d}{dt}(m_i r_i) + \frac{\partial U}{\partial r_i} = 0,$$

or, more familiarly,

$$\mathbf{F}_{\text{net}} = m\mathbf{a}.$$

We now explore the role of geodesics in establishing a different mechanical system—the *Lagrangian* mechanical system.

8.2 The Euler-Lagrange Equation

We start with some definitions:

A *functional* is any mapping from the space of curves to the real numbers. For example, the length of curve γ is a functional given by

$$\Phi(\gamma) = \int_{t_0}^{t} \sqrt{1 + \dot{x}^2}\, dx$$

as where x is a position function of t.

An *increment* to a functional is a slight perturbation to its input. For example, consider the increment to a functional Φ given by $\Phi(\gamma + h) - \Phi(\gamma)$, where h is a function of t. A functional is *differentiable* if the increment $\Phi(\gamma + h) - \Phi(\gamma)$ is equal to $F + R$, where F is a linear function of h and R behaves like an $O(h^2)$ term. The linear part F is called the *differential* of a functional.

Let $\gamma := \{(x, t) : x = x(t), t_0 \leq t \leq 1\}$, and let the *Lagrangian* $\mathcal{L} := \mathcal{L}(a, b, c)$ be a differentiable function of three variables. Define the functional Φ by

$$\Phi(\gamma) := \int_{t_0}^{t} \mathcal{L}(x(t), \dot{x}(t), t) dt$$

Note that for differentiation in $\mathbb{R}^n$, the derivative of a function satisfies

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \mathbf{h} Df(\mathbf{x}) + O(\mathbf{h}^2)$$

.
Similarly, for a functional, we can regard differentiation analogously as:

$$\Phi(\gamma + h) - \Phi(\gamma) = F(h) + R,$$

where R is the nonlinear term and F plays the role analagous to the Jacobian.

Theorem. The functional
$$\Phi(\gamma) = \int_{t_0}^{t_1} \mathcal{L}(x, \dot{x}, t)\, dt$$
is differentiable, and its derivative is given by the formula
$$F(h) = \int_{t_0}^{t_1} \left[\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\right] h\, dt + \left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\, h\right)\Bigg|_{t_0}^{t_1}.$$

Proof.
$$\begin{aligned}\Phi(\gamma + h) - \Phi(\gamma) &= \int_{t_0}^{t_1} \Big[\mathcal{L}(x + h, \dot{x} + \dot{h}, t) - \mathcal{L}(x, \dot{x}, t)\Big]\, dt \\ &= \int_{t_0}^{t_1} \left[\frac{\partial \mathcal{L}}{\partial x} h + \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{h}\right] dt + O(h^2) = F(h) + R,\end{aligned}$$
where
$$F(h) = \int_{t_0}^{t_1} \left(\frac{\partial \mathcal{L}}{\partial x} h + \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{h}\right) dt \quad \text{and} \quad R = O(h^2).$$
Integrating by parts, we get that
$$\int_{t_0}^{t_1} \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{h}\, dt = -\int_{t_0}^{t_1} h \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) dt + \left(h \frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\Bigg|_{t_0}^{t_1}.$$
□

Similar to n-dimensional calculus, to find a curve γ at which the functional Φ achieves an extremum, it is in our interest to set the "derivative" equal to zero. By the above theorem, this leads us to examine the equation
$$\int_{t_0}^{t_1} h \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) dt + \left(h \frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\Bigg|_{t_0}^{t_1} = 0$$

An *extremal* of a differentiable functional $\Phi(\gamma)$ is a curve γ such that $F(h) = 0$ for all h. An extremal is analgous to an extremum in n-variable calculus.

Theorem. The curve $\gamma : x = x(t)$ is an extremal of the functional $\Phi(\gamma)$ on the space of curves passing through the points (x_0, t_0) and (x_1, t_1) iff
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$
along the curve $x(t)$. The proof is done by examining the equation of interest from earlier.

The *Euler-Lagrange equation* for a Lagrangian $\mathcal{L}$ and a functional $\Phi = \int_{t_0}^{t_1} \mathcal{L}(x, \dot{x}, t)\, dt$ is thus given by
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0.$$

We can now compare the governing equations of motion across Newtonian and Lagrangian mechanical systems:

$$\text{Newtonian:} \quad \frac{d}{dt}(m_i r_i) + \frac{\partial U}{\partial r_i} = 0 \qquad\qquad \text{Lagrangian:} \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

Letting T denote kinetic energy and U potential energy, if we define the Lagrangian as $\mathcal{L} = T - U$, it follows that the Newtonian and Lagrangian equations are equivalent across mechanical systems.

8.3 Conclusion

Differential geometry ultimately provides us with the framework to extend the notion of *space*, on which we can do calculus. Different from topology or algebra, the notions of topological spaces and vector spaces are too broad. We need a space we can navigate, and in the case of Riemannian manifolds, a space on which we can also establish the familiar notions of geometry.

Using a second countable Hausdorff topology, charts, and Riemannian metrics, we can ultimately establish Riemannian manifolds as objects with these properties. Such generalizations relax assumptions that we are often bound to by Euclidean space.

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