DRP report

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Abstract

This report describes our study of functions of the form

$$n \mapsto v^*_{u\lambda}(n \cdot v_{w\lambda})$$

arising from highest weight representations and discuss their relationship with biperfect crystal bases of the coordinate ring $\mathbb{C}[N]$. The first section of this report introduces the necessary background on crystals, along with examples. In the second section, we explore the connection between skew shapes and generalized minors, and present a conjecture on the factorization of generalized minors under the assumption that the associated diagram is "split" in a precise geometric sense.

Preliminaries

Crystal structures emerge when studying the combinatorics of seemingly disconnected ares of mathematics. The remarkable feature of crystal theory is that it gives a way to characterize when two counting problems are secretly the same, despite their superficial differences.

Brief overview of Lie theory

Definition

A (complex) Lie algebra $\mathfrak{g}$ is a vector space (over $\mathbb{C}$) with an algebra structure given by the Lie bracket. The Lie bracket is a bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that respects the following conditions:
(i) (skew-symmetry) $[x, y] = -[y, x]$
(ii) (Jacobi Identity) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$

We call a Lie Algebra *simple* if it does not have a non-trivial ideal (as a ring). A Lie algebra is then *semisimple* if it is the direct sum of simple ones.

It turns out that semisimple Lie algebras can be though of as matrix spaces. There is a classification: simple Lie algebras correspond (in a constructive way) to certain Dynkin Diagrams, and the latter parametrize specific families of groups. We will not expose all the details of this correspondence, since for our purpose we will only work over representations of so-called type A_{n-1}".

Its Dynkin diagram is the segment with $n-1$ points, and its group is the permutation group on $n-1$ letters. In this case, the lie algebra is the special linear group

$$\mathfrak{sl}_n(\mathbb{C}) = \{X \in M_{n\times n} : tr(X) = 0\} \qquad [X,Y] = XY - YX$$

Representations of Lie algebras

The representation theory of Lie Algebra is a very broad subject. As we will see, there are helpful in this discipline.

We recall that a *representation* of a Lie Algrebra is a vector space V enriched with a linear $\mathfrak{g}$-action. In other words, a representation is a homomorphism $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$ such that $[\rho(X), \rho(Y)] = \rho([X,Y])$. Representations of Lie algebra are understood through the following construction. First, we will call the Cartan subalgebra $\mathfrak{h}$ the maximal abelian subalgebra of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the Cartan is the subalgebra of diagonal matrices. Then, given a representation V, we call $v \in V$ a weight vector of weight $\lambda \in \mathfrak{h}^*$ if it satisfies the following:

$$\rho(h)v = \lambda(h)v \quad \forall h \in \mathfrak{h}$$

The weight vectors of some fixed weight λ form a subspace of V which we denote V_λ. We have the following theorem.

Theorem
Every representation admits a *weight space decomposition*

$$V = \oplus_\lambda V_\lambda$$

Moreover, we may always take the sum to be finite.

The *roots* of a Lie Algebra are the weights of the representation given by $\mathfrak{g}$ acting on itself (through the bracket). The study of the Cartan, and the roots that live inside of it, are where techniques from combinatorics and linear algebra shine. For example, we define the Weyl group of a Lie algebra to be the group of transformations of $\mathfrak{h}$ generated by the reflections along the roots.

Crystal structures

Definition
Let $\mathfrak{g}$ be a semisimple Lie algebra with index set I corresponding to its simple roots. A crystal is a set B equipped with partial maps

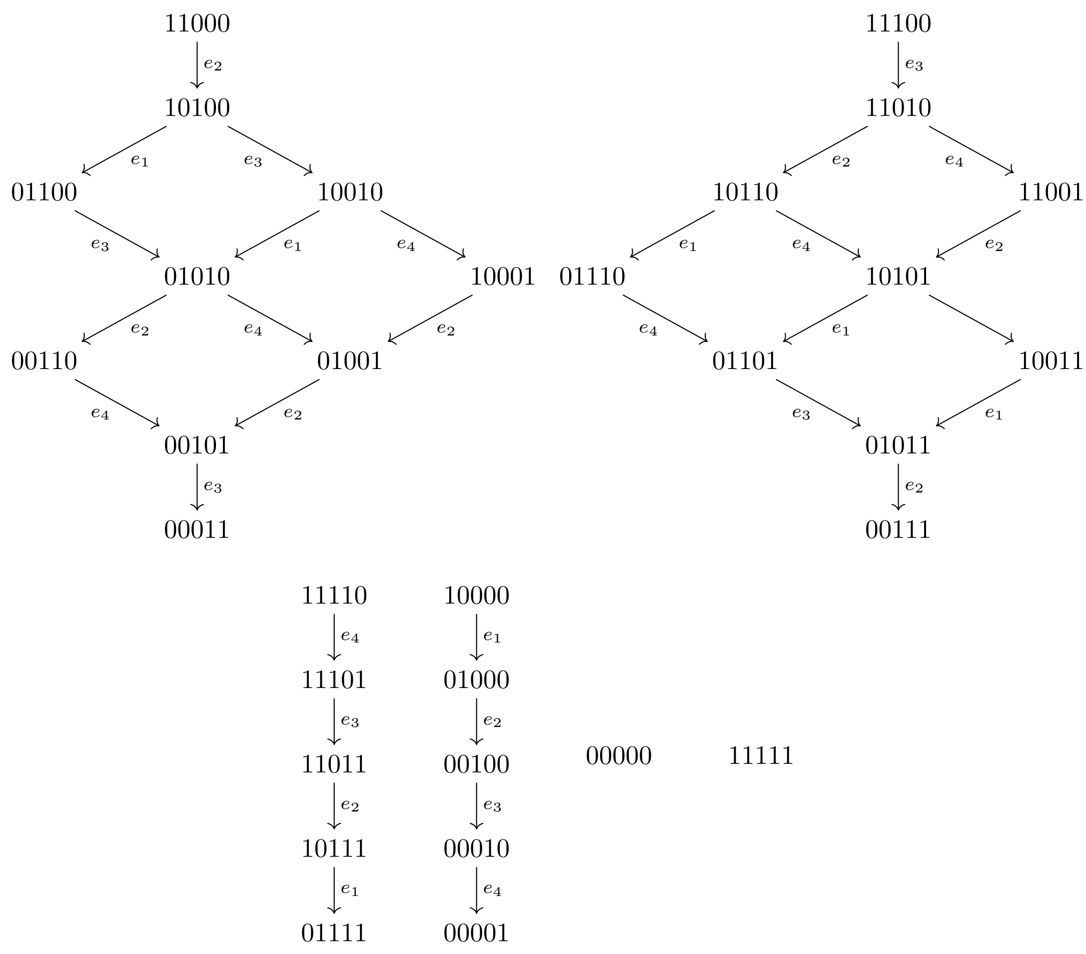
$$e_i, f_i : B \to B \sqcup \{0\}, \quad i \in I$$

and a weight map $\mathrm{wt} : B \to P$ where P is the weight lattice of $\mathfrak{g}$. In addition, the maps are required to obey various compatibility conditions.

Example
Let B denote the set of binary strings of length 5. We define operators e_i and f_i for $i = 1, 2, 3, 4$ by replacing substring in the following way:

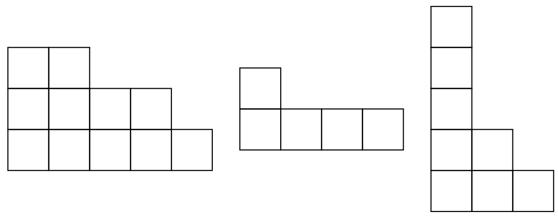
- If $b \in B$ has a substring 01 at the $(i, i+1)$-th position, then $e_i(b)$ replaces that substring with 10. Otherwise, it does nothing.
- If $b \in B$ has a substring 10 at the $(i, i+1)$-th position, then $f_i(b)$ replaces that substring with 01. Otherwise, it does nothing.

The weight of a binary string is determined by the number and positions of its 1's. Applying f_i lowers the weight by the simple root α_i, while e_i raises it For example, $e_3(11010) = 11100$, and $f_1(01000) = 01000$. Starting with a prescribed string b_0, we can draw out all the possible compositions of applications of e_i's and f_i's and obtain a diagram which outlines the structure of the crystal $B(b_0)$. In fact, WLOG we may pick b_0 such that $f_i(b_0) = b_0$ for all $i \in I$; then, b_0 will sit at the top of the crystal. Equivalently, all the 1's appearing in b_0 occur in a row starting at the first entry. Here are all 5 distinct crystals for binary words of length 5:

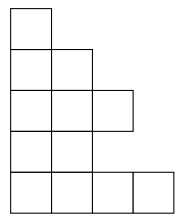


Example
A Young diagram is an array of boxes such that each layer is left-justified and does not include

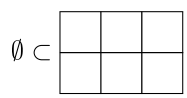
more boxes than the one below. For example, the following drawings are young diagrams:



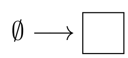
Meanwhile, the following drawing is *not* a valid Young diagram, because row 3 exceeds row 2:



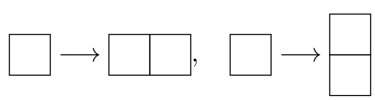
Young diagrams that fit in a $k \times \ell$ grid also give rise to a crystal structures. To illustrate this, let us start with the empty tableau sitting inside a 3x2 grid:



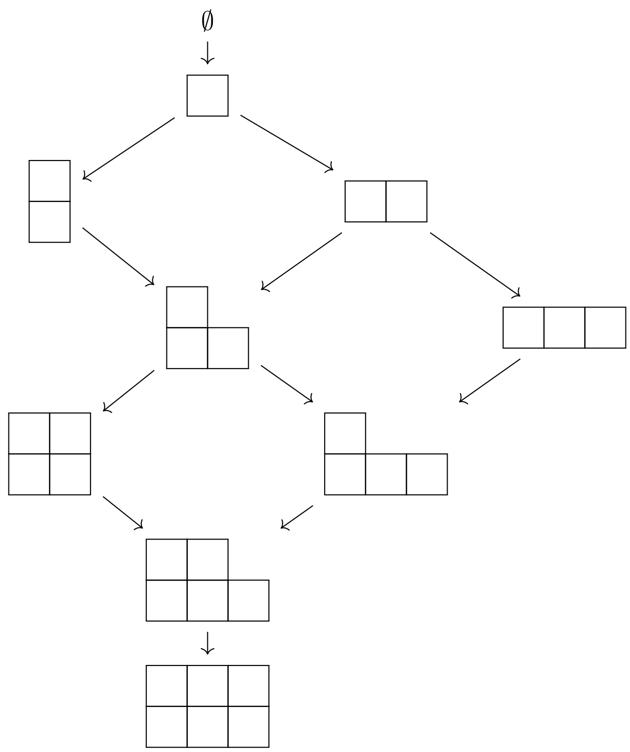
There is exactly one way to extend it by adding one box:



However, there are two ways of extending the single box to a valid diagram



Let us carry out this process until the 3×2 grid is saturated.



This picture is eerily similar to the crystal which came from binary words on 5 letters, and that is not a coincidence; the combinatiorial process is the same in hiding.

Unipotent group N and matrix coefficients

In this section, we will remain in the simplest case $G = SL_n(\mathbb{C})$ and type A_n, although much of what follows generalizes to all semisimple Lie groups. In this case, we define $N \subset G$ to be the group of upper triangular matrices with 1's along the diagonal. Then, the coordinate ring $\mathbb{C}[N]$ consists of all functions which can be written as polynomials over $\mathbb{C}$ in the entries x_{ij} of matrices in N.

Some members of $\mathbb{C}[N]$ will be of particular interest for this paper. Given a highest weight representation of G, which assigns to each weight $w\lambda$ an extremal vector $v_{w\lambda}$, we have functions

$$\Phi_w : N \to \mathbb{C}, \quad n \mapsto v_\lambda^*(n \cdot v_{w\lambda})$$

In fact, they all belong to $\mathbb{C}[N]$. These functions measure the component of $n \cdot v_{w\lambda}$ along the direction of the highest weight vector v_λ. In this sense, we record how the action of N redistributes weight vectors relative to the highest weight vector.

It turns out that representation theory endows $\mathbb{C}[N]$ with a rich combinatorial crystal structure. More precisely, $\mathfrak{g}$ acts on N from the left and the right, and this gives rise to a left and a right crystal structure. It is possible to construct a $\mathbb{C}$-vector space basis $B \subset \mathbb{C}[N]$ to which is attached two different crystal structures which are compatible, making B a bicrystal with maps

$$e_i, f_i, \tilde{e}_i, \tilde{f}_i$$

Broadly speaking, a basis is called *biperfect* if it is compatible with both the left and right crystal structures arising from the $\mathfrak{g}$-actions on $\mathbb{C}[N]$, in the sense that all corresponding operators preserve the basis up to 0, without producing linear combinations. It turns out not only that biperfect bases exist for $\mathbb{C}[N]$, but that different approaches give seemingly different biperfect bases.

Proposition 2.9 and remark 2.10 in "The Mirković-Vilonen Basis and Duistermaat-Heckman Measures" by Baumann, Kamnitzer et al. prove that the functions Φ_w belong to any biperfect basis of $\mathbb{C}[N]$. Note that in type A_n, the Φ_w's correspond exactly to picking out unipotent minors of matrices in N. A natural direction of generalization is to expand the statement of 2.10 to include all possible minors. Exploring this, we reached the following formalism: given two elements u, w of the Weyl group, we define a function

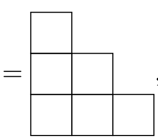
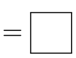
$$\Delta_{u,w}(n) = v^*_{u\lambda}(n \cdot v_{w\lambda})$$

which naturally generalizes Φ_w, since $\Delta_{\mathrm{id},w} = \Phi_w$.

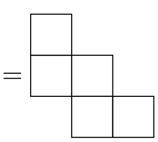
Skew shapes and conjecture

Skew shapes

In some contexts, it may be desirable to generalize Young diagrams in the following way. Suppose we are given two Young diagrams Y_1 and Y_2 such that $Y_2 \subset Y_1$. We say that the pair (Y_1, Y_2) is a skew shape, and we may model a special kind of diagram based on the "difference" of their diagrams. This is best illustrated by example. Suppose that we are given

$Y_1 =$  , $Y_2 =$ 

Substracting Y_2 from Y_1 by erasing the diagram of Y_2 contained in Y_1, we obtain

$Y_1/Y_2 =$ 

In type A_{n-1}, the weight lattice embeds into $\mathbb{R}^n/\langle e_1, \dots, e_n\rangle$, and the k-th fundamental weight is $\varpi_k = e_1 + \cdots + e_k$. The Weyl group in this case is $W = S_n$ and acts by permuting the standard basis vectors. Given an element $w\varpi_k \in W\varpi_k$, we find that

$$w\varpi_k = e_{w(1)} + \cdots + e_{w(k)}$$

and the element $w\varpi_k$ is fully determined by the list $\{w(1), \dots, w(k)\}$. Thus, every element in the orbit $W\varpi_k$ is uniquely determined by a k-sized subset of $\{1, \dots, n\}$. We may create a unique skew shape from a k-sized subset $I = \{i_1 < \cdots < i_k\}$, using the following process: for each $m \in [n]$ in increasing order,

1. if $m \in I$, draw a horizontal step

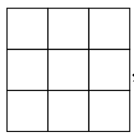
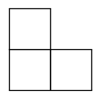
2. if $m \notin I$, draw a vertical step

Since I has k elements, the obtained path has k horizontal steps and $n-k$ vertical step, so it can be placed inside of an $k \times (n-k)$ rectangle and will go from corner to corner. Equivalently, this process associates a Young diagram to any $w\varpi_k \in W\varpi_k$. Now, given a pair $u\varpi_k, w\varpi_k$, if one of the path lies above the other, then the pair gives rise to a skew shape. In fact, if two weights γ, δ satisfy $\gamma - \delta \in Q_+$, then the path will be arranged one above the other so that we are guaranteed to obtain a skew shape. This means that, fixing a fundamental weight ϖ_k, we can connect the generalized minor operators $\Delta_{u,w}(n)$ described before to skew shapes.

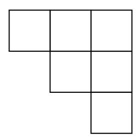
Let us illustrate this with an example. Suppose that we are working over $\varpi = e_1 + e_2 + e_3$, and we have permutations $u = \mathrm{id}$ and $v = (124653)$. Then,

$$u\varpi = e_1 + e_2 + e_3, \qquad w\varpi = e_2 + e_4 + e_6$$

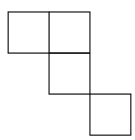
Here, we find that $w\varpi - u\varpi = e_4 + e_6 - e_1 - e_3$, which can be simplified to $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$ by applying the formula $e_a - e_b = \alpha_a + \cdots + \alpha_{b-1}$ for $a < b$. Thus, $w\varpi - u\varpi$ can be written as a positive sum over the root system, which means it belongs to Q_+ by definition. By the previous discussion, we can construct an associated skew shape. Going through the steps described above, we obtain that

$Y(u) =$  , $Y(w) =$ 

and the difference is

$Y(u)/Y(w) =$ 

As a second example, still taking $\varpi = e_1 + e_2 + e_3$, we can choose u and w such that $u\varpi_k = e_1 + e_2 + e_6$ and $w\varpi_k = e_3 + e_4 + e_5$. The associated skew shape is



Notice that this shape is "split" in the sense that it consists of two connected shapes, each of which could be independently obtained as a difference of smaller young diagrams. When we fall in this situation, we will say that the skew shape splits.

The core of our project was to reach and study the following conjecture.

Conjecture

Conjecture
Fix $G = \mathrm{SL}_n(\mathbb{C})$ and ϖ_k for some $k \in \{1, \ldots, n\}$. Let $u, w \in W$ be such that $u\varpi_k - w\varpi_k \in Q_+$. If the associated skew shape splits, then the generalized minor factors as

$$\Delta_{u,w}(n) = \Delta_{u_1,w_1}(n) \cdot \Delta_{u_2,w_2}(n)$$

where u_i, w_i and the weights ϖ_{i_1} and ϖ_{i_2} can be obtained from the skew shape in a natural way.

Remark: since $\Delta_{\bullet,\bullet}(n)$ is a regular function (after all, it lives in $\mathbb{C}[N]$), this factoring is really a statement about factoring polynomials.

Note that the statement is generically false in the non-split case. We wrote code that took as an input two elements of the orbit $W\varpi_k$, and outputs the polynomial associated to $\Delta_{\bullet,\bullet}$. Then, verifying the conjecture for two weights is a simple factorization exercise. This let us check the conjecture for numerous cases and skew shapes of various nature, confirming the split skew shape assumption is the correct condition. Additionally, it was verified that the conjecture generalizes: if a split skew shape consists of k components, then the generalized minor seems to factor into k subminors.

While we were unable to prove the result, we have identified a promising direction: the operator $\Delta_{u,w}$ can be re-interpreted in terms of the canonical bilinear pairing on $V(\varpi_k)^* \times V(\varpi_k)$ given by evaluation of functionals. More precisely, defining

$$\langle \cdot, \cdot \rangle : V(\varpi_k)^* \times V(\varpi_k) \to \mathbb{C}$$

$$\langle \nu, v \rangle = \nu(v)$$

lets us re-define Δ as

$$\Delta_{u,w}(n) = \langle v^*_{u\varpi_k}, n \cdot v_{w\varpi_k} \rangle$$

The key is that this pairing is in general multiplicative over tensor products; if V_1, V_2 are arbitrary finite dimensional vector spaces, then $\langle \cdot, \cdot \rangle$ on $(V_1 \otimes V_2)^* \times (V_1 \otimes V_2) \cong (V_1^* \otimes V_2^*) \times (V_1 \otimes V_2)$ satisfies the following rule on pure tensors

$$\langle \nu_1 \otimes \nu_2, v_1 \otimes v_2 \rangle_{V_1 \otimes V_2} = \langle \nu_1, v_1 \rangle_{V_1} \cdot \langle \nu_2, v_2 \rangle_{V_2}$$

Suppose that we were able to describe an embedding

$$\iota : V(\varpi_k) \hookrightarrow V(\varpi_{i_1}) \otimes V(\varpi_{i_2})$$

which is an N-homomorphism, isometric with respect to the evaluation pairing, and well-behaved in the sense that

$$\iota(v_{s\varpi_k}) = v_{s_1\varpi_{i_1}} \otimes v_{s_2\varpi_{i_2}}, \quad \forall s \in W$$

By abuse of notation, for any $\nu \in (V(\varpi_{i_1}) \otimes V(\varpi_{i_2}))^*$ we let $\iota(\nu)$ denote a choice of a functional that restricts to ν along the image of ι in $(V(\varpi_{i_1}) \otimes V(\varpi_{i_2}))^*$. We lead the reader verify that the choice of extension does not matter for the purpose of the following argument.

Assuming the existence of such a map, we obtain that

$$\begin{aligned} \Delta_{u,w}(n) &= \langle v^*_{u\varpi_k}, n \cdot v_{w\varpi_k} \rangle \\ &= \langle \iota(v^*_{u\varpi_k}), n \cdot \iota(v_{w\varpi_k}) \rangle \\ &= \langle (v_{u_1\varpi_{i_1}} \otimes v_{u_2\varpi_{i_2}})^*, ((n \cdot v_{w_1\varpi_{i_1}}) \otimes (n \cdot v_{w_2\varpi_{i_2}})) \rangle \\ &= \langle v^*_{u_1\varpi_{i_1}}, n \cdot v_{w_1\varpi_{i_1}} \rangle \cdot \langle v^*_{u_2\varpi_{i_2}}, n \cdot v_{w_2\varpi_{i_2}} \rangle \\ &= \Delta_{u_1,w_1}(n) \cdot \Delta_{u_2,w_2}(n) \end{aligned}$$

However, we were unable to show that the split skew shape assumption is sufficient for the construction of the well-behaved embedding ι. The principal difficulty is the requirement that $\iota(v_{s\varpi_k}) = v_{s_1\varpi_{i_1}} \otimes v_{s_2\varpi_{i_2}}$; this is precisely where the skew shape geometry should enter, and working formally with skew shapes requires complicated bookkeeping. The two other conditions are somewhat standard.

Another notable lead we followed is that in type A, all crystals of the form $V(\varpi_k)$ are exterior powers

$$V(\varpi_k) \cong \bigwedge^k \mathbb{C}^n$$

which could give a concrete way of constructing the embedding ι.