Polynomial Embedding and Shooting Methods

for Localized Radial Solutions of the

Swift–Hohenberg Equation

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Abstract

We study localized radially symmetric solutions of the Swift-Hohenberg (SH) equation. The approach has three steps. First, polynomial embedding transforms the PDE into an autonomous 5D first-order ODE system on $\mathbb{R}^5$, whose linearization at equilibrium has five eigenvalues: two positive, two negative, and one zero. Second, a Taylor series expansion in the radial variable r provides a recurrence relation for all coefficients. The problem becomes two free parameters (u_0, u_2) that make up the center-stable manifold. Third, a shooting method, where we integrate the 5D system forward from $r = \delta$ and require decay as r goes to infinity to find the localized solution as a zero of a 2D map. We provide derivations of each part, with eigenvalues, eigenvectors, recurrence relations, and the shooting algorithm.

Contents

1 Introduction and Notation

The Swift–Hohenberg (SH) equation is a polynomial derivation (PDE) used for pattern formation. We focus on the form on $\mathbb{R}^2$, looking for *radially symmetric localized solutions*, functions $U(r)$ that oscillate near the origin and decay to zero as $r \to \infty$.

1.1 Swift–Hohenberg Equation

The SH equation is:

$$0 = -(\Delta + \beta_4)^2 U + \beta_3 U + \beta_1 U^2 + \beta_2 U^3, \tag{1}$$

where the four parameters are:

Parameter	Name	Role
β_4	Wave number	Controls spatial scale of instability; inside $(\Delta + \beta_4)^2$
β_3	Bifurcation parameter	$\beta_3 > 0$ enables localized solutions
β_1	Quadratic nonlinearity	Breaks $U \to -U$ symmetry
β_2	Cubic nonlinearity	Saturating term

The parameter values used are

$$\beta_1 = -\tfrac{1}{2}\sqrt{6}, \quad \beta_2 = -\tfrac{1}{2}, \quad \beta_3 = \tfrac{1}{10}, \quad \beta_4 = -1,$$

which gives the real eigenvalue structure required for the center-stable manifold. (Lessard, 24)

1.2 Polynomial Embedding

Polynomial embedding transforms the singular non-autonomous ODE arising from polar coordinates into a polynomial autonomous system on $\mathbb{R}^5$. We add an equation: $w = 1/r$ that is $w' = -w^2$. This converts the $1/r$ singularity into a new state variable.

1.3 Taylor Series

Because the 5D system has a singularity at $r = 0$ (through $u_5 = 1/r$), not allowing us to integrate at the origin. Then, we substitute $U(r) = \sum_{n\geq 0} u_n r^n$ into the PDE to find a recurrence for all coefficients. Regularity at the origin forces $u_1 = u_3 = 0$, leaving (u_0, u_2) as the only free parameters.

1.4 Shooting

The shooting method converts the boundary value problem into a root-finding problem for a map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. We integrate the 5D ODE forward from $r = \delta$ and look for (u_0^*, u_2^*) with $\Phi(u_0^*, u_2^*) = (0, 0)$.

2 Polynomial Embedding

2.1 Reduction to a First-Order System

The fourth-order stationary SH equation Eq. (1) is split by introducing $V = (\Delta + \beta_4)U$:

$$(\Delta + \beta_4)\, U = V, \tag{2}$$
$$(\Delta + \beta_4)\, V = \beta_3 U + \beta_1 U^2 + \beta_2 U^3. \tag{3}$$

For radial solutions in $\mathbb{R}^2$, the Laplacian becomes $\Delta = \partial_{rr} + (1/r)\partial_r$. Our vector

$$\mathbf{u} = (u_1, u_2, u_3, u_4, u_5) = \big(U,\ U',\ V,\ V',\ 1/r\big),$$

Note: $u_5' = -u_5^2$ has $\frac{d}{dr}(1/r) = -1/r^2$ as a polynomial ODE. The **full 5D polynomial autonomous system $\mathbf{u}' = F(\mathbf{u})$** is:

$$u_1' = u_2, \tag{4}$$
$$u_2' = u_3 - u_5 u_2 - \beta_4 u_1, \tag{5}$$
$$u_3' = u_4, \tag{6}$$
$$u_4' = \beta_3 u_1 - u_5 u_4 - \beta_4 u_3 + \beta_1 u_1^2 + \beta_2 u_1^3, \tag{7}$$
$$u_5' = -u_5^2. \tag{8}$$

No explicit r-dependence is left.

2.2 Nonlinear Map

We can equivalently write the system as $N(U_1, U_2) = 0$, where

$$N(U_1, U_2) = \begin{pmatrix} (\Delta + \beta_4)U_1 - U_2 \\ (\Delta + \beta_4)U_2 - \beta_3 U_1 - \beta_1 U_1^2 - \beta_2 U_1^3 \end{pmatrix}.$$

This splits as $N = L + R$, where R contains the nonlinear terms β_1, β_2 and vanishes at $U_1 = 0$. Therefore $DN(0,0) = L$, so the linearization structure depends on β_3 and β_4.

3 Spectral Analysis

3.1 Linearization at the Origin

Linearizing the 5D system at $\mathbf{u} = \mathbf{0}$ (equivalently, as $r \to \infty$ where $u_5 \to 0$), the Jacobian is

$$A = DN(0,0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -\beta_4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \beta_3 & 0 & -\beta_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The last row and column decouple, giving $\lambda_5 = 0$ from equation Eq. (8).

3.2 Characteristic Equation

The 4×4 block has characteristic polynomial

$$\left(\lambda^2 + \beta_4\right)^2 = \beta_3 \quad \Longrightarrow \quad \lambda^2 = -\beta_4 \pm \sqrt{\beta_3}, \quad \lambda = \pm\sqrt{-\beta_4 \pm \sqrt{\beta_3}}.$$

3.3 Five Eigenvalues

With $\beta_3 = 1/10$ and $\beta_4 = -1$, we get $\lambda^2 = 1 \pm 0.3162$:

#	Symbolic	Numeric	Type
λ_1	$+\sqrt{-\beta_4 + \sqrt{\beta_3}}$	$+1.1473$	Unstable (fast)
λ_2	$-\sqrt{-\beta_4 + \sqrt{\beta_3}}$	-1.1473	Stable (fast)
λ_3	$+\sqrt{-\beta_4 - \sqrt{\beta_3}}$	$+0.8269$	Unstable (slow)
λ_4	$-\sqrt{-\beta_4 - \sqrt{\beta_3}}$	-0.8269	Stable (slow)
λ_5	0	0	Center (from $1/r$)

The spectrum has the structure required by the center-stable manifold theorem: 2 stable, 2 unstable, 1 zero. For each nonzero eigenvalue, the eigenvector is

$$\mathbf{v}(\lambda) = \left(1,\ \lambda,\ \lambda^2 - 1,\ \lambda(\lambda^2 - 1),\ 0\right)^\top.$$

4 Taylor Series Expansion

4.1 Reindexing

We substitute $U(r) = \sum_{n=0}^{\infty} u_n r^n$ into the PDE. The bilaplacian in polar coordinates is

$$\Delta^2 U = U'''' + \frac{2}{r}U''' - \frac{1}{r^2}U'' + \frac{1}{r^3}U'.$$

Using identity $\Delta(r^n) = n^2 r^{n-2}$, giving $\Delta^2(r^n) = n^2(n-2)^2 r^{n-4}$, each group of terms is reindexed to a power r^m:

Terms	Native power	Coefficient of u_{m+k}
U'''', $\frac{2}{r}U'''$, $-\frac{1}{r^2}U''$, $\frac{1}{r^3}U'$	r^{n-4}, $n = m + 4$	$(m+4)^2(m+2)^2\, u_{m+4}$
$2\beta_4 U''$, $\frac{2\beta_4}{r}U'$	r^{n-2}, $n = m + 2$	$2\beta_4(m+2)^2\, u_{m+2}$
$\beta_3 U$, $\beta_1 U^2$, $\beta_2 U^3$	r^n, $n = m$	Cauchy products

4.2 Singular Terms

Terms with negative powers of r vanish:

Regularity conditions at the origin

- At r^{-3}: $u_1 = 0$.
- At r^{-2}: $-2u_2 + 2u_2 = 0$ (automatic; no constraint on u_2).
- At r^{-1}: $9u_3 = 0 \Rightarrow u_3 = 0$.

These cause smoothness of $U(r)$ at $r = 0$. All next odd coefficients $u_5, u_7, \ldots$ are removed by induction.

4.3 Recurrence Relation

Setting the coefficient of each r^m to zero for $m \geq 0$ gives:

$$u_{m+4} = \frac{2\beta_4(m+2)^2\, u_{m+2} - (\beta_3 - \beta_4^2)\, u_m - \beta_1\, (u * u)_m - \beta_2\, (u * u * u)_m}{(m+4)^2(m+2)^2}, \tag{9}$$

where the Cauchy products are:

$$(u * u)_m = \sum_{k=0}^{m} u_k\, u_{m-k}, \qquad (u * u * u)_m = \sum_{k=0}^{m} (u * u)_k\, u_{m-k}.$$

4.4 Taylor Solution

Since all odd coefficients vanish, the solution depends on even powers:

$$U(r) = u_0 + u_2 r^2 + u_4(u_0, u_2)\, r^4 + u_6(u_0, u_2)\, r^6 + \cdots$$

The first explicit coefficients from Eq. (9) are:

$$u_4 = \frac{8\beta_4 u_2 - (\beta_3 - \beta_4^2)u_0 - \beta_1 u_0^2 - \beta_2 u_0^3}{64}, \tag{10}$$

$$u_6 = \frac{32\beta_4 u_4 - (\beta_3 - \beta_4^2)u_2 - 2\beta_1 u_0 u_2 - 2\beta_2 u_0^2 u_2}{576}. \tag{11}$$

The two *free parameters* (u_0, u_2) are the coordinates on the center-stable manifold W^{cs}.

5 Shooting

5.1 Manifold Explanation

The problem is a boundary value problem on $[0, \infty)$:

$$U(0) = u_0, \quad U'(0) = 0, \quad U''(0) = 2u_2, \quad U'''(0) = 0, \quad U(r) \to 0 \text{ as } r \to \infty.$$

The values u_0 and u_2 are not known in advance. Trajectories starting on W^{cs} decay to $\mathbf{0}$ as $r \to \infty$, while any deviation into the unstable directions $\mathbf{v}(+\lambda_1)$ or $\mathbf{v}(+\lambda_2)$ causes exponential blow-up. Shooting finds the precise (u_0, u_2) that keeps the trajectory on W^{cs} for all r.

Because $u_5 = 1/r$ blows up at $r = 0$, integration starts at a small $r = \delta = 0.01$ using the Taylor series of Section 4 as initial data, integrating forward to a truncation radius $L = 15$.

5.2 Missing Boundary Conditions

The 5D system requires 5 boundary conditions. Three are determined immediately; two must be found by shooting:

Condition	Value	Source
$U(0)$	u_0	center-stable manifold
$U'(0)$	0	radial symmetry
$U''(0)$	$2u_2$	center-stable manifold
$U'''(0)$	0	$u_3 = 0$ from recurrence
$u_5(\delta)$	$1/\delta$	$w = 1/r$ definition

This is a $\mathbb{R}^2 \to \mathbb{R}^2$ problem: two free parameters and two decay conditions at $r = L$.

5.3 Shooting Algorithm

The Shooting Map

Define the *shooting map* $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\Phi(u_0, u_2) = \big(U(L;\, u_0, u_2),\ U'(L;\, u_0, u_2)\big), \tag{12}$$

where both components are evaluated by integrating the 5D ODE forward from $r = \delta$ to $r = L$. We look for a root $\Phi(u_0^*, u_2^*) = (0, 0)$.

Building the Initial Condition at $r = \delta$

Using the Taylor coefficients u_0, u_2, and u_4 from Eq. (10), the five components of the state vector at $r = \delta$ are:

$$u_1(\delta) = u_0 + u_2\delta^2 + u_4\delta^4,$$
$$u_2(\delta) = 2u_2\delta + 4u_4\delta^3,$$
$$u_3(\delta) = U''(\delta) + \tfrac{1}{\delta}U'(\delta) + \beta_4 U(\delta) = (2u_2 + 12u_4\delta^2) + (2u_2 + 4u_4\delta^2) + \beta_4(u_0 + u_2\delta^2 + u_4\delta^4),$$
$$u_4(\delta) = 2(12u_4 + \beta_4 u_2)\,\delta,$$
$$u_5(\delta) = 1/\delta.$$

Newton Iteration

Newton's method requires the 2×2 Jacobian $D\Phi$. Let $\mathbf{p}_0 = \partial\mathbf{u}/\partial u_0$ and $\mathbf{p}_2 = \partial\mathbf{u}/\partial u_2$. They satisfy:

$$\frac{d\mathbf{p}_i}{dr} = DF(\mathbf{u})\,\mathbf{p}_i, \qquad i \in \{0, 2\},$$

where $DF(\mathbf{u})$ is the 5×5 Jacobian of the right-hand side Eq. (4)–Eq. (8):

$$DF(\mathbf{u}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -\beta_4 & -u_5 & 1 & 0 & -u_2 \\ 0 & 0 & 0 & 1 & 0 \\ \beta_3 + 2\beta_1 u_1 + 3\beta_2 u_1^2 & 0 & -\beta_4 & -u_5 & -u_4 \\ 0 & 0 & 0 & 0 & -2u_5 \end{pmatrix}.$$

The initial conditions for the vectors are obtained by differentiating the Taylor initial data with respect to u_0 and u_2:

$$\mathbf{p}_0(\delta) = \frac{\partial \mathbf{u}(\delta)}{\partial u_0}, \qquad \mathbf{p}_2(\delta) = \frac{\partial \mathbf{u}(\delta)}{\partial u_2},$$

where the partial derivatives of u_4 with respect to the free parameters are:

$$\frac{\partial u_4}{\partial u_0} = \frac{-(\beta_3 - \beta_4^2) - 2\beta_1 u_0 - 3\beta_2 u_0^2}{64}, \qquad \frac{\partial u_4}{\partial u_2} = \frac{8\beta_4}{64}.$$

The Jacobian of the shooting map is read at $r = L$:

$$D\Phi = \begin{pmatrix} p_{0,1}(L) & p_{2,1}(L) \\ p_{0,2}(L) & p_{2,2}(L) \end{pmatrix},$$

where $p_{i,j}(L)$ denotes the j-th component of $\mathbf{p}_i$ at $r = L$.

Algorithm: Shooting with Newton's Method

Input: initial guess (u_0, u_2); parameters $\delta = 0.01$, $L = 15$, $h = 0.005$, `tol` $= 10^{-10}$, `maxIter` $= 30$.

Step 1. Compute $u_4(u_0, u_2)$ from Eq. (10) and build $\mathbf{u}(\delta)$ from the Taylor series.

Step 2. Integrate $\mathbf{u}' = F(\mathbf{u})$ from $r = \delta$ to $r = L$ using RK4 with step size h.

Step 3. Evaluate the residual $\Phi = (u_1(L), u_2(L))^\top$. If $\|\Phi\| <$ `tol`: **stop**.

Step 4. Apply one Newton step:

$$\begin{pmatrix} u_0 \\ u_2 \end{pmatrix} \leftarrow \begin{pmatrix} u_0 \\ u_2 \end{pmatrix} - [D\Phi]^{-1}\,\Phi.$$

Step 5. Return to Step 1.

RK4 Integration

At each step from r_n to $r_{n+1} = r_n + h$, the state and both sensitivity vectors are advanced together. Writing $F_n = F(\mathbf{u}_n)$ and $J_i = DF(\mathbf{u}_n + \frac{h}{2}k_i^u)$ for the Jacobian at the intermediate stages:

$$\begin{aligned}
k_1^u &= F(\mathbf{u}), & k_1^{p_i} &= DF(\mathbf{u})\,\mathbf{p}_i, \\
k_2^u &= F(\mathbf{u} + \tfrac{h}{2}k_1^u), & k_2^{p_i} &= J_1(\mathbf{p}_i + \tfrac{h}{2}k_1^{p_i}), \\
k_3^u &= F(\mathbf{u} + \tfrac{h}{2}k_2^u), & k_3^{p_i} &= J_2(\mathbf{p}_i + \tfrac{h}{2}k_2^{p_i}), \\
k_4^u &= F(\mathbf{u} + h\,k_3^u), & k_4^{p_i} &= J_3(\mathbf{p}_i + h\,k_3^{p_i}).
\end{aligned}$$

Each quantity is then updated as $(\cdot) \leftarrow (\cdot) + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

Example 5.1 (Starting guess)

A linear approximation gives us the initial guess: $u_0 \approx 0.3$, $u_2 \approx -0.05$. *With the parameter values* $\beta_1 = -\frac{\sqrt{6}}{2}$, $\beta_2 = -\frac{1}{2}$, $\beta_3 = 0.1$, $\beta_4 = -1$, *Newton's method typically converges quadratically in fewer than 10 iterations to a residual* $\|\Phi\| < 10^{-10}$.

6 Conclusion

This project found a way to compute localized radial solutions of the 2D Swift–Hohenberg equation. The three components are polynomial embedding, Taylor series expansion, and shooting. These are not independent techniques, but they make up the same big picture: a center-stable manifold.

Polynomial embedding (Section 2) makes the problem autonomous, allowing the usage of eigenvalues. Spectral analysis (Section 3) identifies the manifold spanned by the two stable modes and the center direction from $1/r$. The Taylor series (Section 4) describes the euqation near the origin by the free coordinates (u_0, u_2). The shooting method (Section 5) then finds the specific (u_0^*, u_2^*) for which the trajectory goes to zero on the horizontal axis.

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