Exploring Turbulence Through the Context of Viscous Fluid Flow at Low Reynolds Number

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1. ACKNOWLEDGMENTS

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2. Introduction

The main objective of this project is to get a handle on the structure of turbulence, guided by the simple question "what is turbulence?" It is most natural to study turbulence in the context of fluids, and for simplicity, fluids in $\mathbf{R}^2$, however turbulence is not restricted to that context, and can be studied without reference to fluids. The incompressible Navier-Stokes equations are equations which are used to model viscous fluid flow, and are given by

$$\begin{cases} \frac{d\mathbf{u}}{dt} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \frac{\mathbf{F}}{\rho} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{2.1}$$

where ρ is the density of the fluid, $\mathbf{u}$ is the velocity field, p is the pressure scalar field, ν is the viscosity, and $\mathbf{F}$ is a forcing term. This report examines this set of equations largely from the point of view of chaotic dynamical systems, then provides a qualitative discussion of features of turbulence and its relationship with symmetries. Before launching into background on dynamical systems, a qualitative physical understanding of terms in the Navier-Stokes equations is presented here.

The first equation can be thought of in terms of Newton's second law $\mathbf{F} = m\mathbf{a}$, the acceleration is $\frac{d\mathbf{u}}{dt}$. There is an arbitrary forcing term $\mathbf{F}$ (think of a pump or similar object which can agitate the fluid). The term $\frac{-1}{\rho}\nabla p$ should be thought of as a Lagrange multiplier; pressure is the force which ensures that the constraint of $\nabla \cdot \mathbf{u} = 0$ holds for all time. The term $\nu\nabla^2\mathbf{u}$ is a viscous damping term; it dissipates energy and is most relevant at small length scales. The reason for the relevance at small scales will be presented more rigorously later, but for now, consider a particulate model of a fluid. In this model, energy is dissipated as heat through collisions between particles. This is essentially a very zoomed in model of a fluid to the molecular level. Finally, the term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is an advection. The part $(\mathbf{u} \cdot \nabla)$ acts to advect any quantity which multiplies it; in this case, the velocity is being advected by the velocity. To fully understand what this means, consider a infinitesimal fluid element dm, at time t_0, this fluid element has position $\mathbf{x}_0$ velocity $\mathbf{u}_0$. At time $t_0 + dt$, the fluid element has moved to $\mathbf{x}_0 + \mathbf{u}_0 dt$, and has velocity $\mathbf{u}_0 + d\mathbf{u}_0$. Hence, one can think of this time stepping as moving the velocity $\mathbf{u}_0$ to a new location in the direction of the fluid flow which is precisely an advection.

The second equation stipulates that the vector field be divergence free. This enforces a continuity equation which corresponds to the conservation of density. In words, the divergence free condition requires that for a volume element of space $dV = \prod_{i=1}^{n} dx_i$, the net flow of fluid into dV is zero. Here, n is the number of dimensions of space.

From here on out, most of the analysis is sourced from Turbulence by Uriel Frisch [1], with the notable exception of section five where the main reference is Turbulent Flows by Stephen B. Pope [2]. Figures in section five are sourced from An Album of Fluid Flow by M. Van Dyke [3].

3. A Physics Student's Summary of Dynamical Systems

It is important to point out that there is a very rigorous definition of a dynamical system, involving probability measures and carefully circumventing difficult set theory often brought about by the axiom of choice. Here, I present a (hopefully) more immediately applicable exposition to dynamical systems.

The state of a system $\psi(t)$ is some representation which includes all of the relevant information about the system at time t. For example, ψ might include the position and velocity of a ball, or might be a scalar temperature field. For the purposes of this report, a dynamical system is any system whose state can change as a function of time. A relationship between the state at time t_0 and time $t_0 + dt$ is investigated as a means of understanding the dynamics. A simple example of such a relationship is the ODE

$$\frac{d\psi}{dt} = \psi \tag{3.1}$$

In that system, $\psi(t + dt) = \psi(t) + \psi(t)dt$. Therefore, given $\psi(0)$, one can evaluate ψ at any time t. Breaking this example into these pieces is certainly contrived as this ODE has the solution $\psi(t) = \psi_0 e^t$, but the point here is to demonstrate that one can write an explicit form of the system state as a function of the previous system states. Other dynamical systems will not have such a nice analytic solution.

While there can be many reasons for a dynamical system to have more complicated solutions, one (pertinent) reason in the case of the Navier-Stokes equations is the presence of chaos. A chaotic system is characterized by the fact that time evolution of chaotic systems with arbitrarily similar initial conditions can have dissimilar solutions. Formally, if $\mathcal{A}$ is the space of possible states of a chaotic system, let $d(\cdot, \cdot)$ be a distance function in $\mathcal{A}$, $a, b \in \mathcal{A}$ and G_t be a time evolution operator which translates a solution $a(t_0) \mapsto a(t_0 + t)$ then

$$\exists \varepsilon > 0 \quad \nexists \delta > 0 : d(a(t_0), b(t_0)) < \delta \implies (d(G_t(a), G_t(b)) < \varepsilon \tag{3.2}$$

Chaos is a vital part of turbulence, and specifically a notion of turbulent mixing is important. It is worth exploring these ideas through the example of a seemingly simple discrete dynamical system which has a very deep structure. Since this system is a discrete dynamical system, we let the time step dt equal one.

$$v_{t+1} = 1 - 2v_t^2 \tag{3.3}$$

This system is clearly deterministic (i.e. given v_0, it is immediately clear that v_t is unique). Consider now the change of variables

$$v_t = \sin(\pi x_t - \frac{\pi}{2}) \tag{3.4}$$

Then using the fact that $1 - 2\sin^2 z = \cos(2z)$, it becomes apparent that examining that dynamical system 3.3 is equivalent to the system

$$x_{t+1} = \begin{cases} 2x_t \text{ if } 0 \leqslant x_t \leqslant \frac{1}{2} \\ 2 - 2x_t \text{ if } \frac{1}{2} \leqslant x_t \leqslant 1 \end{cases} \tag{3.5}$$

This dynamical system is known as the tent map because of the graph of $x_{t+1}(x_t)$.

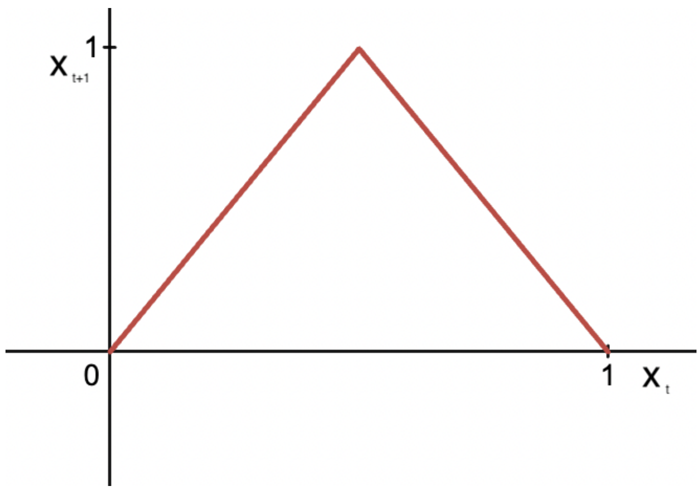

FIGURE 1. Graph of the tent map with equation given by 3.5

This system is well known to be chaotic, and mixes the interval $[0, 1]$. To demonstrate the extent to which the interval is mixed, the first 10 iterations of the tent map on $x_0 = \frac{1}{e}$ is given in the table below

Iteration number	Tent map on $x_0 = \frac{1}{e}$	Tent map on $x_0 = 0.5$	Tent map on $x_0 = \frac{2}{13}$
0	$\frac{1}{e}$	0.5	$\frac{2}{13}$
1	0.735758882343	1	$\frac{4}{13}$
2	0.528482235314	0	$\frac{8}{13}$
3	0.943035529372	0	$\frac{10}{13}$
4	0.113928941257	0	$\frac{6}{13}$
5	0.227857882514	0	$\frac{12}{13}$
6	0.455715765028	0	$\frac{2}{13}$
7	0.911431530055	0	$\frac{4}{13}$
8	0.177136939889	0	$\frac{8}{13}$
9	0.354273879778	0	$\frac{10}{13}$

From this example, one should note a few things. First, 0.5 is not an interesting starting point, it maps immediately into the kernel of the tent map, and since 0 is also in the kernel of the mapping, after two iterations, nothing new happens. For the case of $x_0 = \frac{1}{e}$, we seem to get random values out of iterating the tent map with (up to 10 iterations) no repetition. Finally, the tent map operated on $x_0 = \frac{2}{13}$ does repeat after six repetitions. To explain why this repetition occurs, consider the operation of the tent map denoted by B on an input $x \in [0, 1]$ where we expand x in binary:

$$x = 0.\alpha_1\alpha_2\alpha_3 \cdots = (\alpha_1 2^{-1}) + (\alpha_2 2^{-2}) + (\alpha_3 2^{-3}) + \dots \tag{3.6}$$

where $\alpha_i \in \{0, 1\}$. Let N be the negation map

$$N(\alpha) = \begin{cases} 0 \text{ if } \alpha = 1 \\ 1 \text{ if } \alpha = 0 \end{cases} \tag{3.7}$$

and $N^d(\alpha)$ denotes N applied d times to α, noticing that $N^{2n} = 1$ for $n \in \mathbf{N}$. We can now express the action of the tent map on the interval $[0, 1]$ as a binary shift with negation as follows

$$B(x) = 0.(N^{\alpha_1}\alpha_2)(N^{\alpha_1}\alpha_3)(N^{\alpha_1}\alpha_4)\dots \tag{3.8}$$

Introducing β_t for the sake of notation as follows

$$\beta_t = \sum_{i=1}^{t} \alpha_i \tag{3.9}$$

allows us to succinctly denote the iteration of the tent map as

$$B^t(x) = 0.(N^{\beta_t}\alpha_{t+1})(N^{\beta_t}\alpha_{t+2})(N^{\beta_t}\alpha_{t+3})\dots \tag{3.10}$$

Having done this groundwork, it is now clear why $\frac{1}{e}$ seems never to repeat under the tent map; as an irrational number, the binary expansion of $\frac{1}{e}$ never repeats itself, so the tent map applied on $\frac{1}{e}$ will never repeat. On the other hand, the binary expansion of $\frac{2}{13}$ repeats every twelve digits, but when the negation is included, it actually repeats after only six digits. Finally, $0.5_{10} = 0.1_2$ so $B(B(x)) = 0$. We can now finally fully explore the way that this representation of the dynamical system is inherently chaotic. Consider $x, y \in [0, 1]$ such that for $n \in \mathbf{N}$ the binary expansions of x and y are equal for the first n digits. For each shift, the two numbers get further and further separated. While $|x - y| < 2^{-n}$, $|B^n(x) - B^n(y)| > \frac{1}{2}$. Further iterations of the tent map will result in these two numbers no longer having anything to do with one another (they are in separate orbits). Hence, arbitrarily close initial conditions are insufficient to ensure arbitrarily similar behavior of $B^t(x)$ and $B^t(y)$ $\forall t$. Chaos is usually referred to this incredible sensitivity to initial conditions.

This notion of chaos is vital to a study of turbulence, but there is yet one more takeaway from the shift map. This is the idea of mixing the real line. Consider the intervals $I_1 = [0, \frac{1}{3}]$, $I_2 = [\frac{1}{3}, \frac{2}{3}]$, and $I_3 = [\frac{2}{3}, 1]$.

Iteration number	Tent map on I_1	Tent map on I_2	Tent map on I_3
0	$[0, \frac{1}{3}]$	$[\frac{1}{3}, \frac{2}{3}]$	$[\frac{2}{3}, 1]$
1	$[0, \frac{2}{3}]$	$[\frac{2}{3}, 1]$	$[0, \frac{2}{3}]$
2	$[0, 1]$	$[0, \frac{2}{3}]$	$[0, 1]$
3	$[0, 1]$	$[0, 1]$	$[0, 1]$

Within three iterations of the tent map on these intervals, they are all spread over the entire domain, and fully mixed with one another. To illustrate, consider $y = 0.10111001 = B^3(x)$ which I chose arbitrarily. Since the shift map by definition loses information, we can consider three different cases of what x may be to demonstrate that x could come from any one of the intervals I_i. Using the representation of x as

(3.11) $$x = 0.(\alpha_1\alpha_2\alpha_3)(10111001)$$

We can say the following

(3.12) $$\begin{cases} \alpha_1 = \alpha_2 = \alpha_3 = 0 \implies x \in I_1 \\ \alpha_1 = \alpha_3 = 0, \alpha_2 = 1 \implies x \in I_2 \\ \alpha_1 = \alpha_2 = \alpha_3 = 1 \implies x \in I_3 \end{cases}$$

There are, of course, more possible values of x, but it is clear that in only three iterations of the shift map, the entire interval of $[0, 1]$ has been mixed. The last item of note on the shift map: after subdividing $[0, 1]$ into n intervals $I_k = [\frac{k-1}{n}, \frac{k}{n}]$ for $1 \leqslant k \leqslant n$, we have

(3.13) $$\forall k, B^n(I_k) = [0, 1]$$

Having introduced ourselves to chaos and mixing induced by time evolution of a dynamical system through the lens of the shift map, we can now (finally) leave the shift map in the past.

The last concept required from dynamical systems is the notion of a phase space. A phase space of a system is the set of all possible physical states of the system in a given parameterization. For a particle in $\mathbf{R}^3$, the phase space is six dimensional, with three momentum axes and three position axes. As a dynamical system evolves in time, it follows a trajectory through phase space. For a physical dynamical system, that trajectory is unique. Initial conditions to a second order PDE correspond to a point in phase space which determine the starting point of the trajectory. For the Navier-Stokes equations, the set of possible states for the system is the set of all divergence free vector fields, and therefore the phase space is infinite dimensional. Analysis is readily applied to infinite dimensional spaces, but in the context of a mixing, chaotic, nonlinear system such as the Navier-Stokes equations, pivoting to a probabilistic approach may be more tractable.

4. Statistical Symmetries of Navier-Stokes

4.1. **Motivation for a Statistical Approach.** Consider a probe placed in a wind tunnel measuring the flow velocity in the direction of the stream (i.e. in the direction of the mean fluid flow). Subtracting the mean flow from this measurement reveals the highly fluctuating signal which is unpredictable in detail. This signal is shown in the following two figures taken from Frisch [1].

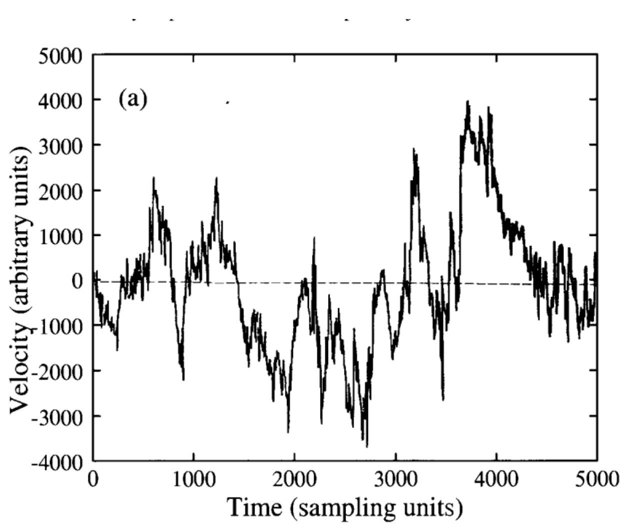

Figure 2. Fluctuations in velocity in the direction of mean flow as a function of time, starting at an arbitrary time t_0 seconds

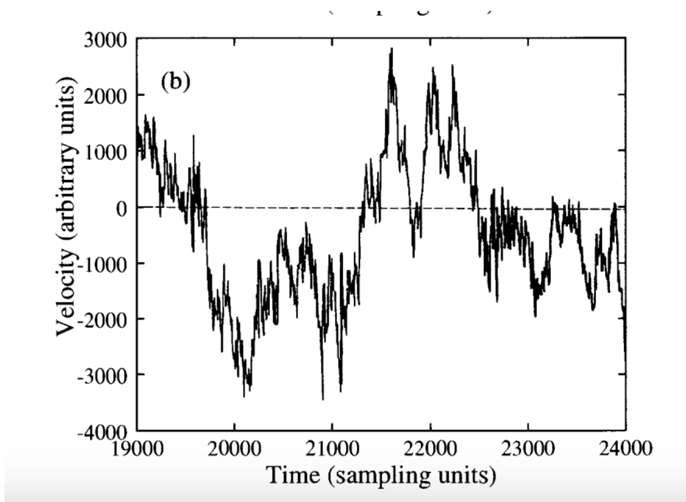

FIGURE 3. Fluctuations in velocity in the direction of mean flow as a function of time, starting at an arbitrary time $t_0 + 4$ seconds

There are two signals because two measurements were taken of the same setup; notice that without being told, there is no way to determine which signal was obtained first. Indeed, this non-determinism indicates a time-symmetry or reproducibility of the result. Given the nature of the fluctuations about $v = 0$, it seems natural to bin the signal into a histogram as is done below, also from Frisch [1].

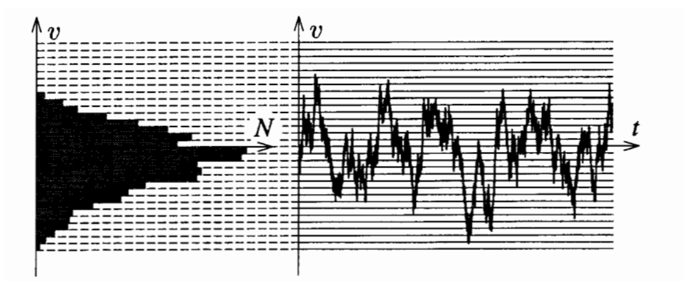

FIGURE 4. Histogram of a signal. Note that the signal in this figure is not the same as the signals presented in 2 or 3.

While the short time scale features seem random, the histogram of the signal is reproducible; it seems then, that statistical properties of turbulent flow are consistent. Thus, a probabilistic model of the Navier-Stokes equations seems justified.

4.2. **Navier-Stokes as a Dynamical System.** Let's write the Navier-Stokes equations in the form of a dynamical system with an initial condition, and apply the tools of dynamical systems to this problem.

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u}(\mathbf{x}, t = 0) = \bar{\omega} \\ \mathbf{H}(\mathbf{u}) = 0 \end{cases} \tag{4.1}$$

Here, $\bar{\omega}$ is the initial condition, and $\mathbf{H}$ is some operator which ensures that $\mathbf{u}$ satisfies any boundary conditions. It is assumed that $\mathbf{H}(\bar{\omega}) = 0$. Finally, suppose there exists a time translation operator G_t such that $G_t(\bar{\omega}(\mathbf{x})) = \mathbf{u}(\mathbf{x}, t)$. The existence of such an operator is tantamount to assuming the existence of unique solutions to the Navier-Stokes equations; this is widely believed to be true but is a famous open problem.

By Birkhoff's Ergodic Theorem, for a random integrable function f,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(z, \bar{\omega}) dz = \langle f \rangle \tag{4.2}$$

Notice that we can write $\mathbf{u}$ as a function of $t, \bar{\omega}$ using the G_t operator

$$\mathbf{u}(t, \bar{\omega}) = G_t \bar{\omega} \tag{4.3}$$

Combining 4.2 and 4.3, along with the observation that $\mathbf{u} - \langle \mathbf{u} \rangle$ has nearly gaussian features in time, represented by $\tilde{g}(t)$, we can write

$$\tilde{g}(\mathbf{x}, T) = \mathbf{u} - \frac{1}{T} \int_0^T \mathbf{u}(t, \bar{\omega}) dt \tag{4.4}$$

It is now time to take a brief physical interlude on the topic of eddies. An eddy is a turbulent feature of a fluid which often has a large vorticity. Through experiments, it is observed that eddies break apart into smaller and smaller eddies until they dissipate due to viscous forces. Supposing that the forcing term $\mathbf{f}$ acts at large length scales, the behavior of eddies seems to suggest that energy is pumped in at large scales, transferred somehow to smaller scales where it is dissipated. This transfer of energy is called the energy cascade, and it is one of the most important results in turbulence. One main tool used to investigate varying length scales is the Fourier transform.

5. Fourier Analysis

Using the standard notation of $\hat{\mathbf{u}}(\mathbf{k}, \bar{\omega})$ is the fourier transform of $\mathbf{u}(t, \bar{\omega})$, we can examine various scales.

$$\mathbf{u}_K^{<}(\mathbf{x}, \bar{\omega}) = \int_{|k| \leqslant K} e^{i(\mathbf{k}\cdot\mathbf{x})} \hat{\mathbf{u}}(\mathbf{k}, \bar{\omega}) d^3\mathbf{k}, \qquad F \geqslant 0 \tag{5.1}$$

Invoking 4.2 in the unbounded spacial domain, and the typical definition of kinetic energy as $\frac{1}{2}mv^2$, we can define a cumulative energy spectrum which has units of energy per unit mass as

$$\mathcal{E}(K) = \frac{1}{2}\langle(\mathbf{u}_K^{<}(\mathbf{x}, \bar{\omega}))^2\rangle \tag{5.2}$$

$\mathcal{E}$ is the energy of all motion with on length scales $l > \frac{1}{K}$, however we still want to isolate the motion of a given length scale. We are then led to differentiate to get the energy spectrum

$$E(\mathbf{k}) = \frac{d}{d\mathbf{k}}\mathcal{E}(\mathbf{k}) \tag{5.3}$$

In Kolmogoroff's seminal work, the energy spectrum is expressed as a power law

$$E(\mathbf{k}) = C||\mathbf{k}||^{-n} \tag{5.4}$$

The mean square of the velocity increment from position $\mathbf{x}'$ to position $\mathbf{x}$ is given by

$$\langle(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))^2\rangle = 2\int_{\mathbf{R^3}}(1 - e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})})E(\mathbf{k})d^3\mathbf{k} \ \propto \ |\mathbf{x}' - \mathbf{x}|^{n-1} \tag{5.5}$$

This restricts the possible values of n to $1 < n < 3$. In fact, through intricate derivation presented in Kolmogoroff's 1941 paper, it can be shown with no reference to fluids that the energy spectrum of turblence is a power law with $n = 5/3$. An immediate qualitative consequence of this result is that most of the energy is in the low frequency modes. The question remains: is it possible to at least get a sense of the math which determines the energy transferring from one frequency mode to another? The answer lies in Fourier Series. In order to simplify the notation slightly, let $\mathcal{F}$ be the Fourier series operator

$$\mathbf{u}(\mathbf{x}, t) = \sum_{k=||\mathbf{k}|| \in \mathbf{Z}} \hat{\mathbf{u}}_k(t)e^{i(\mathbf{k}\cdot\mathbf{x})} = \mathcal{F}(\mathbf{u}) \tag{5.6}$$

Note also that $\mathcal{F}$ is a linear operator. Let $\mathbf{S} = \mathcal{F}(\mathbf{f} - \nabla p)$. In Fourier space, ∇^2 is a diagonal operator. In order to Fourier transform the entire Navier-Stokes equation, we need to express $\nabla\mathbf{u}$ and $\partial_t\mathbf{u}$ in Fourier space. The time derivative is simple:

$$\mathcal{F}(\partial_t \mathbf{u}) = -\sum_{k=||\mathbf{k}||\in\mathbf{Z}} k^2 \hat{\mathbf{u}}_k(t) e^{i(\mathbf{k}\cdot\mathbf{x})} \tag{5.7}$$

Hence, when considering all of the terms of the Navier-Stokes equations *except for* the nonlinear term, different length scales are unable to influence one another. The complication comes when considering the non-linearity $(\mathbf{u}\cdot\nabla)\mathbf{u}$. Fourier expanding the entire term in high dimensions is best left to computers, so I'll present only the one dimensional case, i.e. $\mathcal{F}(u \cdot \partial_x u)$. The terms separately are expressed as

$$u(x,t) = \sum_{k=1}^{\infty} A_k(t) e^{ikx} \qquad \text{and} \qquad \partial_x u(x,t) = \sum_{k=1}^{\infty} ikA_k(t)e^{ikx} = \sum_{k=1}^{\infty} B_k(t) e^{ikx} \tag{5.8}$$

Using the Cauchy Product formula, we express our non-linear term as

$$u \cdot \partial_x u = \sum_{k=1}^{\infty}\sum_{n=1}^{k} A_n(t)e^{inx}B_{k-n}(t)e^{i(k-n)x} = \sum_{k=1}^{\infty} e^{ikx} \sum_{n=1}^{k} A_n(t)B_{k-n}(t) \tag{5.9}$$

While this result seems trivial, we actually have unearthed something quite important; there is a massive amount of interaction between different length scales. In fact, every length scale is interacting with every other length scale. This interaction is the mechanism through which energy can cascade from large to small length scales and then dissipate due to viscous forces.

6. Physical Scales of Turbulence

So far, turbulence in fluids has been discussed in a (relatively) vague way. In this section, we'll dive into the time, space, velocity, and viscosity scales which allow for turbulence. Note that most of the analysis in this section comes from the book Turbulent Flows by Stephen B. Pope [2]. This section refers at length to eddies which, like turbulence, are usually loosely defined. Roughly, an eddy is a turbulent motion with an associated length scale l, which is somewhat coherent over that length scale. There can be smaller eddies inside the region occupied by a larger eddy. Rather than describing eddies further with words, the following are images of eddies. These pictures are taken from An Album of Fluid Flow by M. Van Dyke [3].

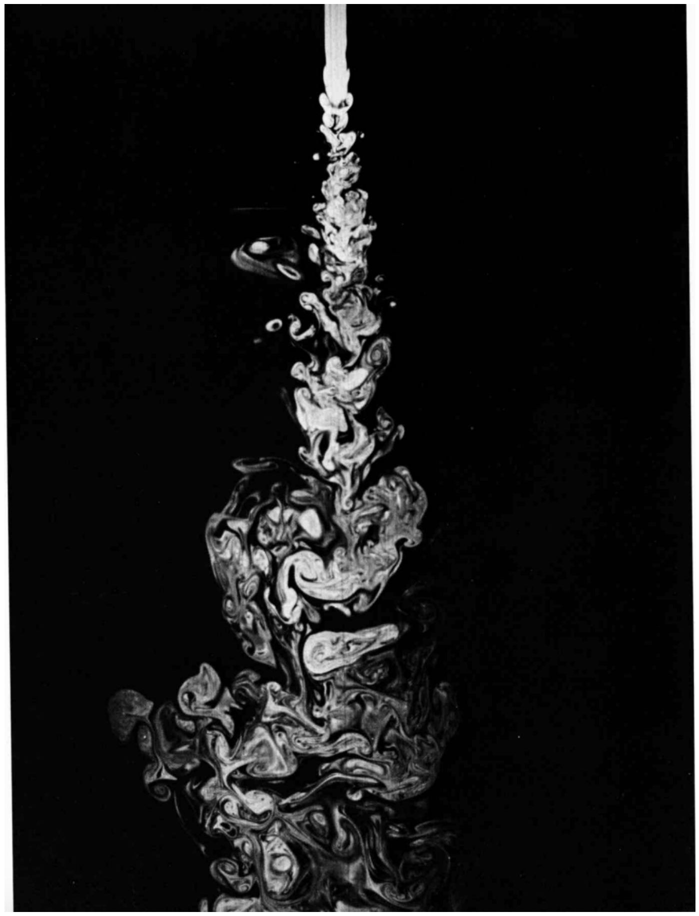

FIGURE 5. Individual eddies are visible in the lower half of the image as deformed ellipses. There are eddies in the upper half of the image as well but they are occluded. Image credit to Dimotakis, Lye & Papantoniou, 1981

FIGURE 6. A large central eddy is present with smaller eddies around the perimeter. This image was taken in the presence of convection on a rotating cylinder. Image credit to Fultz et al. 1959

With slightly more intuition about eddies under our belt, we now turn to the math. Considering a flow of width δ, Kolmogoroff claimed that turbulence can be found up to scales $l \ll \delta$. We then ask: what are the conditions for turbulence to occur in a fluid? One requirement is that inertial forces are stronger than viscous ones. This leads to the definition of the Reynolds number, a dimensionless quantity characterized by the ratio of inertial to viscous forces

$$\text{Re} = \frac{||\mathbf{u}||L}{\nu} \tag{6.1}$$

Here, L is a characteristic length scale. When Re is large, turbulence can (and likely will) occur. Let l be the size of an eddy, and define $\mathcal{U} = ||\mathbf{u}(l)||$ as the characteristic speed of the eddy. Dimensional analysis leads us to $\tau(l) = \frac{l}{\mathcal{U}}$ the characteristic time scale of the eddy. The largest eddies are on the scale $l_0 \sim L$, with $\mathcal{U}_0 \sim \mathcal{U}_{\text{flow}}$. We can then define the Reynolds number for an eddy to be

$$\text{Re}(l) = \frac{\mathcal{U}(l) \cdot l}{\nu} \tag{6.2}$$

The claim to justify the energy cascade and dissipation happening at small length scales is that as the length l decreases, the Reynolds number also decreases (i.e. viscous forces dominate for small eddies). Noting that $\tau(l)$ is (typically) small, we can claim that $\frac{d}{dt} \sim \frac{1}{\tau}$. Using the fact that kinetic energy goes as $\mathcal{U}^2$, we can find the rate of change of energy $\mathcal{R}$ below

$$\mathcal{R} = \frac{dE}{dt} = \frac{d(\mathcal{U}^2)}{dt} \sim \frac{\mathcal{U}}{\tau} = \frac{\mathcal{U}^3}{l} \tag{6.3}$$

Using these scales, we can discuss Kolmogoroff's hypothesis of isotropy. He (roughly) postulates that while the direction and shapes of large scale eddies (of size $\sim l_0$) are effected by boundary conditions, the chaotic transfer to smaller scale eddies loses this influence and therefore small scale eddies are isotropic. For reference, let $l_{IE} \approx \frac{1}{6}l_0$ then $\forall l < l_{IE}$, eddies of size l are isotropic. This is (or should be) a surprising result. It says that asymmetric features become symmetric ones. For physics students reading this, you may be tempted to claim that symmetries are a result of some conserved quantity. In this case, the isotropy is statistical in nature. Additionally, $\tau_{IE} \ll \tau_0$ tells us that the motion of isotropic eddies happens in quasistatic equilibrium with respect to the motion of large scale features. This means that at each time step, the small eddies are able to adjust to the relatively slow motion of the large scale features.

Fascinatingly, $\mathcal{R} \approx \tau_{IE}$ leads to using the dissipation rate in the context of dimensional analysis to define the Kolmogoroff scales $\eta, \mathcal{U}_\eta$, and τ_η, a length scale, velocity scale, and time scale respectively. They are defined as follows:

$$\eta \equiv (\frac{\nu^3}{\mathcal{R}})^{\frac{1}{4}} \qquad \mathcal{U}_\eta \equiv (\mathcal{R}\nu)^{\frac{1}{4}} \qquad \tau_\eta \equiv \sqrt{\frac{\nu}{\mathcal{R}}} \tag{6.4}$$

To get a sense of the magnitude of these scales, we can compare them to the length of the largest eddies:

$$\frac{\eta}{l_0} \sim \text{Re}^{-\frac{3}{4}} \qquad \frac{\mathcal{U}_\eta}{\mathcal{U}_0} \sim \text{Re}^{-\frac{1}{4}} \qquad \frac{\tau_\eta}{\tau_0} \sim \text{Re}^{-\frac{1}{2}} \tag{6.5}$$

Noting that we are discussing turbulence in the presence of high Reynolds number, and therefore the Kolmogoroff scales are small. Lastly, it is worth noticing that by construction $\text{Re}_\eta = 1$, for system Reynolds number sufficiently large, $\exists l$ such that $\eta \ll l \ll l_0$ with characteristic Reynolds number much larger than one, meaning that eddies of scale l are not effected by viscous forces. Let $l_{DI} = 60\eta$, then we have split the lengths in which eddies live in three regions $l_0 < l_{EI} < l_{DI} < \eta$. We can now discuss qualitatively the behavour of eddies in each of these length scales.

6.1. **Dissipative Lengths.** Eddies in this length scale are subject to relatively strong viscous forces. This is the regime in which the non-elastic collisions of particles dissipating energy as heat and sound are most relevant. This regime has low Reynolds number, and motivated the (until now) mysterious subscript l_{DI} which denotes the beginning of the dissipation interval (DI). Eddies in this regime have characteristic length in the interval $[\eta, l_{DI}]$. The factor of 60 involved in calculating l_{DI} is a rough estimate from experimental results, not a derived quantity.

6.2. **Inertial Lengths.** Eddies in this scale are sufficiently large so as to ignore viscous forces. Their motion is therefore mostly driven by inertial forces (hence the unimaginative name). Again, we have an inertial interval (IE) motivating the subscript l_{IE}. This regime is contains the smallest length scales where turbulence can 'freely' live without being subject to much dissipation. Eddies whose lengths are in the interval $[l_{IE}, l_{DI}]$ are in the inertial scale. We will see that energy comes from larger length scales and passes through the inertial scale before being dissipated at smaller scales.

6.3. **Larger Lengths.** Eddies and other fluid motions at scales larger than l_{IE} include features such as mean flow and large eddies. This kind of motion is well characterized by a gust of wind; while there are many small scale features in this phenomenon, the advection of air on the scale of kilometers certainly falls into this large scale bucket. Such motion also contains the majority of the energy of the system (this is compatible with the energy spectrum discussed earlier 5.4 with $1 < n < 3$.

7. Conclusion

Throughout this paper, we have seen turbulence presented through a variety of points of view. First, as a statistical process through the lens of dynamical systems. We explored examples of mixing and chaos in a discrete dynamical system, notions of which are present in any discussion of turbulence. Through experimental results, turbulent flow was then shown to have reproducible time averaging behaviour. This property was expressed mathematically using ergodic results. Proceeding with Fourier analysis, we saw that the energy spectrum follows a power law in Fourier space. Fourier expanding the PDE itself showed that the non-linear term is the contributor to mixing of different energy scales, a process which is also characteristic of turbulence. Finally, we examined the length scales of eddies, separating the regimes in which turbulent flow resides, and regimes where energy is dissipated. With all of this in mind, we can find a non-rigorous answer to the motivating question 'what is turbulence'. Turbulence is a non-linear, chaotic and ergodic motion, in which the energy of any length scale is dependent on the energies of all other scales, and is most prevalent in non-dissipative regimes. Additionally, turbulence is isotropic and statistically reproducible over sufficiently long time scales. The energy spectrum of a turbulent motion follows a power law $E(k) \propto k^{-\frac{5}{3}}$. There is, of course, much more to be said on the topic than what has been presented in this paper, and I highly recommend reading the books the information was sourced from.

References

[1] Uriel Frisch. *Turbulence: The Legacy of A. N. Kolmogorov*. Cambridge University Press, 1995.

[2] Stephen B. Pope. *Turbulent Flows*. Cambridge University Press, 2000.

[3] M. Van Dyke. *An Album of Fluid Motion*. Parabolic Press, 1982. URL: https://books.google.ca/books?id=W4OCvwEACAAJ.