An Introduction to Category Theory

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Abstract

This report was compiled for the 2025 Winter term Directed Reading Program at McGill University and is intended as an undergraduate-friendly introduction to category theory. Algebra involves many abstract structures such as sets, groups, and vector spaces. Category theory introduces a general math structure, category, that encompasses all of the above and more. In fact, many familiar algebraic structures can be interpreted as categories. This paper discusses about basic concepts of category theory, including *objects*, *morphisms*, *functors* and *natural transformation*, developed with an emphasis on intuition and worked examples.

Contents

1 Basic Concepts

Definition 1.1. A **group** is a non-empty set G with a binary operation $* : G \times G \to G$ satisfying:

1. Associativity: $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.
2. Identity: There exists $e \in G$ such that $e * a = a * e = a$ for all $a \in G$.
3. Inverse: For every $a \in G$ there exists $a^{-1} \in G$ with $a * a^{-1} = a^{-1} * a = e$.

If $*$ is commutative, G is an **abelian** (or **commutative**) group.

Definition 1.2. Let G be a group. A non-empty subset $H \subseteq G$ is a **subgroup** if the following holds:

1. $e_G \in H$
2. $a, b \in H \implies a * b \in H$
3. $a \in H \implies a^{-1} \in H$

Clearly, every group is a subgroup of its own.

Definition 1.3. A **ring** R is a non-empty set equipped with two binary operations, called "addition" and "multiplication", satisfying:

1. Commutativity of addition: $x + y = y + x,\ \forall x, y \in R$
2. Associativity of addition: $(x + y) + z = x + (y + z),\ \forall x, y, z \in R$
3. Neutral element for addition: there exists an elements $0 \in R$ such that $0 + x = x,\ \forall x \in R$
4. Inverse with respect to addition: $\forall x \in R,\ \exists y \in R$ such that $x + y = 0$
5. Associativity of multiplication: $(xy)z = x(yz),\ \forall x, y, z \in R$
6. Neutral element for multiplication: there exists an elements $1 \in R$ such that $1x = x1 = x,\ \forall x \in R$
7. Distributivity: $z(x + y) = zx + zy,\ (x + y)z = xz + yz,\ \forall x, y, z \in R$

Definition 1.4. Note that the multiplication operation is not assumed to be commutative in general. If $xy = yx,\ \forall x, y \in R$, then R is a **commutative ring**. If for every non-zero $x \in R$ there is an element $y \in R$ such that $xy = yx = 1$, and also $0 \neq 1$ in R, we call R a **division ring**. A commutative division ring is called a **field**.

Definition 1.5. Let R be a ring. A subset $I \subseteq R$ is a **(two-sided) ideal** of R, denoted as $I \triangleleft R$, if

1. $0 \in I$
2. $\forall\, a, b \in I,\ a + b \in I$
3. $\forall\, r \in R,\ a \in I,\ ra, ar \in I$

Thus I is an additive subgroup of R closed under multiplication by arbitrary ring elements on both sides.

Definition 1.6. Let R be a ring and let $I \triangleleft R$ be a two-sided ideal. A **coset** of I is any subset of R of the form

$$a + I \ := \ \{ a + i \mid i \in I \}, \ a \in R$$

Definition 1.7. Given a ring R and an ideal $I \triangleleft R$, the **quotient ring** R/I is the collection of cosets

$$R/I = \{ a + I : a \in R \},$$

with addition and multiplication defined by

$$(a + I) + (b + I) = (a + b) + I, \quad (a + I)(b + I) = (ab) + I.$$

These operations are well defined precisely because I is an ideal and make R/I into a ring with zero element $0 + I = I$ and identity element $1 + I$.

Definition 1.8. A **homomorphism** is a map between two algebraic structures of the same type, such as two groups, that preserves the operations of the structures.

1. A **group homomorphism** between groups G and H is a map $\varphi : G \to H$ such that
$$\varphi(ab) = \varphi(a)\varphi(b), \ \forall\, a, b \in G.$$
2. A **ring homomorphism** between rings R and S is a map $\psi : R \to S$ satisfying
   - $\psi(a + b) = \psi(a) + \psi(b)$
   - $\psi(a + b) = \psi(a) + \psi(b)$
   - $\psi(1_R) = 1_S$

The **kernel** is $\ker(\varphi) = \{g \in G : \varphi(g) = e_H\}$ (resp. $\ker(\psi) = \{r \in R : \psi(r) = 0_S\}$).

Definition 1.9 (Isomorphism)**.** A bijective homomorphism is called an **isomorphism**.

Two algebraic structures R, S are **isomorphic** if there exists an isomorphism between them, denoted as $R \cong S$. Isomorphic structures are considered algebraically identical.

2 Categories

Definition 2.1. A **category C** consists of

- a collection of **objects**, denoted as $X, Y, Z, ...$
- a collection of **morphisms**, denoted as $f, g, h, ...$

such that:

- Each morphism is assigned to two objects as its **domain** and **codomain** (or **source** and **target**), denoted as $f : X \to Y$ or graphically $X \xrightarrow{f} Y$, where X is domain and Y is codomain of f.
- Each object X has an **identity morphism** $id_X : X \to X$

- For any two morphisms f, g such that the codomain of f equal to the domain of g, there exists a **composite morphism** $g \circ f$(or gf) whose domain equal to the domain of f and codomain equal to the codomain of g.

$$f\colon X \longrightarrow Y, \quad g\colon Y \longrightarrow Z \ \rightsquigarrow \ g \circ f\colon X \longrightarrow Z.$$

Then the structure satisfies the following axioms.
Axiom 2.1.

- Unitality: for each morphism $f : X \to Y$, $f \circ 1_X = f \circ 1_Y = f$
- Associativity: for $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$, $(f \circ g) \circ h = f \circ (g \circ h)$ and we have $hgf : X \to W$

Category is a very general structure that can be used to interpret many other structures in algebra. Here are some examples.
Example 2.1. A **preorder** P is a set equipped with a binary relation such that:

- Reflexive: $x \leq x, \forall x \in P$.
- Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$.

A **partial order** is a preorder that also satisfies:

- Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$.

We can consider a preorder P as a category $\mathbf{C}$ with objects being the elements of P and for objects x, y, there exists a morphism $f : x \to y$ in $\mathbf{C}$ if and only if $x \leq y$ in P. Note that for each such pair x, y, there is at most one morphism from x to y.

Two axioms can be easily verified. $x \leq x$, so we have $id_x : x \to x$ and if $x \leq y$, $y \leq z \implies x \leq z$ by transitivity, then we have $f : x \to y$, $g : y \to z$ and $g \circ f : x \to z$.

Now we introduce another useful construction about groups, which is also an example of how to consider groups as categories.

Definition 2.2. Let G be a group. The **delooping** of G, denoted as $\mathbf{B}G$ defined as follows:

- there is a signal object $\bullet$
- for every element $g \in G$, there is a corresponding morphism $g : \bullet \to \bullet$

We have identity corresponding to $1 \in G$ and composition corresponding to multiplication operation of G, that is composition of morphisms $g \circ f$ is given by $g \cdot f$.

Note that in the construction of delooping, we did not use the inverses of elements in groups, that is, if we drop the inverse, we can still get a valid category. Here is an example:
Example 2.2. A **monoid** is a set M equipped with:

- An identity element $1 \in M$
- An associative binary operation $\cdot : M \times M \to M$

There is no inverse of elements in monoid, and we can still consider M as a category using construction of its delooping. The category $\mathbf{B}G$ is stricter than the delooping of a monoid, since all morphisms in $\mathbf{B}G$ are invertible, while monoids may have non-invertible morphisms.

There are certain classifications within category theory. Within category theory, categories are sometimes classified as small or locally small, depending on the size of their collections of objects and morphisms.

Definition 2.3.

- A category is **small** if the collection of its objects and the collection of morphisms are sets.
- A category is **locally small** if the for any two objects A, B, the collection of morphisms between them. The collection of objects may not be a set.

The reason we distinguish between small and locally small categories is that not all collections can be treated as sets. A well-known example illustrating this limitation is Russell's Paradox: let $R = \{x \mid x \notin x\}$ be the set of all sets that do not contain themselves—does $R \in R$ hold? This contradiction motivated the development of more rigorous foundations, such as Zermelo-Fraenkel set theory and type theory, though we will not explore these in detail here.

3 Duality

Definition 3.1. The **opposite category** of a category $\mathbf{C}$, denoted by $\mathbf{C}^{\mathrm{op}}$, is defined as the category having the same objects as $\mathbf{C}$, but with all morphisms reversed in direction.

If there is a morphism $f : X \to Y$ in $\mathbf{C}$, then the corresponding morphism in $\mathbf{C}^{op}$ is $f^{op} : Y \to X$. The identity and composition defined as follows:

- For each objects X, its identity is $1_X{}^{op}$ in $\mathbf{C}^{op}$.
- For composition, we define $g^{op} \circ f^{op} = (f \circ g)^{op}$

The definition defines $\mathbf{C}^{op}$ if and only if $\mathbf{C}$ is a category. The process of inverting arrows introduces a self-duality, which is a powerful property, for categories: a statement is true in $\mathbf{C}$ if and only if the dual statement is true in $\mathbf{C}^{op}$. The two statements may not look the same.

Corollary 3.1. *X, Y are isomorphic in $\mathbf{C}$ if and only if they are also isomorphic in $\mathbf{C}^{op}$*

4 Morphisms

It is nice to understand and reason about categories in directed graphs. We can consider objects as vertices and morphisms between them as edges.

Example 4.1.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Definition 4.1. A graph **commutes** for each pair of two vertices, compositions formed by paths among them are equal.

In the above example, $f : X \to Y$, $g : Y \to Z$ and the graph commutes if and only if $g \circ f = h$.

Example 4.2.

We can also represent the opposite category(right) in graphs

$$\begin{array}{ccc} X & & X \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^{op}} \\ Y & & Y \\ {\scriptstyle g}\downarrow & & \uparrow{\scriptstyle g^{op}} \\ Z & & Z \end{array}$$

By duality, the graph in $\mathbf{C}$ commutes if an only if the dual graph in $\mathbf{C}^{op}$ commutes.

4.1 Isomorphism

Definition 4.2. An **isomorphism** is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ and $gf = 1_X$, $fg = 1_Y$.

The composition of isomorphisms is an isomorphism. An isomorphism has a two-sided inverse.

Besides the isomorphism, we also have some other kinds of homomorphisms. An **endomorphism** is a morphism whose domain is its codomain and an **automorphism** is a morphism that is both an endomorphism and an isomorphism. Check the graph 1.
Example 4.3.

- Isomorphisms in **Set** are bijections.
- Isomorphisms in **Group**, **Ring** are bijective homomorphisms.

Definition 4.3. A **groupoid** is a category in which every morphism is an isomorphism.

Example 4.4.

- Every group is a groupoid. Delooping of a group is a category, in which the inverses are given by inverse elements in the group and composition is group multiplication, with one object.
- (Database theory) Let objects be entities (e.g. people), and morphisms be equivalence relations such as "is_same_as". If all morphisms are invertible (mutual recognition), the data forms a groupoid.

A groupoid is a natural generalization of both equivalence relations and groups. It not only tells us whether two objects are equivalent, but also records how they are equivalent—retaining the structure of invertible transformations between them.

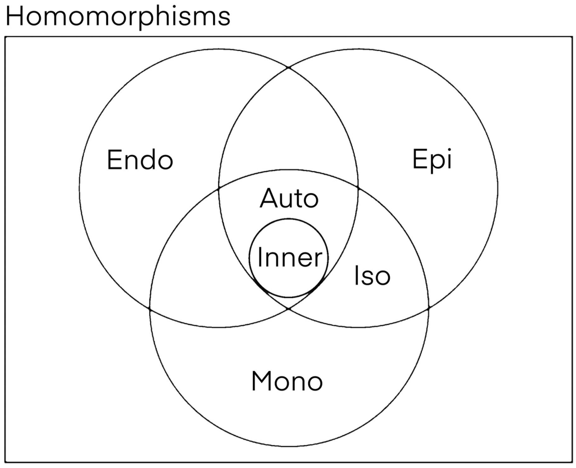

Figure 1: Venn Diagram of Homomorphisms. Image by AbelDrinkingCoffee, via Wikimedia Commons. Licensed under CC BY-SA 4.0

4.2 Monomorphism & epimorphism

Definition 4.4. Let $f : X \to Y$ be a morphism,

- f is a **monomorphism** (or **mono** for short) if for any pair of morphisms $g, h : Z \to X$, the equality $f \circ g = f \circ h$ implies $g = h$. Graphically, the following graph commutes,

$$Z \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

- f is an **epimorphism** (or **epi** for short) if for any pair of morphisms $g, h : Y \to Z$, the equality $g \circ f = h \circ f$ implies $g = h$. Graphically, the following graph commutes,

$$X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$$

Note that if f is a monomorphism or an epimorphism in $\mathbf{C}$, then f is an epimorphism or a monomorphism in $\mathbf{C}^{op}$ respectively.

Example 4.5.

- In **Set**, monomorphisms are injective maps and epimorphisms are surjective maps.
- In **Group**, monomorphisms are injective group homomorphisms.
- In a preorder, every morphism is a monomorphism since there is at most one morphism between each two objects.

Proposition 4.1. *Every isomorphism is both a monomorphism and an epimorphism, but the converse is not necessarily true.*

The forward direction is straightforward to understand and prove. Now, let us consider a counterexample to show that the converse does not necessarily hold.

Example 4.6. In the category of unital rings, the canonical inclusion

$$i : \mathbb{Z} \hookrightarrow \mathbb{Q}$$

of the integers into the rational numbers is an epimorphism.

Proof. Every rational number can be written as the product of an integer and the multiplicative inverse of another integer:

$$\frac{a}{b} = a \cdot b^{-1} \in \mathbb{Q}, \quad \text{with } a, b \in \mathbb{Z}, b \neq 0.$$

Since unital ring homomorphisms preserve multiplicative inverses, for any such homomorphism f, we have:

$$f\left(\frac{a}{b}\right) = f(a) \cdot (f(b))^{-1}.$$

Now, suppose $f, g : \mathbb{Q} \to R$ are two parallel unital ring homomorphisms into some unital ring R, and they agree on the image of $\mathbb{Z} \subseteq \mathbb{Q}$, i.e.,

$$f \circ i = g \circ i.$$

Then for any $\frac{a}{b} \in \mathbb{Q}$, we compute:

$$f\left(\frac{a}{b}\right) = f(a) \cdot f(b)^{-1} = g(a) \cdot g(b)^{-1} = g\left(\frac{a}{b}\right),$$

so $f = g$. Hence, i is an epimorphism. $\square$

Thus we have i which is both mono and epi but not isomorphism.

4.3 split mono and epi

Definition 4.5.

Let $f : X \to Y$ be a morphism,

- **Left inverse** or **retraction** of f is a map $r : Y \to X$ such that $r \circ f = 1_X$.
- **Right inverse** or **section** of f is a map $s : Y \to X$ such that $f \circ s = 1_Y$.

The maps s and r express the object X as a **retract** of the object Y.

Sometimes we need to distinguish left and right sided inverse, and thus we have split monomorphism and split epimorphism.

Definition 4.6.

Let f be a morphism,

- If f admits a left inverse, then f is a **split monomorphism** and the left inverse is its **splitting**.

  Graphically, the following graph commutes, where r is the left inverse of f:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{1_X} & \downarrow r \\ & & X \end{array}$$

- If f admits a right inverse, then f is a **split epimorphism** and the right inverse is its **splitting**.

  Graphically, the following graph commutes, where s is the right inverse of f:

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow^{1_Y} & \downarrow s \\ & & Y \end{array}$$

Every isomorphism is a split monomorphism and a split epimorphism since isomorphism requires a two-sided inverse. Every split monomorphism and split epimorphism is a mono and an epi, respectively.

Example 4.7.

- In **Set**, every injective and surjective map has a left and right inverse respectively, and thus every mono and epi is a split monomorphism and epimorphism respectively.
- inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a monomorphism in **Group**, but not a split mono.

Proposition 4.2. *Let $f : X \to Y$ be a morphism in a category. If f is an epimorphism and also a split monomorphism, then f is an isomorphism. Dually, if f is a monomorphism and also a split epimorphism, then f is an isomorphism.*

This explains why, in the category **Set**, a function is invertible precisely when it is both injective and surjective: such functions are not only monomorphisms and epimorphisms, but also *split*. However, this is not the general case.

5 Functoriality

Definition 5.1. Let **C** and **D** be categories. The **functor** $F : \mathbf{C} \to \mathbf{D}$ consist the following data:

- An object $Fc \in \mathbf{D}$, for each object $c \in \mathbf{C}$
- A morphism $Ff : Fc \to Fc' \in \mathbf{D}$, for each morphism $f : c \to c' \in \mathbf{C}$

such that the following axioms hold:
Axiom 5.1.

- Unitality: for each object $c \in \mathbf{C}$, $F(1_c) = 1_{Fc}$
- Compositionality: for each composable pair $f, g \in \mathbf{C}$, $Fg \circ Ff = F(g \circ f)$

  Graphically, we have the following graph, where X, Y, Z are objects in **C** and FX, FY, FZ are objects in **D**:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$\begin{tikzcd} & FY \arrow[dr, "Fg"] & \\ FX \arrow[ur, "Ff"] \arrow[rr, "F(g\circ f)"'] & & FZ \end{tikzcd}$$

The functor consists of a mapping preserving the structure of categories of objects and morphisms.

Example 5.1. Let $(X, \leq)$ and $(Y, \leq)$ be partially ordered sets viewed as categories. A monotonic function $F : X \to Y$ such that $x \leq x' \Rightarrow F(x) \leq F(x')$ defines a functor $F : (X, \leq) \to (Y, \leq)$. Since there is at most one morphism between any two objects in a poset category, unitality and compositionality hold trivially.

Similarly, for equivalence relations $(X, \sim)$ and $(Y, \sim)$, a function $F : X \to Y$ such that $x \sim x' \Rightarrow F(x) \sim F(x')$ defines a functor $F : (X, \sim) \to (Y, \sim)$. Note that $x \not\sim x'$ does not prevent $F(x) \sim F(x')$. In the target category, there may be more relations than in the source category.

Example 5.2. Let $(X, \sim)$ be a set with an equivalence relation, and $X/\sim$ be the quotient set. Define a category with objects the equivalence classes $[x]$ and only identity morphisms. The quotient map $F : X \to X/\sim$, mapping $x \mapsto [x]$, induces a functor $F : (X, \sim) \to X/\sim$.

Identity morphisms are preserved: $F(1_x) = 1_{[x]}$. There are no nontrivial morphisms, so composition is trivially preserved.

Example 5.3.

- Let G and H be groups. A group homomorphism $f : G \to H$ induces a functor $F : \mathbf{B}G \to \mathbf{B}H$, where $\mathbf{B}G$ is the one-object category.

  $f(1_G) = 1_H$ ensures identities are preserved. $f(g_1 g_2) = f(g_1)f(g_2)$ ensures composition is preserved.

- Let M and N be monoids. A monoid homomorphism $f : M \to N$ induces a functor $F : \mathbf{B}M \to \mathbf{B}N$, in the same way as for groups.

Example 5.4. Let G be a group and **Vect** be the category of vector spaces over a fixed field. A linear representation $R : G \to \mathrm{GL}(V)$, the group of all invertible linear transformations of V, corresponds to a functor $R : \mathbf{B}G \to \mathbf{Vect}$.

$R(\bullet) = V$, a vector space. For each $g \in G$, $R(g) : V \to V$ is a linear isomorphism. Since $gg^{-1} = e \Rightarrow R(g)R(g^{-1}) = R(e) = \mathrm{id}_V$, each $R(g)$ is an isomorphism.

Example 5.5. The power set functor is a functor $P : \mathbf{Set} \to \mathbf{Set}$ defined as follows:

- On objects: for a set X, define $P(X) := \{S \subseteq X\}$, the power set of X.

- On morphisms: for a function $f : X \to Y$, define $Pf : P(X) \to P(Y)$ by

$$(Pf)(S) := \{y \in Y \mid y = f(x) \text{ for some } x \in S\}.$$

This defines a functor since $P(\mathrm{id}_X) = 1_{P(X)}$ and $P(g \circ f) = P(g) \circ P(f)$ for all composable functions f, g.

Example 5.6. A **forgetful functor** drops some structure while preserving the underlying set:

- U : **Group** $\to$ **Set**, mapping a group to its underlying set and a homomorphism f to the same function f.
- V, E : **Graph** $\to$ **Set**, mapping graphs to their vertex and edge sets.
- F : **Ring** $\to$ **Ab**, mapping rings to their additive groups.
- Inclusion functor I : **Ab** $\to$ **Group**, which forgets commutativity.

Example 5.7 (Graphs as presheaves)**.** Let **Par** be the category with two objects V, E and two parallel arrows $s, t : E \to V$:

$$\begin{tikzcd} V & E \arrow[l, bend right, "t"'] \arrow[l, bend left, "s"] \end{tikzcd}$$

A presheaf on **Par**, that is, a functor $F : \mathbf{Par}^{op} \to \mathbf{Set}$, consists of:

- Two sets: FV (vertices) and FE (edges);
- Two functions: $F(s), F(t) : FE \to FV$, giving the source and target of each edge.

This structure defines a directed multigraph: multiple edges and loops are allowed. Note that identities and composition are not present here, unlike in categories.

Example 5.8. *Adapted from [Riehl, 2016, Example 1.3.4].*

Definitions

- **FinMetric**: objects are finite metric spaces, morphisms are distance–non-increasing maps.
- **Cluster**: an object is a partition ("cluster") of a set; a morphism $f : X \to Y$ must refine the partition on X by preimages of the partition on Y.
- Classical clustering algorithms correspond to functors $F : \mathbf{FinMetric} \to \mathbf{Cluster}$.

Kleinberg's impossibility: No non-trivial functor F can satisfy the three reasonable axioms of Kleinberg, so such functors do not exist.

Carlsson–Mémoli's insight: Replace clusters by *persistent clusters*:

$$P : ([0, \infty), \leq) \longrightarrow \mathbf{Cluster},$$

allowing partitions to merge as the scale parameter r grows. This yields a new category **PCluster** whose objects are persistent clusters and whose morphisms are set maps that are morphisms in **Cluster** for every r.

Result: There is a *unique* non-trivial functor

$$\Phi\colon \mathbf{FinMetric} \longrightarrow \mathbf{PCluster},$$

satisfying two natural conditions. Concretely, a two–point metric space with distance r maps to a persistent cluster that has

- one cluster for all $t \geq r$;
- two clusters for $0 \leq t < r$.

Moral. By enlarging the codomain category, we bypass Kleinberg's obstruction and obtain a categorical framework for clustering.

6 Natural transformations

Definition 6.1. Let C and D be categories and given functors $F, G : \mathbf{C} \to \mathbf{D}$. A **natural transformation** α from F to G, $\alpha : F \Rightarrow G$ consist of:

- a morphism $\alpha_C : FC \to GC$ for each object C of $\mathbf{C}$, called **component** of α at C
- for each morphism $f : C \to C'$, the following graph commutes:

$$\begin{tikzcd} FC \arrow[r, "Ff"] \arrow[d, "\alpha_C"] & FC' \arrow[d, "\alpha_{C'}"] \\ GC \arrow[r, "Gf"] & GC' \end{tikzcd}$$

A **natural isomorphism** is a natural transformation $\alpha : F \Rightarrow G$, denote as $\alpha : F \cong G$, in which every component α_C is an isomorphism.

Natural transformations provide a powerful way to compare functors. They do not simply relate individual objects or morphisms, but establish a coherent system of morphisms across the entire category. This coherence is what makes naturality so elegant—it encodes a kind of "structural uniformity" between functors.

From a higher-level perspective, natural transformations can be seen as the morphisms between functors in the functor category, thus enriching category theory with another layer of abstraction.

Example 6.1. Consider a category $\mathbf{Vect}_{\Bbbk}$ whose objects are vector spaces over a field $\Bbbk$. Let V be a finite-dimensional $\Bbbk$-vector space that is an object of $\mathbf{Vect}_{\Bbbk}$. V is isomorphic to its **linear dual** $V^* = \mathrm{Hom}(V, \Bbbk)$ since these vector spaces have the same dimension and this can be proven by a construction of an explicit dual basis. [Riehl, 2016]

A **double dual** $V^{**} = \mathrm{Hom}(V^*, \Bbbk) = \mathrm{Hom}(\mathrm{Hom}(V, \Bbbk), \Bbbk)$, which is also a vector space and thus an object of $\mathbf{Vect}_{\Bbbk}$. We define the **evaluation function** $ev_v : V^* \to \Bbbk$, $ev_v(f) = f(v)$. The map $v \mapsto ev_v$ defines a linear isomorphism and thus $V \cong V^{**}$.

Now we construct the natural transformation. We have an identity endofunctor on $\mathbf{Vect}_{\Bbbk}$ that maps V to itself and a double dual functor that maps V to its double dual V^{**}. Then the map $ev : V \to V^{**}$ that maps $v \in V$ to $ev_v \in V^{**}$ defines the components of a natural transformation from identity endofunctor to double dual functor.

Let V, W be objects of $\mathbf{Vect}_{\Bbbk}$. To check if the following diagram commutes,

$$\begin{array}{ccc} V & \xrightarrow{\text{ev}} & V^{**} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi^{**}} \\ W & \xrightarrow[\text{ev}]{} & W^{**} \end{array}$$

One path is first sending $v \in V$ to $\varphi v \in W$ by φ, and then sending φv to $ev_{\varphi v} : W^* \to \Bbbk$ which maps $f : W \to \Bbbk$ to $f(\varphi v) \in \Bbbk$ by ev.

Another path is first sending $v \in V$ to $ev_v \in V^{**}$ by ev. And by definition, $\varphi^{**} : V^{**} \to W^{**}$, thus we have $\varphi^{**}(ev_v) : W^* \to \Bbbk$ which also maps $f : W \to \Bbbk$ to $f(\varphi v) \in \Bbbk$ by ev.

Example 6.2. For a group G the opposite group G^{op} has the same underlying set as G but the multiplication is reversed: $x \circ y := yx$. This construction extends to a (covariant) endofunctor

$$(-)^{\mathrm{op}} : \ \mathbf{Group} \longrightarrow \mathbf{Group}, \ \varphi \ \mapsto \ \varphi^{\mathrm{op}}, \ \varphi^{\mathrm{op}}(g) = \varphi(g),$$

because a group homomorphism $\varphi : G \to H$ respects the reversed product automatically.

The functor $(-)^{\mathrm{op}}$ is *naturally isomorphic* to the identity on **Group**. For each group G define

$$\eta_G : G \longrightarrow G^{\mathrm{op}}, \ g \mapsto g^{-1}.$$

Although η_G is *not* a group automorphism of G (since $g \mapsto g^{-1}$ does not commute with the original multiplication), it *is* a homomorphism $G \to G^{\mathrm{op}}$.

To check naturality, take any group homomorphism $\varphi : G \to H$. One must verify that the square

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & G^{\mathrm{op}} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi^{\mathrm{op}}} \\ H & \xrightarrow[\eta_H]{} & H^{\mathrm{op}} \end{array}$$

commutes. Indeed, for every $g \in G$,

$$\varphi^{\mathrm{op}}\big(\eta_G(g)\big) = \varphi^{\mathrm{op}}\big(g^{-1}\big) = \varphi\big(g^{-1}\big) = \big(\varphi(g)\big)^{-1} = \eta_H\big(\varphi(g)\big),$$

so $\varphi^{\mathrm{op}} \circ \eta_G = \eta_H \circ \varphi$.

References

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