The Fundamental Group of the Circle

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CONTENTS

1. Introduction

The fundamental group of a topological space X, based on a point $x_0 \in X$, captures information about the shape of the space by studying loops that start and end at x_0. Two loops are considered equivalent if one can be continuously deformed into the other while keeping the endpoints fixed, which is called homotopy. These homotopy classes of loops form a group, denoted $\pi_1(X, x_0)$, where the group operation is given by concatenating loops. Understanding the fundamental group of a space is important because it reveals essential information about the structure of the space, such as the presence of holes, twists, or loops.

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2. Paths and Homotopies

2.1. **Preliminary Definitions.**

Definition 2.1. *A continuous map $f : X \to Y$ is a* ***homeomorphism*** *if its bijective and its inverse function $f^{-1} : Y \to X$ is also continuous.*

Definition 2.2. *A* ***path*** *in a space X is a continuous map $f : I \to X$, where I is the unit interval $[0, 1]$.*

Definition 2.3. *A* ***homotopy*** *of paths in X is a family $f_t : I \to X, 0 \leqslant t \leqslant 1$, such that:*

(1) *The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$, are independent of t.*

(2) *The associated map $F : I \times I \to X$, defined by $F(s, t) = f_t(s)$, is continuous.*

When two paths f_0 and f_1 are connected by way of homotopy, we say that they are ***homotopic****, which we denote by: $f_0 \simeq f_1$.*

2.2. **Homotopy Classes.**

Proposition 2.4. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.*

The equivalence class of a path f under the equivalence relation of homotopy is denoted $[f]$ and is called the ***homotopy class*** *of f.*

Proof. Reflexivity: $f \simeq f$ by the constant homotopy $f_t = f$.

Symmetry: if $f_0 \simeq f_1$ by the homotopy $f_t = (1 - t)f_0 + tf_1$, then $f_1 \simeq f_0$ by the inverse homotopy $f_{1-t} = tf_0 + (1 - t)f_1$.

Transitivity: if $f_0 \simeq f_1$ by $f_t = (1 - t)f_0 + tf_1$, if $f_1 = g_0$, and $g_0 \simeq g_1$ by $g_t = (1 - t)g_0 + tg_1$, then $f_0 \simeq g_1$ by the homotopy:

$$h_t := \begin{cases} f_{2t}, & 0 \leqslant t \leqslant \frac{1}{2} \\ g_{2t}, & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

By assumption, $f_1 = g_0$, so the associated value $t = \frac{1}{2}$ corresponds to the same value in h_t, and is thus well defined. We define the associated map as:

$$H(s, t) := \begin{cases} F(s, 2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ G(s, 2t - 1), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Where F and G are the associated maps of the homotopies f_t and g_t respectively. The continuity of H follows from the continuity of F on $[0, \frac{1}{2}]$ and G on $[\frac{1}{2}, 1]$. So the relation of homotopy on paths with fixed endpoints in any space is an equivalence relation. ☐

3. The Fundamental Group

Definition 3.1. *We define a* ***loop*** *by restricting a path $f : I \to X$, to the same starting and ending point, $f(0) = f(1) = x_0 \in X$, where x_0 is referred to as the* ***basepoint****. The set of all homotopy classes $[f]$ of loops $f : I \to X$ at the basepoint x_0 is denoted as $\pi_1(X, x_0)$.*

Definition 3.2. *The* ***product*** *of two loops, $f, g : I \to X$, where $f(x_0) = g(x_0)$ is defined as follows:*

$$f \cdot g(s) := \begin{cases} f(2s), & 0 \leqslant s \leqslant \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

Note: *The product of loops respects homotopy classes. If f_0 is homotopic to f_1 by the homotopy f_t, and g_0 is homotopic to g_1 by the homotopy g_t, then the product $f_0 \cdot g_0$ is homotopic to $f_1 \cdot g_1$ by the homotopy $f_t \cdot g_t$.*

Proposition 3.3. *$\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \cdot g]$.*

We call $\pi_1(X, x_0)$ the ***fundamental group*** *of X at the basepoint $x_0 \in X$.*

Proof. We begin by defining a **reparameterization** path $\phi : I \to I$, to be any continuous map such that $\phi(0) = 0$ and $\phi(1) = 1$.
Given a path $f : I \to X$, we have that $f \cdot \phi \simeq f$ by the homotopy $f \cdot \phi_t$, where $\phi_t(s) = (1-t)\phi(s) + ts$. We know that the product of homotopies respects homotopy classes (Def. 3.2), so reparameterizing a path will also respect homotopy classes.
Associativity: Let $f, g, h : I \to X$ be loops at the basepoint x_0. We can then define the composition:

$$[f][g \cdot h] := \begin{cases} f(2s), & 0 \leqslant s \leqslant \frac{1}{2} \\ \begin{cases} g(2s'), & 0 \leqslant s' \leqslant \frac{1}{2} \\ h(2s'-1), & \frac{1}{2} \leqslant s' \leqslant 1 \end{cases}, & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

We define the composition:

$$[f \cdot g] \cdot [h] := \begin{cases} \begin{cases} f(2s'), & 0 \leqslant s' \leqslant \frac{1}{2} \\ g(2s'-1), & \frac{1}{2} \leqslant s' \leqslant 1 \end{cases}, & 0 \leqslant s \leqslant \frac{1}{2} \\ h(2s-1), & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

$[f][g \cdot h]$ can be reparameterized to $[f \cdot g] \cdot [h]$ using ϕ, so we have:

$$[f][g \cdot h] \simeq [f \cdot g] \cdot [h]$$

Thus the product of loops of $\pi_1(X, x_0)$ is associative.
Identity element: Given a loop $f : I \to X$ at the basepoint $x_0 \in X$, we define c to be the **constant path** at $f(x_0)$. Specifically, let $c(s) = f(x_0)$ for all $s \in I$. Thus we have that f can be reparametrized to $f \cdot c$ by ϕ, similarly $f \cdot c$ can be reparameterized to f by ϕ, so we have:

$$f \cdot c \simeq f \simeq c \cdot f$$

Thus the homotopy class of the constant path must be the identity element in $\pi_1(X, x_0)$.
Inverses: Given a loop $f : I \to X$ at basepoint $x_0 \in X$, we define an **inverse loop** $\bar{f}$ as $\bar{f}(s) = f(1 - s)$. We define the composition $f \cdot \bar{f}$ as follows:

$$f \cdot \bar{f} := \begin{cases} f(2s), & 0 \leqslant s \leqslant \frac{1}{2} \\ f(2(1-s)), & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

So we have: $f \cdot \bar{f} \simeq c$, where c is the constant path at x_0. Similarly we have $\bar{f} \cdot f \simeq c$. So we have:

$$f \cdot \bar{f} \simeq c \simeq \bar{f} \cdot f$$

Thus $[\bar{f}]$ is the inverse of $[f]$ in $\pi_1(X, x_0)$.
So $\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \cdot g]$.

$\square$

Now we will calculate the fundamental group of the circle, $\pi_1(S^1)$, to show that it is isomorphic to $\mathbb{Z}$ (i.e. $\pi_1(S^1) \simeq \mathbb{Z}$). We will begin with necessary definitions and lemmas for proving $\pi_1(S^1) \simeq \mathbb{Z}$.

Definition 3.4. *Given a space X, a* ***covering space*** *of X consists of a space $\tilde{X}$ and a map $p : \tilde{X} \to X$, satisfying the following condition: For each $x \in X$, there exists an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets, each of which is homeomorphically mapped onto U by p. Such a U will be called* ***evenly covered****.*

3.1. **Path and Homotopy Lifting Lemmas.**

Proposition 3.5. *Let $p : \tilde{X} \to X$ be a covering map, $f : I \to X$ be a map, and $\tilde{f}_1$ and $\tilde{f}_2$ be both lifts of f. Then the lifts $\tilde{f}_1$ and $\tilde{f}_2$ either agree everywhere or nowhere.*

Proof. First note that the unit interval I is a connected as it cannot be expressed as the union of two non-empty disjoint open sets.
Now we will define the following set:

$$K := \{s \in I : \tilde{f}_1(s) = \tilde{f}_2(s)\}$$

We will show that K is both open and closed, and thus since I is a connected space, $K = I$, or $K = \emptyset$.

We will show that K is open: Let $s \in I$ such that $\tilde{f}_1(s) = \tilde{f}_2(s)$. Then there is an evenly covered neighborhood $U \subseteq X$ of $f(s)$ and $V \subseteq \tilde{X}$ such that $\tilde{f}_1(s) \in V$ and the map $p|V : V \to U$ is a homeomorphism. Let $Z := \tilde{f}_1^{-1}(V) \cap \tilde{f}_2^{-1}(V)$. We will show that $\tilde{f}_1 = \tilde{f}_2$ on Z. By construction we have that:

$$p|_V \circ \tilde{f}_1|_Z = p|_V \circ \tilde{f}_2|_Z$$

and since $p|_V$ is a homeomorphism, we have that:

$$\tilde{f}_1|_Z = \tilde{f}_2|_Z$$

.

So $\tilde{f}_1 = \tilde{f}_2$ on Z, where $Z = \tilde{f}_1^{-1}(V) \cap \tilde{f}_2^{-1}(V)$, the intersection of two open sets, so $Z = K$ is open.

Now we will show that K is closed: We will do so by assuming for contradiction that K is **not** closed. In other words, there exists $s \in \bar{K} \backslash K$ (on the boundary of K) such that $\tilde{f}_1(s) \neq \tilde{f}_2(s)$.

Let U be an evenly covered neighborhood of $f(s)$, let $p^{-1}(U) = \sqcup U_\alpha$. Let $\tilde{f}_1(s) \in U_\beta$ and $\tilde{f}_2(s) \in U_\gamma$, where $\beta \neq \gamma$. Then $Z := \tilde{f}_1^{-1}(U_\beta) \cap \tilde{f}_2^{-1}(U_\gamma)$ is an open neighborhood of s and intersects K, since its an open set on the boundary of K. Thus there exists $z \in Z$ such that $\tilde{f}_1(z) = \tilde{f}_2(z)$, but $\tilde{f}_1(z) \in U_\beta$ and $\tilde{f}_2(z) \in U_\gamma$ which is a contradiction. Thus K must be closed.

So since K is both open and closed, $K = I$, or $K = \emptyset$. The lifts $\tilde{f}_1$ and $\tilde{f}_2$ agree everywhere or nowhere.

□

3.1.1. *Path Lifting Lemma.*

Lemma 3.6. *Let $p : \tilde{X} \to X$ be a covering map. For each path $f : I \to X$, starting at a point $x_0 \in X$, such that $p(\tilde{x}_0) = x_0 = f(0)$, there is a unique lift $\tilde{f} : I \to \tilde{X}$, of f, starting at $\tilde{x}_0$.*

Proof. We begin by defining:

$$S := \{s \in I : \tilde{f} \text{ exists and is continuous on } [0, s] \subseteq I\}$$

By construction, $S \neq \varnothing$, since $0 \in S$.

We will now show that S is open: Let $s \in S$, then there exists a continuous lift $\tilde{f} : [0, s] \to \tilde{X}$. We will show that we can extend $\tilde{f}$ on a larger interval, showing that S is open.
We know p is a covering map, so there exists an evenly covered neighbourhood $U \subseteq X$ of $f(s)$ and $V \subseteq \tilde{X}$ such that $\tilde{f}(s) \in V$ and the map $p|V : V \to U$ is homeomorphism.
Since f is continuous, there exists $\varepsilon > 0$ such that for all $t \in [s, s + \varepsilon]$, $f(t) \in U$. We define:

$$\tilde{f}(t) := (p|V)^{-1} \circ f(t), t \in [s, s + \varepsilon]$$

So we have that $\tilde{f}$ is continuous on $[s, s + \varepsilon]$, as $(p|V)^{-1}$ is a homeomorphism, and thus we can extend the continuity of $[0, s]$ to $[0, s + \varepsilon]$. $\tilde{f}$ is continuous on $[0, s]$ by assumption and both intervals agree at $\tilde{f}(s)$, since $p \circ \tilde{f}(s) = f(s)$. So $\tilde{f}$ exists and is continuous on a larger interval $[0, s + \varepsilon]$, and thus S is open.

We will now show that S is closed: We will assume that $\tilde{f}$ exists and is continuous on $[0, s)$, then by taking the limit point of the of the paths as they get arbitrarily close to s, we can extend the continuity of $\tilde{f}$ to $[0, s]$.

We have that p is a covering map for $f(s) \in X$, so there exists an evenly covered open neighborhood $U \subset X$ of $f(s)$ such that $p^{-1}(U) = \sqcup_\alpha V_\alpha$, disjoint open sets in $\tilde{X}$, and $p|V_\alpha : V_\alpha \to U$ is a homeomorphism.

By the continuity of f, there exists $\varepsilon > 0$ such that $f((s - \varepsilon, s]) \subset U$. So the image of $\tilde{f}$ on $(s - \varepsilon, s)$ must lies entirely in a single V_β since each V_α is disjoint and open. So $\tilde{f}$ exists and is continuous on $[0, s)$. Now we will extend $\tilde{f}$ continuously $[0, s]$ by the following:

$$\tilde{f}(t) := \begin{cases} \tilde{f}(t), & 0 \leqslant t < s \\ (p|V_\beta)^{-1}(f(s)), & t = s \end{cases}$$

We restricted map $p|V_\beta : V_\beta \to U$ to a single open set in $\tilde{X}$, and is thus bijective. We know p^{-1} is continuous, so we then have that $p|V_\beta$ is a homeomorphism. We then have that $\tilde{f}$ is continuous arbitrarily close to s, which is then extended to s by the pre-image of $f(s)$. So $\tilde{f}$ exists and is continuous on $[0, s]$, so S is closed.

We have that the set

$$S := \{s \in I : \tilde{f} \text{ exists and is continuous on } [0, s] \subseteq I\}$$

is non-empty, open, and closed, we have that S = I, the entire interval [0,1], so $\tilde{f}$ exists. We can start the lift at $\tilde{x}_0$, by taking the $p^{-1} \circ f(x_0)$, and the uniqueness of the lift follows from Prop. 3.5.

□

3.1.2. *Homotopy Lifting Lemma.*

Lemma 3.7. *For each homotopy $f_t : I \to X$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifted homotopy $\tilde{f}_t : I \to \tilde{X}$ of paths starting at $\tilde{x}_0$, such that $\tilde{f}_t(0) = \tilde{x}_0$ for all t, and $p \circ \tilde{f}_t = f_t$.*

Proof. We will do a sketch relying on the the Path Lifting Lemma 3.6.

(1) We can use compactness of $I \times I$ to partition small neighborhoods U_{ij} in X.

(2) We can then define a lift of f on $U_{1,1}$, using the Path Lifting Lemma. We start this lift at $\tilde{x}_0$, which ensures that $\tilde{f}_t(0) = \tilde{x}_0$.

(3) We can continue this step inductively from U_{ij} to U_{nm} to lift the whole space, $1 < i \leqslant n$ and $1 < j \leqslant m$. Where n and m are the respective partitions of the unit intervals in $I \times I$.

(4) Each lifted path is combined continuously using the uniqueness of path lifting. If any lifts agree at one spot, they will agree over the entire interval.

(5) We can then piece together these lifted paths to form a continuous homotopy $\tilde{f}_t$.

□

3.2. **The Fundamental Group of the Circle.**

Theorem 3.8. $\pi_1(S^1)$ *is an infinite cyclic group generated by the homotopy class of the loop* $\omega(s) = (cos(2\pi s), sin(2\pi s))$ *based at* $(1, 0)$.

Proof. We aim to show that every loop in S^1 based at $(1, 0)$ is homotopic to some integer power of the loop ω, and that this integer is unique. This would imply that $\pi_1(S^1)$ is cyclic, generated by $[\omega]$, and isomorphic to $\mathbb{Z}$.

Let $f : I \to S^1$ be a loop based at $(1, 0)$. Consider the covering map: $p : \mathbb{R} \to S^1$, $p(s) = (cos2\pi s, sin2\pi s)$, and note that $p^{-1}(1, 0) = \mathbb{Z} \subset \mathbb{R}$. By Lemma 3.6, we can lift f uniquely to a path $\tilde{f} : I \to \mathbb{R}$ such that $\tilde{f}(0) = 0$. Then $\tilde{f}(1) = n \in \mathbb{Z}$, since $p(\tilde{f}(1)) = f(1) = (1, 0)$, so the lift starts at 0 and ends at an integer $n \in \mathbb{Z}$. Now define: $\omega_n : I \to S^1$, $\omega_n(s) = (cos2\pi ns, sin2\pi ns)$ for $n \in \mathbb{Z}$. ω_n is a path that loops around the circle n times, where $\omega_n(0) = (1, 0) = \omega_n(1)$. So, $[\omega_n] = [\omega]^n$, where ω^n is the concatenation of ω with itself n times. This means that ω_n is a loop based at $(1, 0)$, and it lifts to the path $\tilde{\omega}_n : I \to \mathbb{R}$, $\tilde{\omega}_n(s) = ns$, starting at 0 and ending at n, since $(p \circ \tilde{\omega}_n)(s) = p(\tilde{\omega}_n(s)) = p(ns) = (cos2\pi ns, sin2\pi ns) = \omega_n(s)$.

We now want to show that $f \simeq \omega_n$.

Define a homotopy $\tilde{f}_t(s) = (1-t)\tilde{f}(s) + t\tilde{\omega}_n(s)$, in $\mathbb{R}$ from $\tilde{f}$ to the path $\tilde{\omega}_n(s) = ns$. Then, $\tilde{f}_t$ is a homotopy of paths that start at 0 and end at n. Composing with p, we get a homotopy of loops $f_t = p \circ \tilde{f}_t$ in S^1, where $f_0 = p(\tilde{f}_0) = p(\tilde{f}) = f$, and $f_1 = p(\tilde{f}_1) = p(\tilde{\omega}_n) = \omega_n$. Therefore, f_t is a homotopy of loops from f to ω_n. Thus, $f \simeq \omega_n$.

Uniqueness of n: Suppose $m \neq n$, and suppose $f \simeq \omega_m$ and $f \simeq \omega_n$. Then, $\omega_m \simeq \omega_n$, so there exists a homotopy f_t with $f_0 = \omega_m$ and $f_1 = \omega_n$. By Lemma 3.7, there is a $\tilde{f}_t : I \to \mathbb{R}$ such that $p \circ \tilde{f}_t = f_t$. By the uniqueness of path and homotopy lifting, $f_0 = \omega_m$ implies $\tilde{f}_0 = \tilde{\omega}_m$, and $f_1 = \omega_n$ implies $\tilde{f}_1 = \tilde{\omega}_n$. The end point of $\tilde{f}_t$ is independent of t, i.e. $\tilde{f}_0(1) = \tilde{f}_1(1)$. But $\tilde{f}_0(1) = \tilde{\omega}_m(1) = m$ and $\tilde{f}_1(1) = \tilde{\omega}_n(1) = n$. Contradiction. Thus, $m = n$.

We have shown that every loop in S^1 based at $(1,0)$ is homotopic to ω_n for a unique $n \in \mathbb{Z}$. Recall that $[\omega_n] = [\omega]^n$. Therefore, every homotopy class of loops in S^1 based at $(1,0)$ is of the form $[\omega]^n$, and different values of n give different classes. This means that $\pi_1(S^1) \simeq \mathbb{Z}$. Thus, $\pi_1(S^1)$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (cos(2\pi s), sin(2\pi s))$. □

4. Applications

Now, we will use the calculation of $\pi_1(S^1)$ to prove the Fundamental Theorem of Algebra.

4.1. **The Fundamental Theorem of Algebra.**

Corollary 4.1. *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. Without loss of generality, let $p(z) = z^n + a_1 z^{n-1} + ... + a_n$ be a nonconstant polynomial with complex coefficients. Suppose, for contradiction, that $p(z)$ has no roots in $\mathbb{C}$. Then $p(z) \neq 0$ for all $z \in \mathbb{C}$. Now define, for each $r \geqslant 0$, the loop

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}, \text{ for } s \in [0,1]$$

This function is a loop on the unit circle $S^1 \subset \mathbb{C}$ based at 1. Simply put, $f_r(s)$ is a function that tracks the direction (or angle) of the polynomial $p(z)$ as $z = re^{2\pi is}$ moves in a circle of radius r. As r varies, f_r is a homotopy of loops based at 1 (since the function is normalized). For $r = 0$ (or very small r), f_0 is the constant loop at 1 (the trivial loop). Hence, the homotopy class $[f_r] \in \pi_1(S^1)$ is zero for small r (since the homotopy class is based on the number of times the loop winds around the circle).

Now, let's look at a large value of r, bigger than both 1 and $|a_1| + \ldots + |a_n|$. For such r, on the circle $|z| = r$ (circle is made up of all $z \in \mathbb{C}$ with a distance r from the origin), we have:

$$|z^n| = |z|^n = r^n > (a_1| + \ldots + |a_n|)r^{n-1} = (a_1| + \ldots + |a_n|)|z^{n-1}|$$

Which implies that,

$$|z^n| > |a_1 z^{n-1}| + \ldots + |a_n| \geqslant |a_1 z^{n-1} + \ldots + a_n|$$

So the entire homotopy $p_t(z) = z^n + t(a_1 z^{n-1} + \ldots + a_n)$, for $t \in [0, 1]$, has no roots on the circle $|z| = r$, since $|z^n| > t(|a_1 z^{n-1}| + \ldots + |a_n|)$. Then, define $f_r^t(s)$ by replacing p with p_t in the formula for $f_r(s)$ above. As t goes from 1 to 0, this gives a homotopy between the original loop f_r and the loop $\omega_n(s) = e^{2\pi ins}$, which winds around the circle n times. By Theorem 3.8, this loop represents the homotopy class $[\omega_n] = n \in \pi_1(S^1) \simeq \mathbb{Z}$. But earlier, we concluded that $[f_r] = 0$, since f_r is homotopic to the trivial loop. So, $[f_r] = [\omega_n] = 0$. This implies $n = 0$, meaning that $p(z)$ is a constant. This is a contradiction, and thus our assumption that $p(z)$ has no roots in $\mathbb{C}$ must be false. Hence, every nonconstant complex polynomial has a root in $\mathbb{C}$. $\square$

5. Conclusion

In this paper, we explored the fundamental group and used it to compute $\pi_1(S^1)$, showing that it is isomorphic to $\mathbb{Z}$. This result reflects the fact that loops around the circle can be classified by how many times they wind around, and in which direction. Beyond this, we saw how calculating the fundamental group of the circle has a surprising application outside of pure topology: by using the structure of $\pi_1(S^1)$, we were able to prove the Fundamental Theorem of Algebra. This connection shows the power of algebraic topology—even simple topological spaces like the circle can provide tools to prove deep and far-reaching results in mathematics.

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