THE TIRING RANDOM WALKER

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ABSTRACT

We consider one-dimensional random walks with unit and decreasing step size, and study the conditions under which these walks converge or diverge. For the family of (convergent) geometric random walks, whose step size at time n is λ^n for some $\lambda \in (0,1)$, we summarize the known results about the limiting distribution. A dichotomy emerges in these distributions, which surprisingly hinges on the algebraic properties of Pisot–Vijayaraghavan numbers.

Section 1 contains an overview of random walks, and the criterion for convergence of random walks with constant step size. Section 2 defines random walks with decreasing step sizes, and provides Kolmogorov's criterion for the convergence of these walks.

1. AN INTRODUCTION TO RANDOM WALKS

Definition 1.1 (Random walk)**.** *Let $(X_n)_{n\geq 1}$ be a sequence of $\mathbb{R}^d$-valued independent random variables. A* ***random walk*** *started at $s_0 \in \mathbb{R}^d$, with step sizes $(X_n)_{n\geq 1}$, is the sequence $(S_n)_{n\geq 0}$ defined recursively as follows. $S_0 \coloneqq s_0$, and*

$$S_n \coloneqq S_{n-1} + X_n, \quad n \geq 1.$$

Throughout this report, we consider one-dimensional random walks.

One can interpret the process $(S_n)_{n\geq 0}$ as a person starting at s_0 and taking steps $X_1, X_2, \ldots, X_k$ to get to the position S_k at time k. Even though the steps $(X_n)_{n\geq 1}$ are independent from one another, the position S_k is crucially dependent on the position S_{k-1}.

In order to study the distribution of a given random walk $(S_n)_{n\geq 0}$, we define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are working. Ω is defined to be the set of all sequences $(X_n)_{n\geq 1}$ of step sizes. We set $\mathcal{F} \coloneqq \sigma(\Omega)$. To define $\mathbb{P}$, we'll first define a bijection $f : \Omega \to [0,1]$. For a given set of step sizes $X = (X_n)_{n\geq 1}$, we define $f(X)$ to be the binary representation of the binary sequence $\left(\mathbb{1}_{\{X_n=1\}}\right)_{n\geq 1}$. We set $\mathbb{P} \coloneqq f_*^{-1}(\nu) = \nu \circ f$, where ν is the Lebesgue($[0,1]$) measure.

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The simplest example of a random walk is called the *simple walk*, where $s_0 = 0$ and each X_n is distributed according to the random variable

$$X := \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases} \tag{1.1}$$

In this scenario, one can picture a person standing at the origin and moving left or right by 1 with each step, flipping a coin each time to decide on the direction. Intuitively, we see that there is a sense of "fairness" to this random walk. The person has as much a chance to move left as right, and with equal distance. We can formalize this with the following definition.

Definition 1.2. *Let $(S_n)_{n\geq 0}$ be a random walk with steps $(X_n)_{n\geq 1}$. We say that $(S_n)_{n\geq 0}$ is **balanced**, or **unbiased**, if $\mathbb{E}[X_n] = 0$ for all $n \geq 1$. We say $(S_n)_{n\geq 0}$ is **left-biased** (resp. **right-biased**) if $\mathbb{E}[X_n] < 0$ (resp. $\mathbb{E}[X_n] > 0$) for all $n \geq 1$.*

Definition 1.3 (Recurrence and transience)**.** *We say that a random walk is **recurrent** if it visits its starting position infinitely often with probability one and **transient** if it visits its starting position finitely often with probability one.*

Chung and Fuchs [CF51] characterized the asymptotic behaviour of one-dimensional random walks with IID steps, based on their bias. Note that Theorem 1.4 can also be stated in terms of recurrence and transience.

Theorem 1.4 (Chung-Fuchs)**.** *For a random walk $(S_n)_{n\geq 0}$ with* IID *steps $(X_n)_{n\geq 1}$ such that $\mathrm{Var}(X_1) \in (0, \infty)$, the following holds.*

(1) $\mathbb{E}[X_1] = 0 \iff \limsup_{k\to\infty} S_k = \infty$ *and* $\liminf_{k\to\infty} S_k = -\infty$ *a.s.*

(2) $\mathbb{E}[X_1] > 0 \iff \lim_{k\to\infty} S_k = \infty$ *a.s.*

(3) $\mathbb{E}[X_1] < 0 \iff \lim_{k\to\infty} S_k = -\infty$ *a.s.*

Proof. Let $\mu := \mathbb{E}[X_1]$ and $\sigma^2 := \mathrm{Var}\,(X_1)$. By rescaling, we can assume without loss of generality that $\sigma^2 = 1$.

We first prove that (1) holds. By the Central Limit Theorem, it holds that

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ is the standard Gaussian distribution. Since $\mu = 0$ and $\sigma = 1$, we have that

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1). \tag{1.2}$$

Let $\phi : \mathbb{R} \to [0,1]$ denote the CDF of the $\mathcal{N}(0,1)$ distribution, and let $\varepsilon := 1 - \phi(1) > 0$. By (1.2), for all n large enough, it holds that

$$\begin{aligned}\left|\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \le 1\right) - \phi(1)\right| < \varepsilon &\implies -\varepsilon < \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \le 1\right) - \phi(1) < \varepsilon \\ &\implies \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \le 1\right) < \varepsilon + \phi(1) = 1 \\ &\implies 1 - \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \le 1\right) > 0 \\ &\implies \mathbb{P}\left(\frac{S_n}{\sqrt{n}} > 1\right) =: \delta > 0.\end{aligned}$$

Now for all $k \ge 0$, let E_k be the event $\{\frac{S_k}{\sqrt{k}} > 1\}$, or equivalently, $\{S_k > \sqrt{k}\}$. We know that $\mathbb{P}(E_k) \ge \delta$ for all $k \ge n_0$. By the measure-exhaustion lemma (A.5), it follows that

$$\mathbb{P}(E_k \text{ occurs infinitely often}) \ge \delta.$$

Therefore there is a positive-measure set of random walks on which $\limsup_{k\to\infty} S_k = \infty$. Since $\{\limsup_{k\to\infty} S_k = \infty\}$ is a tail event, by the Kolmogorov 0–1 Law (A.4), it follows that $\limsup_{k\to\infty} S_k = \infty$ almost surely. The proof that $\liminf_{k\to\infty} S_k = -\infty$ almost surely is analogous.

The other direction of (1) follows from (2) and (3) by contrapositive; we show that if $\mathbb{E}[X_1] > 0$ (resp. $\mathbb{E}[X_1] < 0$), we have $\limsup_{k\to\infty} S_k = \liminf_{k\to\infty} S_k = \lim_{k\to\infty} S_k = \infty$ (or $-\infty$). We will only prove that $\mathbb{E}[X_1] > 0 \implies \liminf_{k\to\infty} S_n = \infty \ a.s.$ as the other case is identical. Note that since $\liminf_{k\to\infty} \le \limsup_{k\to\infty}$, it suffices to show $\liminf_{k\to\infty} = \infty$. To this end, we will use the Strong Law of Large Numbers (SLLN). Let $\mu = \mathbb{E}[X_1]$. By the SLLN, since X_i have finite moments, we know that $\overline{X_n} \overset{a.s.}{\to} \mu$. Thus picking $\varepsilon = \frac{\mu}{2}$, we have for all n large enough,

$$\begin{aligned}\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) = 1 &\iff \mathbb{P}\left(-\varepsilon < \frac{S_n}{n} - \mu < \varepsilon\right) = 1 \\ &\iff \mathbb{P}\Big(n(\mu - \varepsilon) < S_n < n(\mu + \varepsilon)\Big) = 1 \\ &\iff \mathbb{P}\left(\frac{n\varepsilon}{2} < S_n < \frac{3n\varepsilon}{2}\right) = 1\end{aligned}$$

Thus with probability one, for all n large enough, we have $S_n > \frac{n\varepsilon}{2}$. We can conclude that $\liminf_{n\to\infty} S_n = \lim_{n\to\infty} S_n = \infty$.

□

2. RANDOM WALKS WITH DECREASING STEP SIZE

Let $(S_n)_{n\geq 0}$ be a balanced random walk with steps $(X_n)_{n\geq 1}$, and suppose there is a sequence $(c_n)_{n\geq 1}$ such that $X_n \in \{-c_n, c_n\}$ almost surely for all $n \geq 1$. Then we call $(c_n)_{n\geq 1}$ the set of step sizes of $(S_n)_{n\geq 1}$. In this section, we consider random walks with step sizes $(c_n)_{n\geq 1}$, where $\lim_{n\to\infty} c_n = 0$. A natural question arises when studying these walks: when does $(S_n)_{n\geq 1}$ converge or diverge?

Rademacher's convergence result is central to analyzing random walks with decreasing step sizes. It is proved with the aid of Kolmogorov's inequality. The inequality gives a useful statement for a sequence of independent random variables, each with mean zero and finite variance, that we can apply to balanced random walks.

Theorem 2.1 (Kolmogorov's Inequality)**.** *Let $X_1, ..., X_n$ be independent random variables, each with zero expectation and finite variance. Then for each $\lambda > 0$, it holds that*

$$\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| \geq \lambda\right) \leq \frac{1}{\lambda^2}\operatorname{Var}(S_k),$$

where $S_k = X_1 + ... + X_k$.

Proof. We know for all $k = 1, .., n$,

$$\operatorname{Var}(X_k) = \mathbb{E}\left[X_k^2\right] + \mathbb{E}\left[X_k\right]^2 = \mathbb{E}\left[X_k^2\right].$$

Define $A_k := \{|S_k| \geq \lambda \text{ and } S_i < \lambda \text{ for all } i < k\}$. Then the disjoint union of the A_k's is equivalent to the event $\{\max_{1\leq k\leq n} |S_k| \geq \lambda\}$. For all $k \geq 1$, we have that

$$\mathbb{P}(A_k) \leq \mathbb{P}\left(|S_k| \mathbb{1}_{A_k} \geq \lambda\right) \;\leq\; \lambda^{-2}\mathbb{E}\left[S_k^2\, \mathbb{1}_{A_k}\right]$$

due to Chebyshev's inequality and the fact that $\mathbb{E}\left[X_k^2\right] \geq \mathbb{E}\left[X_k^2 \mathbb{1}_{A_k}\right]$. Furthermore,

$$\begin{aligned}
\lambda^{-2}\mathbb{E}\left[S_k^2\, \mathbb{1}_{A_k}\right] \leq\; & \lambda^{-2}[\mathbb{E}\left[S_k^2 \mathbb{1}_{A_k}\right] \;+\; \mathbb{E}\left[(S_n - S_k)^2 \mathbb{1}_{A_k}\right] \\
&= \lambda^{-2}\mathbb{E}\left[(S_k^2 + (S_n - S_k)^2)\mathbb{1}_{A_k}\right] \\
&= \lambda^{-2}\mathbb{E}\left[(S_k^2 + 2S_k(S_n - S_k) + (S_n - S_k)^2)\mathbb{1}_{A_k}\right] \\
&= \lambda^{-2}\mathbb{E}\left[(S_k + (S_n - S_k))^2 \mathbb{1}_{A_k}\right] \\
&= \lambda^{-2}\mathbb{E}\left[S_n^2 \mathbb{1}_{A_k}\right]
\end{aligned}$$

where the first equality is from independence of S_k and $S_n - S_k$. Now taking the disjoint union over the A_k's, we have that

$$\begin{aligned}\mathbb{P}\left(\max_{1\le k\le n}|S_k|\ge\lambda\right) &= \mathbb{P}\left(\bigsqcup_{k=1}^{n}A_k\right)\\ &\le \lambda^{-2}\mathbb{E}\left[S_n^2\bigsqcup_{k=1}^{n}A_k\right]\\ &\le \lambda^{-2}\mathbb{E}\left[S_n^2\right]\\ &= \frac{\mathrm{Var}[S_n]}{\lambda^2}.\end{aligned}$$

□

With the extra condition that $\mathbb{P}(|X_i|\le c)=1$, for $i\le n$, we also have Kolmogorov's second inequality:

$$\mathbb{P}\left(\max_{1\le k\le n}|S_k|\ge\varepsilon\right)\ge 1-\frac{(c+\varepsilon)^2}{\mathbb{E}\left[S_n^2\right]}.$$

We will omit the proof of this second inequality in this paper.

Theorem 2.2 (Rademacher's criterion for convergence)**.** *A (balanced) random walk S_k with step sizes $(c_n)_{n\ge1}$ converges almost surely if and only if*

$$\sum_{n=1}^{\infty}c_n^2<\infty.$$

Proof. We first show that the square-summability of $(c_n)_{n\ge1}$ implies the almost sure convergence of the random walk. To this end, we will show that the sequence $\{S_n\}_{n\ge1}$ is almost surely Cauchy. Fix $n,m\in\mathbb{N}$ and suppose W.L.O.G. that n > m; then,

$$|S_n-S_m|=\left|\sum_{i=1}^{n}X_i-\sum_{j=1}^{m}X_j\right|=\left|\sum_{i=m+1}^{n}X_i\right|.$$

Let $\widetilde{S}_k := S_{m+k}-S_m$, for all $k\ge1$. In other words, $(\widetilde{S}_k)_{k\ge1}$ is the random walk $(S_n)_{n\ge1}$ ignoring the first m steps. Similarly, we define $\tilde{c}_i$ to be c_{i+m}. Note that the variance of $\widetilde{\widetilde{S}}_k$ is $\sum_{i=0}^{n}\widetilde{c}_i^2$. We then have

$$\mathbb{P}\left(\left|\sum_{i=m+1}^{n} X_i\right| > \varepsilon\right) = \mathbb{P}\left(\left|\widetilde{S}_n\right| > \varepsilon\right) \overset{(*)}{\leq} \frac{1}{\varepsilon^2}\sum_{i=0}^{n}\widetilde{c}_i^2 \leq \frac{1}{\varepsilon^2}\sum_{i=0}^{\infty}\widetilde{c}_i^2,$$

where $(*)$ comes from Kolmogorov's Inequality (Theorem 2.1). Note that the last series goes to zero as $m \to \infty$ since the original series $\sum_{i=0}^{\infty} c_i$ converges (the tail goes to zero). Thus for any $\varepsilon > 0$, for m and n big enough, $|S_n - S_m|$ will be smaller than ε. The sequence $(S_n)_{n\geq 1}$ is then almost surely Cauchy, and hence almost surely convergent since $\mathbb{R}$ is a complete metric space.

For the other direction, we show that the c_n are square-summable if the walk converges almost surely. Indeed, for any $\varepsilon > 0$, we know that for n large enough,

$$\mathbb{P}\left(\sup_{k\geq n}|S_k - S_n| \geq \varepsilon\right) < \frac{1}{2} \tag{2.1}$$

However, we also know by the second part of Kolmogorov's inequality that

$$\mathbb{P}\left(\sup_{k\geq n}|S_k - S_n| \geq \varepsilon\right) \geq 1 - \frac{(c+\varepsilon)^2}{\sum_{k\geq n}^{\infty}\mathbb{E}[X_k^2]}. \tag{2.2}$$

Now if the series $\sum_{i=1}^{\infty} c_n^2 = \sum_{i=1}^{\infty}\mathbb{E}[X_i^2]$ diverges, we have that for any n, no matter how large, the series $\sum_{k\geq n}^{\infty}\mathbb{E}[X_k^2]$ starting from n will diverge and thus the right-hand side in (2.2) tends to 1, contradicting (2.1). We conclude that S_n converging almost surely implies the square summability of the c_n's.

□

Definition 2.3 (Singular measure)**.** *Two measures μ and ν defined on the measurable space (Ω, Σ) are called **singular** if there exist disjoint measurable sets $A, B \in \Sigma$ such that $A \cup B = \Omega$, and μ is zero on all measurable subsets of B while ν is zero on all measurable subsets of A. If μ and ν are singular we write $\mu \perp \nu$.*

Definition 2.4 (Absolutely continuous measures)**.** *A measure μ on (Borel) subsets of the real line is **absolutely continuous** with respect to the Lebesgue measure λ if*

$$\lambda(A) = 0 \Rightarrow \mu(A) = 0$$

for every λ-measurable set A. If μ is absolutely continuous with respect to ν, we write $\mu \ll \lambda$.

Definition 2.5 (Geometric random walks)**.** *Let $(S_n)_{n\geq 0}$ be a random walk with steps $(X_n)_{n\geq 1}$. Then $(S_n)_{n\geq 0}$ is called a* ***geometric random walk with parameter*** *λ if there exists $\lambda > 0$ such that*

$$X_n = \begin{cases} \lambda^n, & w.p. \frac{1}{2} \\ -\lambda^n, & w.p. \frac{1}{2}. \end{cases}$$

We say such a walk has distribution $GRW(\lambda)$. For a $GRW(\lambda)$-distributed random walk, the limiting distribution is denoted by

$$S_\lambda := \sum_{n=1}^{\infty} X_n.$$

Let $(S_n)_{n\geq 0}$ be a $GRW(\lambda)$-distributed random walk. If $\lambda > 1$, then $(S_n)_{n\geq 0}$ will not converge, and if $\lambda = 1$, then $(S_n)_{n\geq 0}$ is a simple random walk. Hence, we will focus on the case $\lambda \in (0, 1)$. The limiting distributions of geometric walks in $\lambda \in (0, 1)$ has been extensively studied, including by [insert sources]. The following theorem summarizes their efforts to characterize the distribution of S_λ, for $\lambda \in (0, 1)$.

Theorem 2.6 (Characterization of geometric random walks)**.** *The limiting distribution S_λ of a $GRW(\lambda)$-distributed random walk can be characterized as follows, for $\lambda \in (0, 1)$.*

(1) If $\lambda \in (0, \frac{1}{2})$, then S_λ is distributed uniformly over the Cantor set generated by λ.

(2) If $\lambda = \frac{1}{2}$, then S_λ is uniformly distributed over the interval $[-1, 1]$.

(3) There is a Lebesgue-conull subset $A \subset [0, 1]$ such that for all $\lambda \in A$, S_λ is absolutely continuous with respect to the Lebesgue measure. [Solomyak]

(4) If $\lambda \in (1/2, 1)$ is the reciprocal of a Pisot–Vijayaraghavan number, then S_λ is singular with respect to the Lebesgue measure.

It has been shown by Jessen and Wintner [JW35] that $GRW(\lambda)$ must be either singular or absolutely continuous.

Definition 2.7 (Pisot-Vijayaraghavan numbers)**.** *A* ***Pisot-Vijayaraghavan number*** *is a real algebraic integer (a root of a monic polynomial with integer coefficients) greater than one whose Galois conjugates all have absolute value less than one.*

Appendix A. Measure theory

Definition A.1 (σ-algebra)**.** *For a set X, a set $\Sigma \subset \mathcal{P}(X)$ is a* ***σ-algebra*** *if it satisfies the following properties:*

(1) $\emptyset \in \Sigma$

(2) Σ is closed under complements, i.e. $A \in \Sigma \Rightarrow A^C \in \Sigma$

(3) Σ is closed under countable unions, i.e. $(A_n)_{n\geq 1} \subset \Sigma \Rightarrow \bigcup_{n\geq 1} A_n \in \Sigma$.

It follows immediately from conditions 2 and 3 that σ-algebras are also closed under countable intersections.

Generating a σ-algebra: For a set $A \subseteq \mathcal{P}(X)$, we let $\sigma(A)$ denote the **minimal σ-algebra containing** A. That is to say, $\sigma(A)$ is such that for any other σ-algebra Σ such that $A \subseteq \Sigma$, we have that $\sigma(A) \subseteq \Sigma$.

One can generate $\sigma(A)$ through induction as follows:

Let $B_0 \coloneqq A \cup \{\emptyset\}$.

For $n \geq 1$, we set $B_n \coloneqq \{\text{countable unions of elements of } B_n\} \cup \{E^C : E \in B_n\}$.

Finally, we set $B_\infty \coloneqq \bigcup_{n\geq 0} B_n$.

By construction, B_∞ is clearly closed under countable unions and complements, and $\emptyset \in B_\infty$, so B_∞ is a σ-algebra. Since $A \subseteq B_\infty$, we have that $\sigma(A) \subseteq B_\infty$. Since any σ-algebra containing A must include the elements of B_∞ in order to be a σ-algebra, we conclude that $B_\infty = \sigma(A)$.

Definition A.2. *Fix $d \geq 1$. We call a set $B \subset \mathbb{R}^d$ a **box** if B is of the form*

$$(a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_d, b_d)$$

for $a_1, b_1, \ldots, a_d, b_d \in \mathbb{R}$. (For instance, if $d = 1$ then each box is simply an open interval on $\mathbb{R}$).

*Let $A \subseteq \mathbb{R}^d$ be the set of all finite unions of boxes. We write $\mathcal{B}(\mathbb{R}^d) \coloneqq \sigma(\mathbb{R}^d)$, and call $\mathcal{B}(R^d)$ the **Borel σ-algebra** of $\mathbb{R}^d$.*

The **Lebesgue measure** λ is defined on $\mathcal{B}(\mathbb{R}^d)$, and on sets $Z \subset A$ where $A \in \mathcal{B}(\mathbb{R}^d)$ is such that $\lambda(A) = 0$.

Definition A.3. *Let $(Y_n)_{n\geq 1}$ be a sequence of independent random variables. The **tail σ-algebra** of $(Y_n)_{n\geq 1}$ is defined as*

$$\mathcal{T}((Y_n)_{n\geq 1}) = \bigcap_{m\geq 1} \sigma((Y_n)_{n\geq m}),$$

*i.e. it is the set of events whose occurrence does not depend on a finite subset of $(X_n)_{n\geq 1}$. We call an event $E \in \mathcal{T}((X_n)_{n\geq 1})$ a **tail event**.*

Theorem A.4 (Kolmogorov Zero-One Law). *Let $(Y_n)_{n\geq 1}$ be a sequence of independent random variables. For any tail event $E \in \mathcal{T}((Y_n)_{n\geq 1})$, we have that $\mathbb{P}(E) \in \{0, 1\}$.*

Proof sketch. Fix a tail event $E \in \mathcal{T}((Y_n)_{n\geq 1})$. Then for any $n \geq 1$, $E \notin \sigma(Y_1, \ldots, Y_n)$, so E is independent of the random variables $Y_1, \ldots, Y_n$. Hence, E is independent from every Y_n, so indeed E is independent from every $S \in \sigma((Y_n)_{n\geq 1})$. However, $E \in \sigma((Y_n)_{n\geq 1})$, so E is independent of itself. Therefore,

$$\mathbb{P}(E) = \mathbb{P}(E \cap E) = \mathbb{P}(E)^2,$$

so we must have that $\mathbb{P}(E) \in \{0, 1\}$. □

Lemma A.5 (Measure Compactness Lemma). *Let $(A_n)_{n\geq 1}$ be a sequence of events. Suppose there exists $\epsilon > 0$ such that for all $n \geq 1$, we have that $\mathbb{P}(A_n) \geq \epsilon$. Then*

$$\mathbb{P}(A_n \textit{ occurs infinitely often}) \geq \epsilon.$$

Proof. Recall that the event $\{A_n$ occurs infinitely often$\}$ is equivalent to

$$\bigcap_{n\geq 1} \bigcup_{k\geq n} A_k.$$

For $n \geq 1$, we set

$$B_n := \bigcup_{k\geq n} A_k.$$

Notice that the B_ns are decreasing events, i.e. $B_1 \supseteq B_2 \supseteq B_3 \supseteq$ .... We set

$$A_\infty := \{A_n \text{ occurs infinitely often}\}$$

for ease of notation. Then

$$\mathbb{P}(A_\infty) = \mathbb{P}(\bigcap_{n\geq 1} B_n) = \lim_{n\to\infty} \mathbb{P}(B_n). \tag{A.1}$$

For all n,

$$\mathbb{P}(B_n) \geq \mathbb{P}(A_n) \geq \epsilon,$$

so it follows from (A.1) that

$$\mathbb{P}(A_\infty) = \lim_{n\to\infty} \mathbb{P}(B_n) \geq \lim_{n\to\infty} \epsilon = \epsilon.$$

□

References

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