Recurrence and Transience on the Frog Model
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1 Introduction

The frog model is an infinite system of interacting random walks that takes place on a d-ary tree.

Definition 1: A **random walk** is a stochastic process that describes the path of a particle. The path consists of a random sequence of steps.

Definition 2: A **d-ary tree** (denoted T_d) is an infinite rooted graph which starts with a root (origin) and has d branches from each node.

The frog model begins with an awake frog at the root, and a sleeping frog at every node below it. The awake frog moves along the tree in a simple random walk and wakes the frog at every node it stops at. The initially sleeping frog at that node is now awake and goes on its own random walk on the tree, waking all frogs it reaches as well. Frogs can go both forward and backward. Each frog not starting at the origin has a $\frac{d-1}{d}$ probability of going down the tree, and a $\frac{1}{d}$ probability of going up the tree.

Note: Sometimes the frogs are also called particles. Branches that come from a node of the tree are called that node's children.

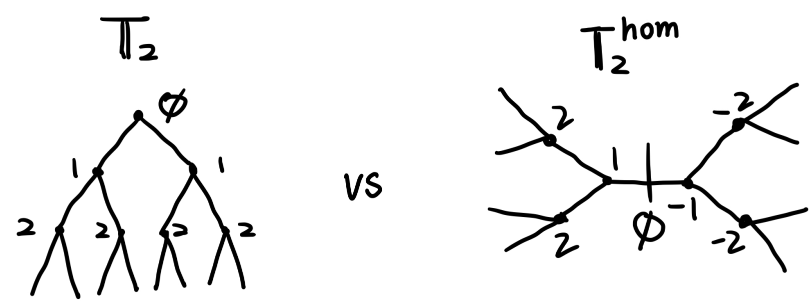

Figure 1: $\mathbb{T}_2$ tree and $\mathbb{T}_2^{hom}$ tree.

Note: We will have a further explanation in section 3.2. Briefly speaking, in this example, the rooted $2-ary$ binary tree ($\mathbb{T}_2$) has a degree-2 root and degree-3 other nodes, enabling recursive frog model analysis through generating functions. And the homogeneous tree ($\mathbb{T}_2^{hom}$) gives all nodes degree 3, and these are also the main differences between these two model structures.

The frog model has applications in modeling the spread of disease and the spread of rumours. The frog model is part of a familiar of self-interacting random walks that has garnered interest in the mathematical community.

This report summarizes key results from the paper *Recurrence and Transience for the Frog Model on Trees* by Hoffman, Johnson and Junge. The paper was published in the 2017 edition of the Annals of Probability

The question investigated is:

What is the probability the frog model on a d-ary tree will revisit the origin an infinite amount of times?

Definition 3: If the frog model visits the root an infinite amount of times, it is called **recurrent**. If it does not visit the root an infinite amount of times (the root is visited a finite amount of times), it is called **transient**

Theorem 1:
i) The frog model is recurrent when $d \leqslant 2$.
ii) The frog model is transient when $d \geqslant 5$.

There is strong evidence that the frog model is recurrent when d= 3 and transient on d= 4 but those results are outside the scope of this report.

Definition 4: An event A **stochastically dominates** another event B if $\mathbb{P}(A \geqslant x) \geqslant \mathbb{P}(B \geqslant x) \forall x$.
In simpler terms, this indicates that the event A is "more probable" than event B.
We will prove theorem 1 using the notion of stochastic dominance.
For the proof of theorem 1 i) we will couple the frog model with a process where the root is visited less often. If we let V denote the amount of visits to the root, we will that V dominates a Poisson with any mean, and thus goes to ∞

For the proof of theorem 1 ii) we will dominate the frog model by a branching random walk and show that the random walk is transient.

Definition 5: A **branching random walk** is a random walk where each particle produces a number of offspring at every step. The number of offspring produced is often governed by a probability distribution. After branching, all particles and offspring continue on the random walk.

We will also prove a 0-1 law for the frog model on T_d.

Theorem 2:
The probability a frog model is recurrent on T_d for any d and any i.i.d. (independent and identically distributed) initial conditions is either 0 or 1.

2 Recurrence

2.1 The nonbacktracking frog model

In this section, we simplify the frog model by restricting frogs to *nonbacktracking random walks* to make recurrence analysis tractable. The model is different from the original one since the frogs move randomly but cannot return to their immediately previous vertex, in other words, the first step is uniformly random neighbor while the later steps refer to be the uniformly random neighbor except the last visited vertex. In this case, the frog model will preserve *recurrence/transience* while reducing path dependencies.

2.2 The self-similar frog model

The purpose of this section is to define the *self-similar frog model* that have the similar behaves like the original process.

Firstly, denote $T_d(v)$ as the subtree of T_d which rooted at v, such that we have the nonroot vertex v' and its parent v.

By the image of the original tree and the self-similar frog model, we conclude that:

1. Only one frog is allowed to enter subtree.
2. Each subtree is a smaller copy of the full tree.
3. Arbitrarily choose one keep moving and the rest of them stop at v.

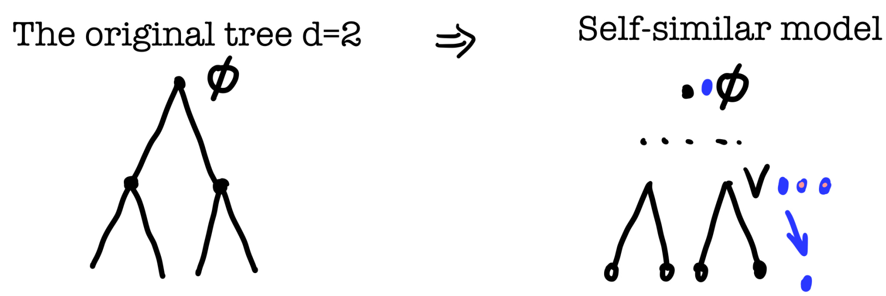

Figure 2: self-similar frog model

Proposition 1: The self-similar frog model behaves identically in every subtree.
Proof: Denote an arbitrary subtree $T_d(v')$ and ignore the frog outside it (which is just the rule: If there are more than one frogs move from v to v', stop all of them but arbitrary one at v').

By the results above, the self-similar frog model allows only one frog each time to enter the subtree rooted by v', and all frogs outside the subtree is no longer moving and thus preventing external interference.

2.3 Coupling the models

In this section, we have a step by step coupling, which is:

$$\text{Original model} \rightarrow \text{Nonbacktracking model} \rightarrow \text{Self-similar model}$$

Let's start with the first step: For each original model, construct a nonbacktracking model by taking a random walk with deleting the 'backtracks', and in this case, there will be fewer return to the root and thus if we can show that the nonbacktracking visits (to the root) are finite, then can directly conclude that the original one must be too.

Then for the second step, if we artificially stop frogs to enforce the rule in nonbacktracking model, then we can just get the self-similar model. Generally speaking, this is a further reduce of root visit, which means if we can show that even this restricted model has infinite visits, then the recurrence one must hold.

To sum up, the whole idea is like proving the 'lower case' in Analysis, i.e, if the simplified model (self-similar model) is recurrent, then the original one must be too!

Proposition 2: The nonbacktracking and self-similar model can be coupled with the original frog model, such that every root visits in the simplify model corresponds to the distinct root visit in the original one.

Before doing the proof, we need to clarify a definition first.

Definition 6: A **geodesic** refers to the unique shortest path between two vertices on the tree (specifically, the infinite $d - ary$ tree T_d. In other words, the geodesic between two vertices on the tree is the unique path connecting them without cycles. And in mathematics, for a surface given parametrically by $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, the geodesic can be found by minimizing the *arc length*

$$I \equiv \int ds = \int \sqrt{dx^2 + dy^2 + dz^2}.$$

And, for a surface of revolution in which $y = g(x)$ is rotated about the *x-axis* so that the equation of the surface is

$$y^2 + z^2 = g^2(x)$$

and thus the equation of the geodesics is

$$v = c_1 \int \frac{\sqrt{1 + [g'(u)]^2} du}{g(u)\sqrt{[g(u)]^2 - c_1^2}}$$

Proof: After coupling the original frog model with nonbakctracking rule, the new simplified path is a subset of the original path, and every root visit in the simplified path must appear in the original path at the same step, and thus if the simplified model has infinitely many root visits, the so does the original model.

2.4 Generating function recursion

In this section, we will develop a generating function that analyze the number of root visit (v) in the self-similar frog model.

Definition 7: The **probability generating function** in this model is a mathematical tool that encodes the *probability distribution* of the number of visit V to the root in the frog model. i.e, the generating function is defined as:

$$f(x) = \mathbb{E}[x^V] = \sum_{k=0}^{\infty} \mathbb{P}(V = k) \cdot x^k$$

where $V =$ the number of root visits, and x is a variable ($0 \le x \le 1$).
For $x = 0$:$f(0) = \mathbb{P}(V = 0)$, which represents the probability of no visits.
For $x = 1$: $f(1) = \sum_{k=0}^{\infty} \mathbb{P}(V = k) = 1$.

Proposition 3: If $V = \infty$, then $f(x) = 0$.
Proof: if $V = \infty$ with probability 1, then the original function becomes

$$f(x) = \mathbb{E}[x^{\infty}] = \sum_{k=0}^{\infty} \mathbb{P}(V = \infty) \cdot x^{\infty}$$

For $x \in [0, 1)$, $x^{\infty} = 0$, so $f(x) = 0$,
Then the special case is when $x = 1$, as ti maintain the continuity, we set $f(1) = \sum_{k=0}^{\infty} \mathbb{P}(V = k) = 0$, which implies if $V = \infty$, then $f \equiv 0$ in both cases.

Definition 8: A **functional operator** (denoted by A in this paper) is a mathematical object that takes a function as input and outputs another function by some computation. And it acts as a 'bridge' to transforms the generating function.

Proposition 4: Define A which is a operator on functions on $[0, 1]$ by

$$Ag(x) = \frac{x+2}{3} g\left(\frac{x+1}{2}\right)^2 + \frac{x+1}{3} g\left(\frac{x}{2}\right)\left(1 - g\left(\frac{x+1}{2}\right)\right)$$

and the generating function f satisfies $f = Af$.

Definition 9: A is also defined as a **fixed point** in this paper. More generally, a fixed point of an operator (or function) T is an output x that satisfies:

$$T(x) = x$$

Proof: By the proposition above we conclude that if $V = \infty$, then $f(x) = 0$, and thus assume form now on that $\mathbb{P}[V = \infty] = 0 < 1$. Recall in the previous results we have that the subtrees have the identical behavior as the original one, then for the recursive structure of V, which is the root visits that can be decomposed into the visits from the initial active frog and the visits from awakened frogs in subtrees, and meanwhile, each subtree's contribution is an independent copy of V due to self-similarity. $\square$

We define v as a *new root* compared to the *real root* $\varnothing'$, and let u to be the children that to be visited. From the results we got previously, define V_u and V_v to be the number of frogs which visit $\varnothing'$ in the subtrees $\mathbb{T}_2(u)$ and $\mathbb{T}_2(v)$ respectively. Since self-similar, then we can say that both V_u and V_v have the same distribution as the original V, and also define V' which is distributed as V, and we can rewrite V as a composition of a pair of independent stuffs:

$$V = 1\{\text{frog at } \varnothing' \text{ visit} \varnothing\} + 1\{u \text{ is visited}\} Bin(v', \frac{1}{2}) + Bin(V_v, \frac{1}{2}) \tag{1}$$

Recall: Let X follows the *binomial distribution* with parameters $n \in \mathbb{N}$, and $p \in [0, 1]$, we write $X \backsim Bin(n, p)$. The probability of getting exactly k successes in n independent Bernoulli trials (with the same rate p) is given by the probability mass function:

$$f(k, n, p) = \mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

and for

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Now back to function (1) and look at the three terms, we can conclude that the first term accounts for a possible visit to $\varnothing$ by frog started at $\varnothing'$, the second term represents if u is visited, then the number of times that frog from $\mathbb{T}_2(u)$ visit $\varnothing$ with binomial distribution $Bin(V', \frac{1}{2})$, and the third term is just the number of times the frog from $\mathbb{T}_2(v)$ that visit $\varnothing$ also with Binomial distribution $Bin(V_v, \frac{1}{2})$.

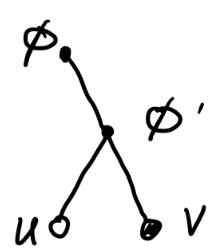

Figure 3

Then we will firstly consider three disjoint events as A, B, C:
A: Initial frog moves to a child and activates its subtree (with probability $\frac{1}{3}$;
B: Initial grog doesn't activate but another frog does (with probability$\frac{2(1-q)}{3}$;
C: There is no frogs activate the other subtree (with probability $\frac{2q}{3}$, where $q = \mathbb{P}(\text{subtree isn't activated})$.

When the initial frog activates both subtrees (Event A), we get terms involving products of generating functions, representing subtree contributions. And when there is only one subtree gets activated (Event B), the expression involves conditional terms that account for this restricted activation. The precise form of A carefully weights these scenarios by their probabilities.

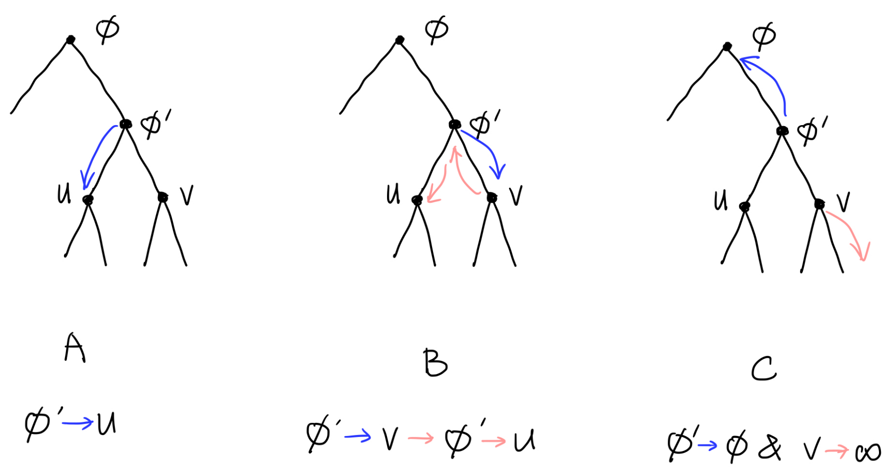

Figure 4

By the image above, we can combine event A, B, C with function (1), or in other words, recompose event A, B, C with conditions in three terms in function (1).

1.For event A, the frog starting st $\varnothing'$ visit u directly, so term 1 is 0 and term 2 and 3 are distributed as independent $Bin(V, \frac{1}{2})$.

2.For event B, the frog at $\varnothing'$ does not visit u, and a frig returns to $\varnothing'$ through v ad visit u, so term 1 is Bernoulli distributed with $p = \frac{1}{2}$, term 2 is $Bin(V, \frac{1}{2})$, and for term 3, by the rule of nonbacktracking, if there is a frog moves from v to u, then it cannot go back to $\varnothing'$ (even $\varnothing$ again, thus we can say that term 3 is $Bin(V, \frac{1}{2})$ conditional on being strictly less than V.

3. For event C, there is no frog visit u, so term 1 is also Bernoulli distributed with $p = \frac{1}{2}$ and the special thing in this case is that term 2 is 0 since u will never be visited, and thus every frog will return to $\varnothing$ (since u cannot be visited), so term 3 is exactly $Bin(V, \frac{1}{2})$ conditional on being equal to V.

In order to get a further result, define notations as follows to summarize:
Let X' and X be the $Bin(V, \frac{1}{2})$ for term 2 and 3 respectively. Let Y be conditional Binomial distribution that $< V$. Let Z be distributed as $Bin(V, \frac{1}{2})$ conditional on $Bin(V, \frac{1}{2}) = V$. Let $I \backsim Bernouli(\frac{1}{2})$, and since the three events A, B, C are independent of each other, we can then write V as A $\cup$ B $\cup$ C, or in other words,

$$V = \begin{cases} X' + X & \text{with probability } \frac{1}{3} \text{ by event A} \\ I + X' + Y & \text{with probability } \frac{2(1-q)}{3} \text{ by event B} \\ I + Z & \text{with probability } \frac{2q}{3} \text{by event C} \end{cases}$$

where $q = 1 - p$ represents the probability that no frog ever visits the subtree rooted at node u.

Plug this into the expression of expectation of x^V and then we get:

$$\begin{aligned}\mathbb{E}\left(x^V\right) &= \frac{1}{3}\mathbb{E}\left(x^{X'+X}\right) + \frac{2(1-q)}{3}\mathbb{E}\left(x^{I+X'+Y}\right) + \frac{2q}{3}\mathbb{E}\left(x^{I+Z}\right) \\ &= \frac{1}{3}\mathbb{E}\left(x^{X'}\right)\mathbb{E}\left(x^{X}\right) + \frac{2(1-q)}{3}\mathbb{E}\left(x^{I}\right)\mathbb{E}\left(x^{X'}\right)\mathbb{E}\left(x^{Y}\right) + \frac{2q}{3}\mathbb{E}\left(x^{I}\right)\mathbb{E}\left(x^{Z}\right)\end{aligned}$$

Recall: If X is a random variable with a **Bernoulli distribution**, then:

$$\mathbb{P}(X=1) = p = 1 - \mathbb{P}(X=0) = 1 - q.$$

The probability mass function (pdf) f of this distribution over possible outcomes k is,

$$f(k;p) = \begin{cases} p & \text{if } k = 1, \\ q = 1-p & \text{if } k = 0. \end{cases}$$

or it can be expressed as

$$f(k;p) = p^k(1-p)^{1-k}, \text{ for } k \in \{0,1\}$$

Moreover, the Bernoulli distribution is a special case of the **Binomial distribution** with $n = 1$, i.e. if $X_1, ..X_n$ are independent. identically distributed (i.i.d.) random variables, all Bernoulli trails with success probability p, then their sum is distributed according to a *Binomial distribution* with parameters n and p:

$$\sum_{k=1}^{n} X_k \backsim B(1,p).$$

To derive the final expression of operator A, we want to find the expectations one by one, set $\mathbb{E}x^I$ as an example and the rest are for the similar reason. Here is the detailed breakdown.
We define:
$I = 1$ if the frog at $\varnothing'$ jumps back to $\varnothing$;
$I = 0$ if it moves away. Then for any Bernoulli random variable $I \backsim Bernoulli(p)$:

$$\mathbb{E}\left[x^I\right] = \sum_{k=0}^{1} x^k \mathbb{P}(I=k) = x^0(1-p) + x^1 p = 1 - p + xp.$$

For $p = 1/2$:

$$\mathbb{E}\left[x^I\right] = 1 - \frac{1}{2} + x \cdot \frac{1}{2} = \frac{x+1}{2}$$

Thus,

$$\mathbb{E}\left[x^I\right] = \frac{x+1}{2}$$

This is a direct consequence of I being Bernoulli with $p = \frac{1}{2}$.
Going back to three events above, they can be simplified as:
For **Event A**:
Since V is identically distributed in subtrees, then the generating function becomes $f(\frac{x+1}{2})^2$. Thus,

$$\mathbb{E}\left[x^V | \text{Event A}\right] = f(\frac{x+1}{2})^2.$$

For **Event B**:
The first term is still contribute as $\frac{x+1}{2}$, and since by previous we have shown that second term gives $Bin(V, \frac{1}{2})$ conditional on being $< V$, i.e. the generating function now becomes $\frac{f(\frac{x+1}{2})-f(\frac{x}{2})}{1-q}$, and the third term is the same as usual. Thus,

$$\mathbb{E}\left[x^V \middle| \text{Event B}\right] = \frac{x+1}{2} \cdot \frac{f(\frac{x+1}{2}) - f(\frac{x}{2})}{1-q} \cdot f\left(\frac{x+1}{2}\right).$$

For **Event C**:
By previous results, the generating function of term 3 now becomes $\frac{f(\frac{x}{2})}{q}$ since all frogs in subtree v must return to the root. Thus,

$$\mathbb{E}\left[x^V \middle| \text{Event C}\right] = \frac{x+1}{2} \cdot \frac{f(\frac{x}{2})}{q}$$

Now, we combine all events with their probabilities:

$$\begin{aligned} f(x) &= \frac{1}{3} f\left(\frac{x+1}{2}\right)^2 + \frac{2(1-q)}{3} \cdot \frac{x+1}{2} \cdot \frac{f\left(\frac{x+1}{2}\right) - f\left(\frac{x}{2}\right)}{1-q} \cdot f\left(\frac{x+1}{2}\right) \\ &+ \frac{2q}{3} \cdot \frac{x+1}{2} \cdot \frac{f\left(\frac{x}{2}\right)}{q} \\ &= \frac{x+2}{3} f\left(\frac{x+1}{2}\right)^2 + \frac{x+1}{3} f\left(\frac{x}{2}\right)\left(1 - f\left(\frac{x+1}{2}\right)\right) \end{aligned}$$

Hence, we finally derive the operator A:

$$f(x) = \frac{x+2}{3} f\left(\frac{x+1}{2}\right)^2 + \frac{x+1}{3} f\left(\frac{x}{2}\right)\left(1 - f\left(\frac{x+1}{2}\right)\right) = Af(x)$$

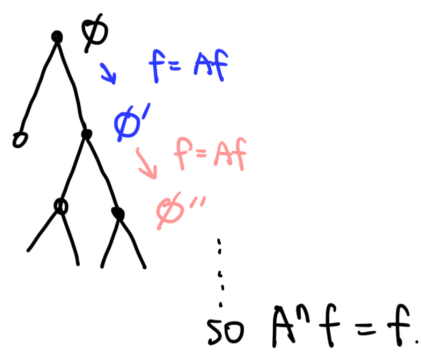

Figure 5: $f = Af$

2.5 Proving recurrence

In this section, we will complete the proof of recurrence for the frog model on the binary tree ($d = 2$) by analyzing the generating function $f(x) = \mathbb{E}[x^V]$, where V is the number of visits to the root, and we can construct it into some key steps, such as the operator A by got by last proposition, and the limit analysis.

Definition 10: A function f defined on a subset of the real numbers with real values is called **Monotonic** if it is either entirely non-decreasing, or entirely non-increasing. In other words, suppose W.L.O.G. that for all x and y that $x \leq y$, we then have $f(x) \leq f(y)$, then such a function is called monotonically decreasing.

Lemma : The operator A is monotonic for functions belongs to the set $\mathscr{S} = \{g : [0, 1] \to [0, 1], \text{nondecreasing}\}$.
Proof: Let $g, h \in \mathscr{S}$ and assume W.L.O.G. that $g(x) \leq h(x)$ for all $x \in [0, 1]$, now we analyze the operatoe term by term:
The first term: Since $g \leq h$, and by definition g, h are nondecreasing, and since $\frac{x+1}{2} \geq 0$ which preserves the inequality, then:

$$g\left(\frac{x+1}{2}\right)^2 \leq h\left(\frac{x+1}{2}\right)^2$$

The second term: For $g \leq h$, we can conclude that $-g \geq -h$, and moreover $1 - g \geq 1 - h$, since there are both $\leq$ and $\geq$ in this case, let

$$a = g\left(\frac{x}{2}\right), b = g\left(\frac{x+1}{2}\right), c = h\left(\frac{x}{2}\right), d = h\left(\frac{x+1}{2}\right)$$

and we are given that $a \leq c$, and $b \leq d$ (since$\frac{x}{2} \leq \frac{x+1}{2}$). After do something on inequality and then we can get

$$c(1 - d) - a(1 - b) = (c - a) + (ab - cd) \geq (c - a) + 0 \geq 0$$

Therefore,

$$a(1 - b) \leq c(1 - d)$$

That is,

$$g\left(\frac{x}{2}\right)\left(1 - g\left(\frac{x+1}{2}\right)\right) \leq h\left(\frac{x}{2}\right)\left(1 - h\left(\frac{x+1}{2}\right)\right)$$

To sum up, since both $\frac{x+2}{3}$ and $\frac{x+1}{3}$ are ≥ 0, so multiply them will not effect the sign of the final answer, so we get

$$\frac{x+2}{3}g\left(\frac{x+1}{2}\right)^2 + \frac{x+1}{3}g\left(\frac{x}{2}\right)\left(1 - g\left(\frac{x+1}{2}\right)\right) \leq \frac{x+2}{3}h\left(\frac{x+1}{2}\right)^2 + \frac{x+1}{3}h\left(\frac{x}{2}\right)\left(1 - h\left(\frac{x+1}{2}\right)\right)$$

which finally implies

$$g(x) \leq h(x)$$

This shows that A is monotonic on the set $\mathscr{S} = \{g : [0, 1] \to [0, 1], \text{nondecreasing}\}$ $\square$

Lemma: If $g \in \mathscr{S} = \{g : [0, 1] \to [0, 1]$, then $Ag \in \mathscr{S} = \{g : [0, 1] \to [0, 1]$.
Note: This is really similar to the definition of *T-invariant* in Linear Algebra.
(Let $T : V \to V$ be an operator. A subspace $U \subseteq V$ is called T-invariant if $T(U) \subseteq U$, that is, $T(u) \in U$ for every vector $u \in U$.)
From this perspective, we can say that $\mathscr{S}$ is 'A-invariant' because A applied to any function in $\mathscr{S}$ results in another function in $\mathscr{S}$.
Proof: Recall that

$$f(x) = \frac{x+2}{3}f\left(\frac{x+1}{2}\right)^2 + \frac{x+1}{3}f\left(\frac{x}{2}\right)\left(1 - f\left(\frac{x+1}{2}\right)\right) = Af(x)$$

and we want to show that:
1. *Nondecreasing*: $Ag(x)$ is not decrease as x increases;
2. *Bounded*: $Ag(x) \in [0, 1]$ for $x \in [0, 1]$.

Cliam: $Ag(x)$ is nondecreasing.
Let $g \in \mathscr{S}$ for $x \in [0,1]$, then since both $\frac{x+1}{2}$ and $\frac{x}{2}$ increases as x increases, thus

$$Ag(x) = \frac{x+2}{3} g\left(\frac{x+1}{2}\right)^2 + \frac{x+1}{3} g\left(\frac{x}{2}\right)\left(1 - g\left(\frac{x+1}{2}\right)\right)$$

is also nondecreasing. (Notice that the sum of two nondecreasing functions is also nondecreasing).
Claim: $Ag(x)$ is bounded in [0,1]. When consider the first term, $\frac{x+2}{3} \in [\frac{2}{3}.1]$ and $g(\frac{x+1}{2})^2 \in [0,1]$, so their product $\in [0,1]$, and for the second term, $\frac{x+1}{3} \in [\frac{1}{3}, \frac{2}{3}]$, $g(\frac{x}{2}) \in [0,1]$, and $1 - g(\frac{x}{2}) \in [0,1]$, so their product also $\in [0,1]$. Combine two terms together and come to a conclude that $Ag(x)$ is a convex combination of terms in [0,1], so $Ag(x) \in [0,1]$.
To sum up, we have ensured that if $g \in \mathscr{S}$, then $Ag(x) \in \mathscr{S}$ as well. $\square$

Note: Define $g_a(x) = e^{a(x-1)}$ for all $a \geq 0$, then for all $x \in [0,1]$, there will be $Ag_a(x) \leq g_{a+c_a}(x)$, where

$$c_a = \begin{cases} \frac{1}{3}e^{-2} & 0 \leq a \leq 4, \\ \frac{1}{3}e^{-\frac{a}{2}} & a \geq 4. \end{cases}$$

Lemma: For all $x \in [0,1)$

$$\lim_{n\to\infty} A^n g_0(x) = 0.$$

Proof: We ca start by $a_0 = 0$, and this implies $g_0(x) = e^{a\cdot(x-1)} = e^0 = 1$. And we define

$$a_{n+1} = a_n + c_{a_n},$$

where c_a is defined by last note.

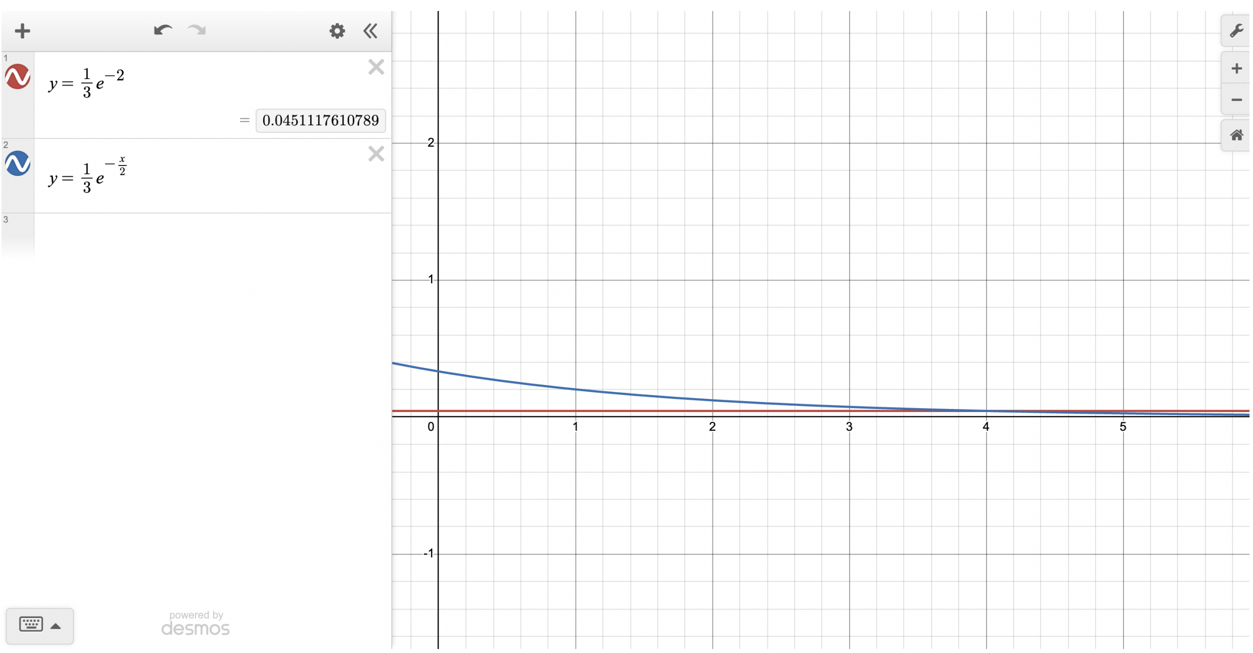

Figure 6: The graph of c_a

from above we observe that $c_a > 0$ for all $a \geq 0$, so a_n is strictly increasing.
Also by the previous note we conclude that $Ag_a(x) \leq g_{a+c_a}(x)$, then in order to show $A^n g_0(x) \leq g_{a_n}(x) =$

$e^{a_n(x-1)}$, we will use the Induction proof. *Base case*(n=0): By definition, $A^0 g(x) = g_0(x) = 1$, thus the inequality holds for $A^0 g(x) = 1 \leq 1 = g_{a_0}(x)$.
Inductive Hypothesis: Assume for $k \geq 0$, we have

$$A^k g_0(x) \leq g_{a_k}(x)$$

then apply A to both sides of the equation above, we get:

$$A^{k+1} g_0(x) = A(A^k g_0)(x) \leq A g_{a_k}(x),$$

And by the notes above we know:$A g_{a_k}(x) \leq g_{a_k + c_{a_k}} = g_{a_{k+1}}$, by combining these we finally get:

$$A^{k+1} g_0(x) \leq g_{a_{k+1}}(x).$$

Thus, this statement also holds for $n = k+1$. So by induction, the inequality holds for all $n \geq 0$, and then as $a_n \to \infty$, and $x \in [0, 1)$, we have:

$$\lim_{n\to\infty} A^n g_0(x) \leq \lim_{n\to\infty} e^{a_n(x-1)} = 0,$$

and since $A^n g_0(x) \geq 0$, by the squeeze theorem, we conclude that:

$$\lim_{n\to\infty} A^n g_0(x) = 0.$$

In the whole section 2, we proved the recurrence for the frog model on the binary tree, which give a strong evidence **Theorem 1**: 'The frog model is recurrent when $d \leqslant 2$.'.

3 Transience

3.1 Transience for $d \geqslant 14$

In order to prove transience, we want to stochastically dominate the frog model by a transient process. We will dominate the frog model a branching random walk on T_d whose particles split in 2 at every step. This random walk has more particles than the frog model. The larger number of particles means that the probability of recurrence is higher because there are more particles that could revisit the root. Therefore by proving the branching random walk is transient, we will have proved the frog model on T_d is transient.

Let C_n be the nth Catalan number.

Definition 11: A **Catalan number** is a sequence of natural numbers often used in recursively defined objects, defined as follows:

$$\begin{aligned} C_n &= \frac{1}{n+1}\binom{2n}{n} \\ &= \frac{(2n)!}{(n+1)! \cdot n!} \\ &= \frac{n(n+1)}{2} \\ C_n &\approx \frac{4^n}{n^{3/2}} \end{aligned}$$

C_n is the number of Dyck paths of length $2n$.

Definition 12: The origin of **Dyck paths** was enumerating paths from (0, 0) to (n, n) on a grid. However, this can be generalized to tree graphs. A Dyck path of length $2n$ starts at (0, 0) and ends at (n, n). It consists up up steps (1,1) and down steps (1, -1) and never goes below y = 0.

Definition 13: A **union bound** can be shown as: $P(A) + P(B) \geqslant P(A \cup B)$
or more generally when $P(A_1) + P(A_2) + \cdots + P(A_n) \geqslant P(A_1 \cup A_2 \cdots A_n)$

Definition 14: $O(f(n))$ **notation** (also known as big O notation) is an upper bound on the growth or decay rate of a function as $n \to \infty$

At the nth step of the branching random walk, there are 2^{2n} particles. Using a union bound, the probability that any of the 2^{2n} particles is at the root is at most

$$\begin{aligned} 2^{2n}\left(\frac{d}{d+1}\right)^n\left(\frac{1}{d+1}\right)^n C_n &\approx 2^{2n}\left(\frac{d}{d+1}\right)^n\left(\frac{1}{d+1}\right)^n \frac{4^n}{n^{3/2}} \\ &= \left(\frac{16n}{(d+1)^2}\right)^n \frac{1}{n^{3/2}} \end{aligned}$$

$$O\left(\left(\frac{16n}{(d+1)^2}\right)^n \frac{1}{n^{3/2}}\right) = O\left(\left(\frac{16d}{(d+1)^2}\right)^n\right)$$

$$\text{If} \quad \frac{16d}{(d+1)^2} < 1 \Rightarrow \left(\frac{16d}{(d+1)^2}\right)^n \quad \text{decays to 0,}$$

$$\begin{aligned} \frac{16d}{(d+1)^2} &< 1 \\ 16d &< d^2 + 2d + 1 \\ 0 &< d^2 - 14d + 1 \\ d &= 7 + 4\sqrt{3} \\ d &\approx 13.9 \end{aligned}$$

We can omit the negative root. When $d \geqslant 14$ the the probability the branching random walk visit the root decays to 0.

Thus the frog model is transient when $d \geqslant 14$.
We will further refine this result through branching random walks to prove transience on $d \geqslant 6$, then using an alternating 5-6 ary tree to prove transience for $d = 5$.

3.2 Couplings

We will now generalize the T_d. Let λ be any infinite rooted graph and H be any graph. Enumerate copies of λ as $\lambda(i)$. Form a larger graph G by adding an edge from the root of the $\lambda(i)$ into H. We will show that regardless of how many sleeping frogs are in H, a frog model is less transient on G than on λ. We will then set $\lambda = T_d$.

Lemma: Consider model 1, a frog model on λ with i.i.d. v initial conditions for any v in N. Then consider model 2, a second frog model on G with one initially active frog at the root of $\lambda(1)$ and i.i.d sleeping frogs at all other vertices of $\lambda(i)$ for all i, and any configuration of sleeping frogs in H.
Assume H is a random walk on G that escapes.

Let V_g be the number of times the root of any $\lambda(i)$ is visited in model 2, not counting steps from H to a root. Let V_λ be the number of times the root is visited in model 1.
Then the models can be coupled so that $V_g \geqslant V_\lambda$.

Proof: Let x be the awake frog in model 1 and x' be the awake frog in model 2. Have x copy x', pausing at the root when x' enters H. Every time x wakes sleeping frog so does x'. Couple the newly awoken frogs as well. x' can visit any λ(i) and H, which allows it to wake more frogs than x which can only visit λ. Every visit to the root in model 2 corresponds to a visit to any root in model 1, showing $V_g \geqslant V_\lambda$.

Definition 15: T_d^{hom} is the homogeneous tree, where there is the same amount of children coming from each node (including the origin). This means that at the origin there is a branch going backwards, to level -1. At this level, the tree continues as normal.

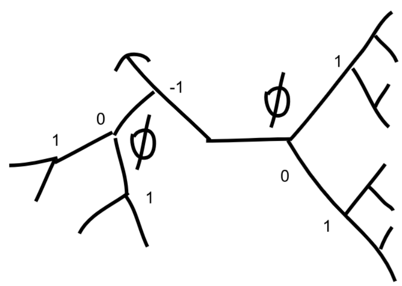

Figure 7: T_d^{hom}

Corollary: Let G = T_{d+1}^{hom} and $\lambda = T_d$. There is an awake frog at the root of G, and no sleeping frogs at direct ancestors of the root. Following directly from the previous lemma, if any roots in G are a.s visited finitely often, then the frog model on T_d is a.s transient.

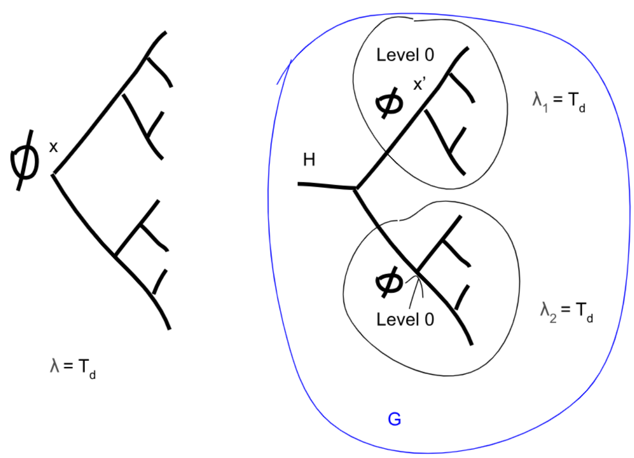

Figure 8: λ and G

Corollary: Run the frog model on T_d , starting with an awake frog at level k instead of the root. Assume that there are no sleeping frogs at levels 0,..., $k-1$ and i.i.d.-v sleeping frogs at level k and beyond, (except the the initial frog). The probability that the root is visited infinitely often in this model is at least the probability that the root is visited infinitely often in the usual frog model on T_d with i.i.d.-v initial conditions.

Proof: Let $G = T_d$, viewing it as d^k copies of T_d connected to levels 0 to $k-1$ of the original T_d. Let p be the probability that the root is visited infinitely often in the usual frog model. Let Y be the number of visits from level $k+1$ to k in our alternative model. If we consider $\lambda(i)$ to be the d^k copies of T_d and H as the levels 0 to $k-1$ of the original T_d, it follows from Lemma 15 that P(Y = infinity) $\geqslant p$.

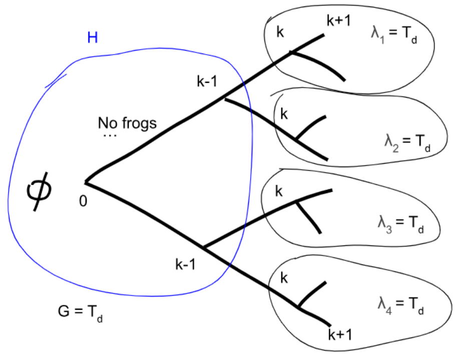

Figure 9: G

Let X be the number of visits to the root. We want to show that $P((Y = \infty) \cap (X < \infty)) = 0$ which implies that if $Y = \infty, X$ must also be equal to ∞. Thus, $P(X = \infty) \geqslant p$.

We define a dash as the event of a frog moving from level $k+1$ to a vertex v at level k, then going directly to the root then, directly back to v.

Let X' be the number of dashes. Since there are no sleeping frogs on levels 0 to $k-1$, if a frog goes from $k+1$ to k, the process of making a dash is independent. This process is also independent of any future dashes.

At each visit from $k+1$ to k, there is an independent $\frac{1}{d(d+1)^{2k-1}}$ chance of a dash.
$X' = \frac{1}{d(d+1)^{2k-1}} Y$ so as $Y \to \infty, X' \to \infty$
Thus $P(Y = \infty \cap X' < \infty) = 0$
and since $X' < X$ this implies $P(Y = \infty \cap X < \infty) = 0$ and $P(X) \geqslant p$, as desired.

Proof of Theorem 2: Let the probability the root is visited infinitely often in a frog model on T_d with i.i.d.-v initial conditions be $p > 0$. We want to show that $p = 1$. We will accomplish this by during this statement into a finite event, then showing there are infinitely many independent opportunities for this event to occur. We will fix a constant N and show that at least N frogs visit the root with probability 1.

Claim: For any k and N, there is a constant $K = K(k, N)$ such that the following holds. Consider the frog model on T_d starting with an awake frog at level k, with i.i.d. - v sleeping frogs at levels $k, k+1, K-1$ with the exception of the vertex of the initial frog, and with no sleeping frogs outside of this range. With probability at least $p/2$, this process makes at least N visits to the root.

Proof: Consider the frog model in corollary 17. Let E_k be the event that there are at least N visits to the root by frogs woken by frogs from levels $K-1$ and above. As $K \to \infty, P(E_k) \to P($ at least N visits to the root by any frog$) \geqslant p$ by corollary 17.

Thus if we take K sufficiently large, $P(E_k) \geqslant p/2$.

Let k_0 and $k'_0 = 0$. Let $k'_i = K(k_{i-1}, N)$ from the claim. Let k_i be the level of the first frog that wakes up at a level k'_i or below.
Run a frog process starting with the frog at k_i with no sleeping frogs below level k_i or at level $k+1$ or below.
We represent these processes as bands in the figure.

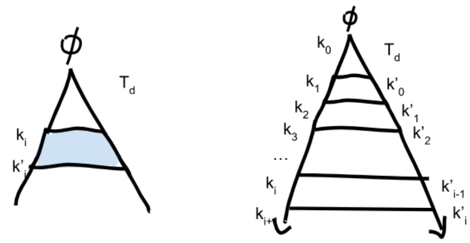

Figure 10: Levels on T_d

Let the shaded band be all levels between levels k_i and k'_i. There are N visits to the root by frogs woken by frogs in levels k'_i and above with probability $p/2$.

We can embed these bands into the original process process on T_d. Each band independently has a $p/2$ probability of visiting the root at least N times by the claim.

P(root visited N times) as $N \to \infty = P$(root visited infinitely many times)

With a sufficiently large K, you also have a corresponding number of levels k, so
P(root visited infinitely many times) $= k \cdot P$(root visited N times) $= k \cdot p/2 \approx 1$
Thus we have proved the 0-1 law for recurrence on T_d.

3.3 Proving Transience

Now we move to proving transience on $d \geqslant 6$. We will define ξ_n, a branching random walk to stochastically dominate the frog model.
At every step, this branching random walk will birth one child to its left or two children to its right.
Start with a single particle ξ_0 at 0. The probability of moving left is $\frac{1}{d+1}$ and the probability of moving right is $\frac{d}{d+1}$.
Thus with probability $\frac{1}{d+1}\xi_1$ consists of a single particle at -1, and with probability $\frac{d}{d+1}\xi_1$ consists of two particles at 1. Each particle in ξ_n independently produces children in ξ_{n+1} in the same way at its position.

Proposition 5: For $d \geqslant 6,$ the frog model on T_d is a.s. transient.

Proof: Consider the frog model on T_d^{hom} with no sleeping frogs at direct ancestors of the root (see Figure in Corollary 16). If a frog jumps backwards, it does not spawn a new frog, and when it moves forward, it sometimes does. Thus if we project the frog model with the levels corresponding to the integers, we can couple the frog model with $(\xi_n, n \geqslant 0)$ so each frog has a corresponding particle. By Corollary 16, ξ_n stochastically dominates the frog model, so it suffices to prove ξ_n visits the root finitely many times.
We will define a weight function to determine the behaviour of ξ_n. The position of particle i is $P(i)$. The weight function w is defined as follows.

$w(\xi) = \sum_{i \in \xi} e^{-\theta P(i)}$, with θ to be chosen later.

Let $\mu = \mathbb{E}w(\xi_1)$, and let's solve for a value for μ:
$\mathbb{E}w(\xi_1) = e^{-\theta P(1)}$, where

$$P(1) = \begin{cases} 1 & \text{with probability } \frac{2d}{d+1} \\ -1 & \text{with probability } \frac{1}{d+1} \end{cases}$$

($2d$ accounts for the two particles birthed going right instead of 1)

So, $\mathbb{E}w(\xi_1) = e^{-\theta P(1)}$
$= \frac{2d}{d+1}e^{-\theta} + \frac{1}{d+1}e^{\theta}$

Definition 16: The **conditional expectation** of a variable is its expected value evaluated with respect to a conditional probability distribution. The conditional expectation is also a random variable!
For example, if we let X be the number rolled on a fair six-sided die

$$\mathbb{E}(X \Big| \text{number rolled is larger than 3}) = \begin{cases} 4 & \text{with probability } \frac{1}{3} \\ 5 & \text{with probability } \frac{1}{3} \\ 6 & \text{with probability } \frac{1}{3} \end{cases}$$

Since each step of ξ_n is independent:

$$\begin{aligned} \mathbb{E}(w(\xi_{n+1}) \mid \mathbb{E}w(\xi_n)) &= \frac{2d}{d+1}e^{-\theta P(i)+1} + \frac{1}{d+1}e^{\theta P(i)-1} \\ &= \left(\sum_{i\in\xi} e^{-\theta P(i)}\right)\left(\frac{2d}{d+1}e^{-\theta} + \frac{1}{d+1}e^{\theta}\right) \\ &= w(\xi_n)\mu \end{aligned}$$

Definition 17: A **martingale** is a sequence of random variables X_n for which $\mathbb{E}(X_{n+1}|X_n) = X_n$.

It can be verified that $\frac{w(\xi_n)}{\mu^n}$ is a martingale. By martingale properties, since all elements are positive, it converges a.s.

When $\mu < 1 \Rightarrow \mu^n \to 0$ as $n \to 0 \Rightarrow w(\xi_n) \to 0$ in order for the martingale to converge.

In order for the martingale to converge, there cannot be a particle at the origin at step n. If particle j in ξ_n is at the origin, then $P(j) = 0$. $e^{-\theta \cdot 0} = 1$ so
$\sum_{i\in\xi_n(i\neq j)} e^{-\theta P(i)} + 1 \geqslant 1$
since $\sum_{i\in\xi_n(i\neq j)} e^{-\theta P(i)} > 0$.
Infinitely many visits to the origin prevent $w(\xi_n)$ from converging.
Thus $\mu < 1$ implies that ξ_n is a.s. transient.

Now we want to minimize θ to find the smallest d that guarantees transience under this model.
$\mu = (\frac{2d}{d+1}e^{-\theta} + \frac{1}{d+1}e^{\theta})$
This is minimized by setting $\theta = \frac{log(2d)}{2}$.
This value of θ makes $\mu = \frac{2\sqrt{2d}}{d+1}$. Now we can use this value to solve for d

$$\begin{gathered} \mu = \tfrac{2\sqrt{2d}}{d+1} \\ 1 < \tfrac{2\sqrt{2d}}{d+1} \\ d+1 < \sqrt{2d} \\ d > 3 + 2\sqrt{2} \approx 5.83 \\ d \geqslant 6 \end{gathered}$$

This proves that the frog model on T_d with $d \geqslant 6$ is a.s. transient.
Proposition 6: Let $T^{hom}_{5,6}$ be the homogeneous tree that alternates between 5 and 6 children at every level. The root will have either 5 or 6 children. The frog model on this tree is transient a.s.

Proof: $T_{5,6}^{hom}$ contains $T_{5,6}$. Place a sleeping frog at each vertex except for direct ancestors of the root of $T_{5,6}$. Lemma 15 implies that it suffices to prove the transience of this frog model on $T_{5,6}^{hom}$.

A frog at a vertex with 5 children has different probabilities of moving forward and backward than a frog at a vertex with 6 children. The frog alternates between the two states as the tree alternates between the two levels.

There are certain cases where every frog's movement doesn't result in the birth of a new frog. If a frog moves backwards, there is a chance it immediately jumps forward to the same vertex, which never births a new frog. If two frogs occupy the same site, they could jump forward to the same vertex which births 1 frog instead of 2. We will define a new branching random walk that will reflect this.

We will create a multitype branching random walk on Z with 6 particle types; $F_5, D_5, B_5, F_6, D_6, B_6$. If the frog is at a vertex with 5 children, the subscript is 5 and likewise for 6. The letters indicate different cases.

F - single frogs with sleeping frogs present at all children.
D - represents 2 frogs at once, the waker and the wakee at a vertex where a frog has just been awoken.
B - frogs that have just stepped backward
Visualize the frogs in the diagram below:

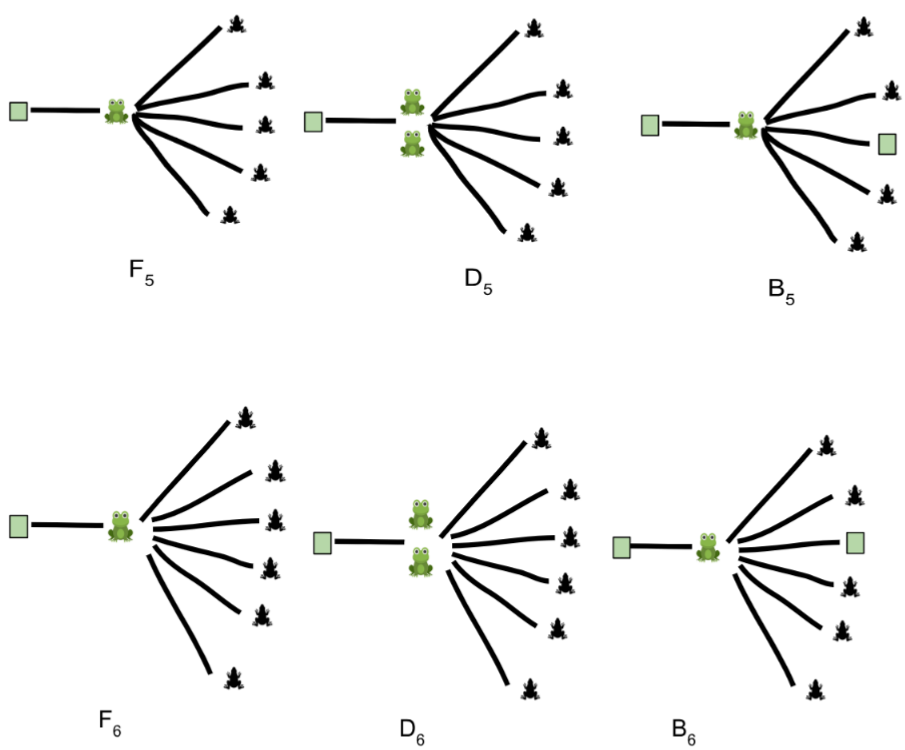

Figure 11: Frog particle types: Green frogs denote awake frogs, squares deonte an empty vertex and black frogs denote sleeping frogs.

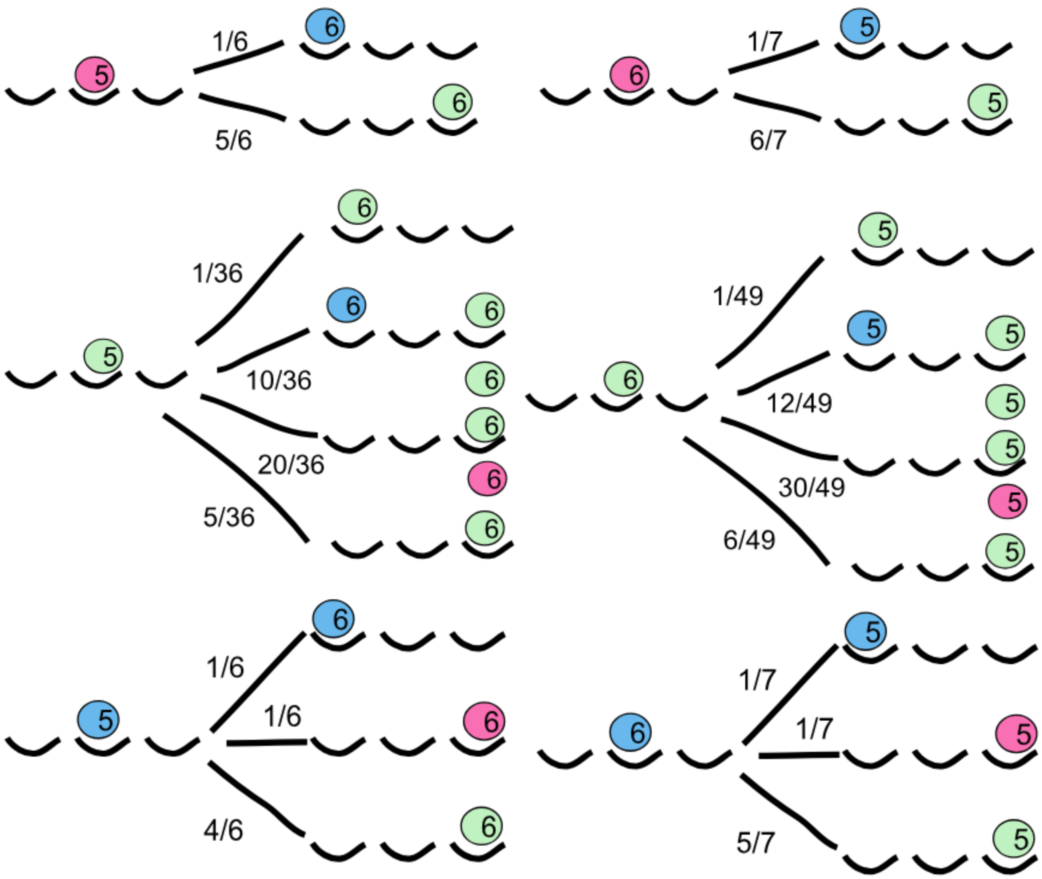

Figure 12: Distribution of children for each frog on a fresh tree: The colour represents the type of each frogs. Pink denotes a type F frog, green denotes a type D frog and blue denotes a type B frog. The numbers inside each circle represents the subscript of each frog. The probability of each potential result is shown.

Let ζ_n be the branching random walk where the particles independently reproduce following the given children distributions. If a frog jumps back, it becomes type B. When a new frog wakes up, it and its waker consolidate into a type D particle. All other particles are type F. These particles then reproduce independently on a "fresh" tree. On a "fresh" tree, all children will have sleeping frogs (except type B particles where one child will not have a sleeping frog). This ensures that the particles in the random walk always generate at least as many frogs as the actual frog model. We can couple ζ_n with $T_{5,6}^{hom}$, and prove the transience of $T_{5,6}^{hom}$ by proving transience of ζ_n.

We will use the results of Proposition 5 to show that ζ_n is transient. Let $\zeta_n = \sum_i \zeta_n^i$.
i will go through the 6 particle types that we previously defined, and ζ_n^i restricts to particles of type i. We will use the weight function w defined in Proposition 5 to set the position of the particles at time n. Define the matrix $\phi(\theta)$ by: $\phi_{ij}(\theta) = \mathbf{E}_i[w(\zeta_1^j)]$

This represents the expected weight of type j particles after 1 step given that you start with one type i particle at the origin.

Let w_n denote a row vector whose ith entry is $w_n\phi(\zeta_n^i)$. Then by martingale properties as proved earlier, $\mathbf{E}[w_{n+1}] = w_n\phi(\theta)$.

Then for any eigenvalue λ and associated eigenvector v of $\phi(\theta)$, $\mathbf{E}[w_{n+1}|z_n] = w_n\phi(\theta)v = \lambda w_n v$.

This shows that $\frac{w_n v}{\lambda^n}$ is a martingale.

Since $\phi(\theta)$ is a nonnegative and irreducible matrix, there is a positive eigenvalue $\phi(\theta)$ and the eigenvector $v(\theta)$ has strictly positive values. This results from Perron Frobenius, which we will take for granted in this paper.

Therefore the martingale $\frac{w_n v(\theta)}{\phi(\theta)^n}$ is positive. If the eigenvalue $\phi(\theta)$ ¡ 1, then by Proposition 5 the branching random walk visits 0 finitely often, and thus the frog model is a.s transient.

Therefore we must find a value of θ such that $\phi(\theta)$ ¡ 1. We will order the rows and columns $F_5, D_5, B_5, F_6, D_6, B_6$ and find the values of the matrix $\mathbf{E}[w(\zeta_i^j)]$ from Figure.

$z_n = \{z_n \text{ particles in state } F_5\} \bigcup \{z_n \text{ particles in state } F_6\} \bigcup \cdots \bigcup \{z_n \text{ particles in state } B_6\}$

$\phi_{ij}(\theta) = \mathbf{E}_i[w(\zeta_1^j)]$

We will look at the expected weight of a particle being in each state j after 1 step, given we are starting with a frog of type i at the origin (see Figure). So $\phi_{ij}(\theta)$ is the expected weight of a type j particle given we are starting with a type i particle at the origin. Notice that if a frog is at a node with 5 children, the probability it goes to a state with 5 children is 0, and vice versa with 6.

We will provide the calculation of row 1 as an example, which represents having a frog of type F_5 at the origin.

Thus $\phi_{11}(\theta) = \phi_{12}(\theta) = \phi_{13}(\theta) = 0$.
Since there is no way an F_5 particle can become a F_6 particle because it will either hop backwards (becoming a B_6 particle), or hop forwards and wake one of its children (becoming a D_6 particle. Thus $\phi_{14}(\theta) = 0$.

$\phi_{15}(\theta) = \mathbf{E}_1[w(\zeta_1^5)] = w(z_1) = \sum_{1 \in z_5} e^{-\theta P(1)} = \frac{5}{6}e^{-\theta}$

$\phi_{16}(\theta) = \mathbf{E}_1[w(\zeta_1^6)] = w(z_1) = \sum_{1 \in z_6} e^{-\theta P(1)} = \frac{1}{6}e^{-\theta}$

The other calculations follow similarly. Notice for entries $\phi_{52}(\theta)$ and $\phi_{25}(\theta)$, the value is a summation, because D_i particles are made up of two frogs that hop independently.

$\phi(\theta) =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{5}{6}e^{-\theta} & \frac{1}{6}e^{-\theta} \\ 0 & 0 & 0 & \frac{5}{36}e^{-\theta} & \frac{1}{36}e^{-\theta} + \frac{55}{36}e^{-\theta} & \frac{5}{18}e^{-\theta} \\ 0 & 0 & 0 & \frac{1}{6}e^{-\theta} & \frac{2}{3}e^{-\theta} & \frac{1}{6}e^{-\theta} \\ 0 & \frac{6}{7}e^{-\theta} & \frac{1}{7}e^{-\theta} & 0 & 0 & 0 \\ \frac{6}{49}e^{-\theta} & \frac{1}{49}e^{-\theta} + \frac{78}{49}e^{-\theta} & \frac{12}{49}e^{-\theta} & 0 & 0 & 0 \\ \frac{1}{7}e^{-\theta} & \frac{5}{7}e^{-\theta} & \frac{1}{7}e^{-\theta} & 0 & 0 & 0 \end{bmatrix}$$

Using a computer to compute the eigenvalues, we find that there indeed exists a value of θ that makes $\phi(\theta) < 1$. If we take $\theta = log(3)$ then $\phi(\theta) \approx 0.9937$. Note that the value of $log(3)$ does not hold any significance, it is simply a value that works. Further calculations outside the scope of this paper show that all eigenvalues of $\phi(log(3))$ are less than 1.

Thus, we have shown that there exists a θ such that all eigenvalues of $\phi(\theta)$ ¡ 1, which proves that ζ is transient, and therefore the frog model on $T_{5,6}^{hom}$ is transient.

To prove that T_5 is transient, a similar process is used with 27 particle types instead of the 6 used in the previous proof. This refinement creates more complex calculations that are outside the scope of this paper.

However, the structure of the proof is similar, and $\theta = log(3)$ can again be used to show that T_5 is transient.

4 Conclusion

In this paper, we explored the transience and recurrence of the frog model on d-ary trees by using the concept of stochastic dominance. The frog model was a very interesting model of self-interacting random walks to explore. It has many applications on the spread of information and disease that are hopefully further investigated in the future. The usage of branching random walks to stochastically dominate the frog models was vital in proving its recurrence and transience. While we have shown the model is transient on $d \geqslant 5$ and recurrent on $d = 2$ it remains to prove that the model is recurrent on T_3 and transient on T_4. In the whole process, we highlights phase transitions in stochastic systems via recursive distributional equations and generating functions, revealing how recursive structures simplify complexity. These results are still conjectures that require a full proof in the future.

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