Ultrametric Spaces: fixed points and examples

Directed Reading Program

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1 Introduction

It is well-know from Banach Fixed Point Theorem that for any complete metric space (X, d) with a contraction map $T : X \to X$ on X (that is, there exists $0 \leq q < 1$ such that $d(T(x), T(y)) \leq qd(x, y)$ for any $x, y \in X$), T has a unique fixed point. Naturally, this extends to ultrametric spaces (Z, d). Since ultrametric spaces are more restrictive then usual metric spaces, it could be interesting to see if some conditions of Banach Fixed Point Theorem on the mapping $T : Z \to Z$ could be dropped in order to still ensure the existence of a fixed point. The main goal of this report is to summarize a result from [1] which presents a new class of functions with fixed points in ultrametric spaces and also properties some of these fixed points have.

Definition 1.1. *Let X be a field, a valuation on X is:*

a map $|\cdot| : X \to \mathbb{R}$ such that for some real number $C \geq 1$, the following holds:

(1) $|x| \geq 0$ for any $x \in X$ with equality if and only if $x = 0$
(2) $|xy| = |x| \cdot |y|$ for any $x, y \in X$
(3) for $x \in X$: if $|x| \leq 1$, then $|x + 1| \leq C$

Definition 1.2. *A valuation $|\cdot|$ on X satisfies the ultrametric inequality iff:*

$$\text{for any } x, y \in X,\ |x + y| \leq max\{|x|, |y|\}$$

Definition 1.3. *(X, d) is an ultrametric space iff it is a metric space and satisfies the strong triangle inequality, that is:*

$$\text{for any } x, y, z \in X,\ d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

Definition 1.4. *for an ultrametric space (X, d), balls in (X, d) are:*

$$B(a, r) = \{x \in X : d(x, a) < r\} \quad \text{for } a \in X \text{ and } r \in \mathbb{R}_{>0}$$
$$\overline{B(a, r)} = \{x \in X : d(x, a) \leq r\} \quad \text{for } a \in X \text{ and } r \in \mathbb{R}_{>0}$$

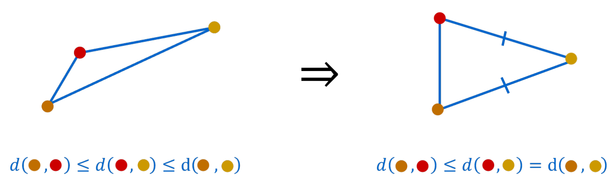

Figure 1: Result the strong inequality infers

2 Properties of balls

***For this section, we let X be a field and (X, d) an ultrametric space with:

$$d(x, y) := |x - y| \text{ for } |\cdot| \text{ a valuation}$$

Lemma 2.1. *for any $a \in X$ and for any $r > 0$:*

$$\text{(1) } B(a, r) = a + B(0, r) \text{ and (2) } \overline{B(a, r)} = a + \overline{B(0, r)}$$

proof of (1). Let $x \in B(a, r)$, then $d(a, x) < r$. Let $y = x - a$, note that $y \in B(0, r)$ as $d(0, y) = |0 - y| = |y| = |x - a| < r$. So, $x = a + y$ and thus $x \in a + B(0, r)$ and $B(a, r) \subseteq a + B(0, r)$. Now, let $x \in a + B(0, r)$, then $x = a + y$ with $y \in B(0, r) \Rightarrow |y| < r \Rightarrow |x - a| < r$. So, $x \in B(a, r)$ and $a + B(0, r) \subseteq B(a, r)$. We thus have, $B(a, r) = a + B(0, r)$ □

Lemma 2.2. *For (X, d) an ultrametric space and $x_0 \in X$:*

$$\text{if } |x_0| \geq r, \text{ then } B(x_0, |x_0|) \subseteq X \setminus B(0, r)$$

Proof. Let $z \in B(x_0, |x_0|)$, then (by the def. of a ball)

$$|z - x_0| < |x_0| \tag{1}$$

suppose for a contradiction $z \in B(0, r)$ (in particular $|z| < r \leq |x_0|$)
$\Rightarrow |z - x_0| \leq \max\{|z|, |x_0|\} = |x_0|$, we then have:

$$|x_0| = |x_0 - z + z| \leq \max\{|x_0 - z|, |z|\} \leq |x_0| \Rightarrow |x_0 - z| = |x_0|$$

which contradicts (1).
We thus get, $|z| \geq r$, and so $z \in X \setminus B(0, r) \Rightarrow B(x_0, |x_0|) \subseteq X \setminus B(0, r)$ □

Theorem 2.3. *Every ball is both open and closed*

Proof. By Lemma 2.1, we consider the open ball $B(0, r)$ (centered at 0). We show that $X \setminus B(0, r)$ is open. This will infer that $B(0, r)$ is closed.
Let $x \notin B(0, r)$, then by Lemma 2.2 we have $B(x, |x|) \subseteq X \setminus B(0, r)$.
Then, for any $x \in X \setminus B(0, r)$, x is contained in an open set contained in $X \setminus B(0, r)$, which means that $X \setminus B(0, r)$ is open, and thus $B(0, r)$ is closed.
Now, we consider the ball $\overline{B(0, r)}$.
Let x be such that $|x| \leq r$. Consider $y \in B(x, r)$ (i.e. $|y - x| < r$). Then,

$|y| = |x + (y - x)| \leq \max\{|x|, |y - x|\}$. We have two cases:
(1) $|x| < r \Rightarrow |y| < r \Rightarrow B(x, r) \subseteq B(0, r) \subseteq \overline{B(0, r)}$
(2) $|x| = r \Rightarrow |y| \leq r \Rightarrow B(x, r) \subseteq \overline{B(0, r)}$
And so each $x \in \overline{B(0, r)}$ is contained in an open set contained in $\overline{B(0, r)}$, thus $\overline{B(0, r)}$ is open. $\square$

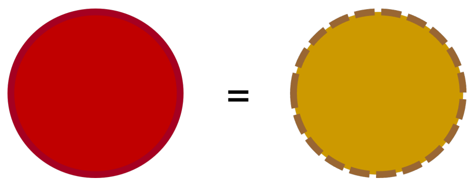

Figure 2: Theorem 2.3

Remark. *It is not necessarily the case that $B(a, r) = \overline{B(a, r)}$. Look at $X = \mathbb{Z}^+$ and $d : \mathbb{Z}^+ \times \mathbb{Z}^+ \to [0, 1]$ defined as:*

$$d(m, n) := \begin{cases} 0 & \textit{if } m = n \\ \max\left\{\frac{1}{m}, \frac{1}{n}\right\} & \textit{if } m \neq n \end{cases}$$

Then, for the ball $B(1, 1)$ in (X, d):

$$B(1, 1) = \{n \in \mathbb{Z}^+ : d(n, 1) < 1\} = \{1\} \neq \mathbb{Z}^+ = \{n \in \mathbb{Z}^+ : d(n, 1) \leq 1\} = \overline{B(1, 1)}$$

Theorem 2.4. *Let $Y \subset X$ and $B \subset X$ where B is a ball, then*

if $B \cap Y \neq \emptyset$, then it is a ball in the subspace (Y, d)

Proof. Let $B = B(b, r)$ Suppose $B \cap Y \neq \emptyset$, then consider $a \in B \cap Y$. Denote

$$\overline{B_Y(a, r)} := \{y \in Y : d(a, y) \leq r\}$$

with

$$s := \sup\{r : \overline{B_y(a, r)} \subseteq B \cap Y\}$$

We show that $B \cap Y = \overline{B_Y(a, s)}$ (and so $B \cap Y$ is a ball in (Y, d)). Note that, by definition of s, we already have $\overline{B_Y(a, s)} \subseteq B \cap Y$. Now, let $x \in B \cap Y$, but suppose $d(a, x) > s$.
claim: $\overline{B_Y(x, d(a, x))} \subseteq B \cap Y$
for $z \in \overline{B_Y(x, d(a, x))}$, $d(z, b) \leq \max\{d(z, x), d(x, a), d(a, b)\}$. But we also have:

$$d(x, a) \leq \max\{d(x, b), d(b, a)\} < r$$

since $a, b, x \in B$. And so, $d(z, b) < r \Rightarrow z \in B \cap Y$
But this contradicts the definition of s, and so we have $d(a, x) \leq s$, in particular, $x \in \overline{B_Y(a, s)}$. So

$$\overline{B_Y(a, s)} = B \cap Y$$

as desired. $\square$

Proposition 2.5. *For any ball $B(a,r)$, for any $x \in B(a,r)$,*

$$B(a,r) = B(x,r)$$

Proof. Since $x \in B(a,r)$, $|x-a| < r$.
Now look at $y \in B(x,r)$, then $|y-x| < r$. So, $|y-a| = |y-x+x-a| \leq \max\{|y-x|, |x-a|\} < r$.This gives us: $B(x,r) \subseteq B(a,r)$.
For $y \in B(a,r)$, $|y-a| < r$. So, $|y-x| = |y-a+a-x| \leq \max\{|y-a|, |x-a|\} < r$.
Thus, $B(x,r) = B(a,r)$ as desired. $\square$

Remark. *The proof of Theorem 2.4 becomes trivial by 2.5. For $B = B(a,r)$, we can assume $a \in Y$,*

$$B \cap Y = \{x \in X : d(a,x) < r\} \cap Y = \{x \in Y : d(a,x) < r\} = B_Y(a,r)$$

In particular, $B \cap Y$ is a ball in (Y,d).

Proposition 2.6. *Every two balls are either disjoint or contained one in another.*

Proof. For $B(a_1,r_1) \cap B(a_2,r_2) \neq \emptyset$. Consider $x \in B(a_1,r_1) \cap B(a_2,r_2)$. Then, by Proposition 2.5, $B(a_1,r_1) = B(x,r_1)$ and $B(a_2,r_2) = B(x,r_2)$. In the case where $r_1 \leq r_2$ we get $B(a_1,r_1) \subseteq B(a_2,r_2)$, otherwise $B(a_2,r_2) \subset B(a_1,r_1)$. $\square$

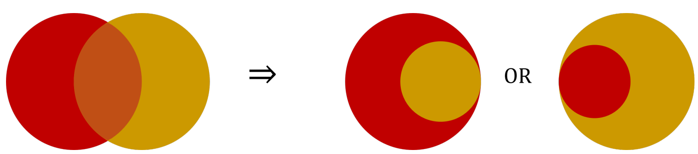

Figure 3: Proposition 2.6

3 Spherical completeness

Definition 3.1. (X,d) *is said to be spherically complete iff:*

for any decreasing sequence of nested balls $(B_n)_{n\in\mathbb{N}}$ in (X,d), $\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset$

Theorem 3.1. *Let $X \neq \emptyset$ be spherically complete, and let (Z,d) be an ultrametric space such that $X \subset Z$, then*

for any $z \in Z$,there exists $x_0 \in X$ such that
$$d(z,x_0) = d(z,X) = \inf\{d(z,x) : x \in X\}$$

Proof. Fix $z_0 \in Z$, $d(z_0,X) = inf\{d(z_0,x) : x \in X\}$ does exist. Therefore, there exists a decreasing sequence $(r_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\lim_{n\to\infty} r_n = d(z_0,X)$.
Let $B_n = B(z_0,r_n) = \{z \in Z : |z_0 - z| < r_n\}$. Then, $B_i \supset B_{i+1}$ for all $i \in \mathbb{N}$.
So, $B_1 \cap X \supset B_2 \cap X \supset ...$ is a sequence of nested balls in X (by Theorem 2.4).

Since X is spherically complete, we have that there is some $a \in \cap_{n\in\mathbb{N}} B_n \cap X$.
$\Rightarrow |z_0 - a| < r_n$ for any $n \in \mathbb{N}$ and in particular, $|z_0 - a| = d(z_0, X)$.
Since $a \in X$, we indeed have that there exists some $a \in X$ such that $d(z_0, a) = d(z_0, X) = inf\{d(z_0, x) : x \in X\}$. $\square$

Proposition 3.2. *A sequence $(x_n)_{n\in\mathbb{N}}$ in ultrametric space (X, d) is Cauchy iff*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

Example $(\star)$**.** *Define $d : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ as:*

$$d(n,m) = \begin{cases} 0 & \text{if } n = m \\ \max\left\{1 + \frac{1}{n}, 1 + \frac{1}{m}\right\} & \text{if } n \neq m \end{cases}$$

Then, $(\mathbb{Z}^+, d)$ is an ultrametric space which is complete but not spherically complete:
(i) $d(n, m) \geq 0$ with equality iff $n = m$, and $d(n, m) = d(m, n)$ are easy, we show the strong triangle inequality.
Let $k, m, n \in \mathbb{Z}^+$, then

$$\begin{aligned} \max\{d(k,m), d(m,n)\} &= \max\left\{\max\left\{1+\frac{1}{k}, 1+\frac{1}{m}\right\}, \max\left\{1+\frac{1}{m}, 1+\frac{1}{n}\right\}\right\} \\ &= \max\left\{1+\frac{1}{k}, 1+\frac{1}{m}, 1+\frac{1}{n}\right\} \\ &\geq \max\left\{1+\frac{1}{k}, 1+\frac{1}{n}\right\} \\ &= d(k,n) \end{aligned}$$

as desired. We thus have $(\mathbb{Z}^+, d)$ is indeed an ultrametric space.
(ii) We want to show that for $(x_n)_{n\in\mathbb{N}}$ a Cauchy sequence, its limit exists and is in $\mathbb{Z}^+$. By Proposition 3.2, consider a sequence such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. So there exists $N \in \mathbb{N}$ such that for any $n > N$, $x_n = x_{n+1} = k \in \mathbb{Z}^+$. In particular, $\lim_{n\to\infty} x_n = k \in \mathbb{Z}^+$. So $(\mathbb{Z}^+, d)$ is indeed complete.
(iii) To show that $(\mathbb{Z}^+, d)$ is not spherically complete, we find a sequence of nested balls $(B_n)_{n\in\mathbb{N}}$ such that $\cap_{n\in\mathbb{N}} B_n = \emptyset$.
Let $B_n := \overline{B(n, 1 + \frac{1}{n})}$, then

$$\begin{aligned} B_n &= \left\{m \in \mathbb{N} : d(m,n) \leq 1 + \frac{1}{n}\right\} \\ &= \left\{m \in \mathbb{N} : \max\left\{1+\frac{1}{m}, 1+\frac{1}{n}\right\} \leq 1 + \frac{1}{n}\right\} \\ &= \left\{m \in \mathbb{N} : 1 + \frac{1}{m} \leq 1 + \frac{1}{n}\right\} \\ &= \{m \in \mathbb{N} : n \leq m\} \end{aligned}$$

Note that indeed, $B_n \supset B_{n+1}$ for all $n \in \mathbb{N}$. But then if there exists $n_0 \in \cap_{n\in\mathbb{N}} B_n$, for all $n \in \mathbb{N}$ we have $n_0 \geq n$, a contradiction. So

$$\cap_{n\in\mathbb{N}} B_n = \emptyset$$

Thus, $(\mathbb{Z}^+, d)$ is not spherically complete

Definition 3.2. *Let (X, d) be an ultrametric space. A subset $A \subset X$ is said to be proximinal if given any $x \in X$, there exists $a_0 \in A$ such that:*

$$d(x, a_0) = d(x, A) = \inf\{d(x, a) : a \in A\}$$

Such an $a_0 \in A$ is called a best approximation to x in A.

Remark. *Theorem 3.1 can be reformulated as:*
Let $X \neq \emptyset$ be spherically complete and (Z, d) be an ultrametric space containing X, then, X is proximinal in Z.

Definition 3.3. *For $A, B \subset X$, we define the distance between A and B:*

$$d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}$$

The diameter of A is : $\delta(A) := \sup\{d(a_1, a_2) : a_1, a_2 \in A\}$

Remark. *For any ball $B = B(a, r)$ in (X, d), $\delta(B) \leq r$.*

Proof. Let $x, y \in B$, then

$$d(x, y) \leq \max\{d(a, x), d(a, y)\} \leq r \Rightarrow \delta(B) \leq r$$

□

The following two sets will be very useful in results to come.

Definition 3.4. *For (X, d) a metric space and $A, B \subset X$ nonempty, we define:*

$$A_0 = \{x \in A\text{: there is some } y \in B \text{ with } d(x, y) = d(A, B)\}$$
$$B_0 = \{y \in B\text{: there is some } x \in A \text{ with } d(x, y) = d(A, B)\}$$

$(x, y) \in A_0 \times B_0$ is called a best proximity pair for A and B if $d(x, y) = d(A, B)$
We say (A, B) is proximinal if A and B are proximinal

Definition 3.5. *Let (X, d) be an ultrametric space. For a map $f : X \to X$ and $B = B(x, r), r > 0$ a ball in (X, d). We say that B is a minimal f-invariant ball iff:*

$$\begin{aligned} &(1) \quad f(B) \subseteq B \text{ and} \\ &(2) \quad \text{for any } b \in B,\ d(b, f(b)) = r \end{aligned}$$

Remark. *$\{x_0\}$ for $x_0 \in X$ is not minimal f-invariant as, by (1), we would have $f(\{x_0\}) = \{x_0\}$, and by (2) $d(x_0, f(x_0)) = d(x_0, x_0) = 0 = r$ which contradicts the assumption that $r > 0$.*

Definition 3.6. *Let $A \subset X$, $f : A \to A$ is said to be nonexpansive iff:*

$$d(f(x), f(y)) \leq d(x, y) \text{ for all } x, y \in A$$

Proposition 3.3. *For (X, d) spherically complete and $f : X \to X$ nonexpansive, let $\mathbf{B_T}$ be the set of all balls B that are f-invariant, that is $f(B) \subseteq B$, then:*

$B \in \mathbf{B_T}$ is minimal f-invariant iff for every $B_0 \in \mathbf{B_T}$, $B_0 \subseteq B$ implies $B_0 = B$

Proof. We only show ($\Rightarrow$), the proof of the other implication is very similar to that of Theorem 3.4 using Proposition 3.5.
Let $B \in \mathbf{B_T}$ be minimal f-invariant and B_0 be f-invariant with $B_0 \subseteq B$. Consider $x \in B_0$, then $B_0 = B(x, r)$ and $B = B(x, d(x, f(x)))$, with $r \leq d(x, f(x))$. Suppose $r < d(x, f(x))$, then $f(x) \notin B_0 \Rightarrow f(B_0) \nsubseteq B_0$, a contradiction. We thus have $r = d(x, f(x))$, and so $B = B_0$. □

Theorem 3.4. *(from [3]) Suppose (X, d) is spherically complete and $f : X \to X$ is a nonexpansive map, then:*

$$\text{every ball } \overline{B(x, d(x, f(x)))} = \{y \in X : d(x, y) \leq d(x, f(x))\}$$

contains either a fixed point of f or a minimal f-invariant ball.

Proof. We denote for any $a \in X$,

$$B_a = \overline{B(a, d(a, f(a)))}$$

Consider the collection of all such balls

$$A = \{B_a : a \in X\}$$

and define the partial order, for $a, b \in X$

$$B_a \leq B_b \text{ iff } B_b \subseteq B_a$$

Fix a chain C of balls in A, which corresponds to a decreasing sequence of nested balls, denote

$$B := \bigcap_{B_a \in C} B_a \neq \emptyset$$

(such B exists by spherical completeness). Let $b \in B$, then for some $a \in X$, $b \in B_a$ (and so $d(a, b) \leq d(a, f(a))$). If $x \in B_b$, then

$$d(b, x) \leq d(b, f(b)) \leq \max\{d(b, a), d(a, f(a)), d(f(a), f(b))\} \leq d(a, f(a))$$

(since $d(f(a), f(b)) \leq d(a, f(a))$, because f is nonexpansive), and so

$$d(x, a) \leq \max\{d(x, b), d(b, a)\} \leq d(a, f(a)) \Rightarrow x \in B_a$$

In particular, for any $B_a \in C$, $B_b \subseteq B_a$, so B_b is a maximal element of C. By Zorn's Lemma, A has a maximal element (w.r.t $\leq$), call it B_z for $z \in X$ (i.e. for all $a \in X$, $B_z \subseteq B_a$). We now show that for any $b \in B_z$, $d(b, f(b)) = d(z, f(z))$ and that $f(B_z) \subseteq B_z$. Consider $b \in B_z$,

$$d(b, f(b)) \leq \max\{d(b, z), d(z, f(z)), d(f(z), f(b))\} \leq d(z, f(z))$$

Then $f(b) \in B_z$, which means that $f(B_z) \subseteq B_z$. Suppose, for a contradiction, that $d(b, f(b)) < d(z, f(z))$. By the maximality of B_z (w.r.t. $\leq$), $z \in B_b$ and so $d(z, b) \leq d(b, f(b))$. We have

$$d(b, f(b)) < d(z, f(z)) \leq \max\{d(z, b), d(b, f(b)), d(f(b), f(z))\} \leq d(b, f(b))$$

a contradiction. We conclude that it must be that for all $b \in B_z$, $d(b, f(b)) = d(z, f(z))$. □

Definition 3.7. *For (X, d) an ultrametric space and a map $f : X \to X$,*
(1) f is strictly contractive iff

$$d(f(x), f(y)) < d(x, y) \text{ for } x \neq y$$

(2) f is strictly contractive on orbit iff

$$f(x) \neq x \text{ implies } d(f^2(x), f(x)) < d(f(x), x) \text{ for any } x \in X$$

Proposition 3.5. *For (X, d) an ultrametric space, and a ball $\overline{B(a, r)}$ in (X, d),*

$$\text{if } B(a, r) \text{ is } f\text{-invariant, then } \overline{B(a, d(a, f(a)))} \subseteq \overline{B(a, r)}$$

Proof. Since $f(a) \in \overline{B(a, r)}$, $d(a, f(a)) \leq r$. By Proposition 2.6, $\overline{B(a, d(a, f(a)))} \subseteq \overline{B(a, r)}$. □

The following Theorem will ensure the existence of a fixed point for a mapping f in an ultrametric space without requiring it to be a contraction. Note that the fixed point is not necessarily unique.

Theorem 3.6. *(from [1]) Let X be spherically complete with $f : X \to X$ a nonexpansive map satisfying:*

$$\text{if } x \in X \text{ and } x \neq f(x), \text{ then } \liminf_{n \to \infty} d(f^n(x), f^{n+1}(x)) < d(x, f(x))$$

Then f has a fixed point in any f-invariant closed ball

Proof. Consider B an f-invariant ball such that for any $x \in B$, $x \neq f(x)$. By Proposition 3.5, for $a \in B$, $\overline{B(a, d(f(a), a))} \subseteq B$, and so by Theorem 3.4, B contains a minimal f-invariant ball B_0 with radius $r > 0$, with $0 < r = d(x, f(x))$ (by definition of minimal f-invariant ball) and since $f(x) \in B_0$, we actually have:

$$0 < r = d(x, f(x)) = d(f(x), f^2(x)) = ... = \liminf_{n \to \infty} d(f^n(x), f^{n+1}(x)) < d(x, f(x))$$

in particular, $d(x, f(x)) < d(x, f(x))$, which is a contradiction. So it must be that B actually had a fixed point. □

Remark. *Theorem 3.6 implies that any $f : X \to X$ which is strictly contractive on orbit, with X spherically complete, will admit a fixed point.*

4 Other results

The following results are from [1] and will be useful in order to characterize certain fixed points.

Lemma 4.1. *Let (A, B) be proximinal w.r.t. (X, d) nonempty (i.e. for any $x \in X$ there are $a_0 \in A$ and $b_0 \in B$ such that $d(x, a_0) = d(x, A)$ and $d(x, b_0) = d(x, B)$):*

$$\text{if } \delta(B) \leq d(A, B), \text{ then } A_0 \neq \emptyset \text{ and } B_0 = B$$

Proof. We first show that for any $a \in A$ and $b_1, b_2 \in B$

$$d(a, b_1) = d(a, b_2) \tag{2}$$

$$\begin{aligned} d(a, b_2) &\leq \max\{d(a, b_1), d(b_1, b_2)\} \text{ (by strong triangle inequality)} \\ &\leq \max\{d(a, b_1), \delta(B)\} \text{ (by def. of } \delta \text{)} \\ &\leq \max\{d(a, b_1), d(A, B)\} \text{ (by assumption)} \\ &\leq d(a, b_1) \text{ (by def. of } d(A, B)) \end{aligned}$$

$d(a, b_1) \leq d(a, b_2)$ is proved as above by interchanging b_1 and b_2.
Now, let $b_0 \in B$, then we have:

$$d(A, B) = \inf_{a \in A, b \in B} d(a, b) = \inf_{a \in A} d(a, b_0) = d(A, b_0)$$

Since A is proximinal, there is $a_0 \in A$ such that $d(A, b_0) = d(a_0, b_0)$. In particular by the above equation, $d(A, B) = d(a_0, b_0)$.
That is for any $b_0 \in B$, there is $a_0 \in A$ such that $d(a_0, b_0) = d(A, B)$. And so, $a_0 \in A_0$ and $b_0 \in B_0$, in particular $A_0 \neq \emptyset$ and $B_0 = B$ $\square$

Remark. *In the previous theorem, if we had $\delta(B) > d(A, B)$ instead, then A_0 and B_0 could have both been empty. We show this by the following example:*
Let $X = \mathbb{Z}^+$ and $d : \mathbb{Z}^+ \times \mathbb{Z}^+ \to [0, 1]$ be defined as:

$$d(n, m) = \begin{cases} 0 & \text{if } n = m \\ \max\left\{\frac{1}{n}, \frac{1}{m}\right\} & \text{otherwise} \end{cases}$$

(X, d) is indeed an ultrametric space. Consider $A = 2\mathbb{Z}^+$ and $B = 2\mathbb{Z}^+ - 1$.
note that both A and B are proximinal:
consider $x \in X$, if $x \in A$, then there exists $a \in A$ (x itself) such that $d(x, A) = \inf\{d(x, z) : z \in A\} = 0$, otherwise $x \in B$, so $d(x, A) = \inf\{d(x, z) : z \in A\} = \frac{1}{x}$ which is attained by any $z > x, z \in A$. Similarly we prove B is proximinal.
We also have $\delta(B) = 1 > \delta(A) = \frac{1}{2} > 0 = d(A, B)$.
We can now observe that $A_0 = B_0 = \emptyset$.

Lemma 4.2. *Let (A, B) be proximinal w.r.t. (X, d) nonempty satisfying $\delta(B) \leq d(A, B)$. Let $b_0 \in B$ and write $r := d(A, B)$, then we have:*

$$A_0 = A \cap S(b_0, r)$$

where $S(b_0, r) := \{x \in X : d(x, b_0) = r\}$ (i.e. the sphere centered at b_0 with radius r).

Proof. case 1: $r = 0$
We have $\delta(B) = 0$, and so $B = \{b_0\}$. By definition,

$$A_0 = \{a \in A : d(a, b_0) = 0\}$$

Note that by Lemma 4.1, $A_0 \neq \emptyset \Rightarrow A_0 = \{b_0\}$.
Also, $S(b_0, 0) = \{b_0\}$ and so $A \cap S(b_0, r) = \{b_0\} = A_0$ as desired.
case 2: $r > 0$
Consider the following sets

$$C(b_0, r) := \{x \in X : d(x, b_0) < r\}$$

$$E(b_0, r) := \{x \in X : d(x, b_0) > r\}$$

Then, $X = C(b_0, r) \cup S(b_0, r) \cup E(b_0, r)$ and, so

$$A_0 = [A_0 \cap C(b_0, r)] \cup [A_0 \cap S(b_0, r)] \cup [A_0 \cap E(b_0, r)]$$

We thus show $A_0 \cap C(b_0, r) = \emptyset = A_0 \cap E(b_0, r)$.
First note that for $a \in A$,

$$d(a, b_0) \geq d(a, B) \geq d(A, B) = r$$

and thus for all $a \in A$, $d(a, b_0) \not< r$, in particular, $A_0 \cap C(b_0, r) = \emptyset$
Now, suppose $A_0 \cap E(b_0, r) \neq \emptyset$, then there is $a_0 \in A_0$ such that

$$d(a_0, b_0) > r = d(A, B)$$

By Lemma 4.1 (2), we have that for any $b \in B$, $d(a_0, b) = d(a_0, b_0)$.
But recall that as $a_0 \in A_0$, by definition, there is $b_1 \in B$ such that $d(a_0, b_1) = d(A, B)$. Which implies:

$$d(A, B) = d(a_0, b_1) = d(a_0, b_0) > d(A, B)$$

a contradiction. □

Lemma 4.3. *Let $B = B(a, r)$ be a ball in (X, d), then for any $x \in X \setminus B$ and for any $b_1, b_2 \in B$:*

$$d(x, b_1) = d(x, b_2)$$

Proof. Note that $d(x,b) \geq r$ for any $x \in X \setminus B$ and $b \in B$, and $d(b_1,b_2) < r$ (so $d(b_1,b_2) < d(x,b)$ for any $b \in B$). Then

$$\begin{aligned} d(x,b_1) &\leq \max\{d(x,b_2), d(b_1,b_2)\} \\ &= d(x,b_2) \\ d(x,b_2) &\leq \max\{d(x,b_1), d(b_1,b_2)\} \\ &= d(x,b_1) \\ \Rightarrow d(x,b_1) &= d(x,b_2) \end{aligned}$$

$\square$

Theorem 4.4. *Let (A,B) be proximinal w.r.t. (X,d) nonempty. The following are equivalent:*
(1) there is $a_0 \in A$ such that for every $b \in B$,

$$d(a_0,b) = d(A,B)$$

(2) $\delta(B) \leq d(A,B)$
(3) The following hold:

$$\begin{array}{cl} (i) & A_0 \text{ and } B_0 \text{ are proximinal w.r.t. } (X,d) \\ (ii) & B = B_0 \\ (iii) & \text{for any } (a_0,b_0) \in A_0 \times B_0,\ d(a_0,b_0) = d(A_0,B_0) = d(A,B) \end{array}$$

Proof. (3)$\Rightarrow$(1) : let $b \in B$, then $b \in B_0$ by assumption. Take any $a_0 \in A_0$, then $d(a_0,b) = d(A,B)$, as desired.
(1)$\Rightarrow$(2) : fix $a_0 \in A$ such that for any $b \in B$, $d(a_0,b) = d(A,B)$. Consider $b_1,b_2 \in B$, then

$$d(b_1,b_2) \leq \max\{d(b_1,a), d(a,b_2)\} = d(A,B)$$

and so $\delta(B) \leq d(A,B)$
(2)$\Rightarrow$(3) : assume $\delta(B) \leq d(A,B)$, by Lemma 4.1 $B = B_0$ and since B is proximinal, so is B_0. We show that A_0 is also proximinal (i.e. for all $x \in X$, there exists $a_0 \in A_0$ such that $d(x,a_0) = d(x,A_0)$). Let $b_0 \in B_0 = B$, by Lemma 4.2,

$$A_0 = A \cap S(b_0,r) \text{ where } r = d(A,B)$$

but, from the proof we also had $A \cap B(b_0,r) = \emptyset$, so

$$A_0 = A \cap \overline{B(b_0,r)}$$

Now, let $x \in X$
Case 1: $x \notin \overline{B(b_0,r)}$
$d(x,b_0) > r$ and by Lemma 4.3, for any $a_0 \in A_0$, $d(x,b_0) = d(x,a_0)$. And so, $d(x,a_0) = d(x,A_0)$
Case 2: $x \in \overline{B(b_0,r)}$

Since A is proximinal, there exists $a \in A$ such that $d(x,a) = d(x,A)$. Note that if $a \notin \overline{B(b_0,r)}$, $d(x,a) > r$, while for all $a_0 \in A_0$ (since $a_0 \in \overline{B(b_0),r)}$, $d(x,a_0) \leq r$. We get

$$r < d(x,a) = d(x,A) \leq d(x,A_0) \leq d(x,a_0) \leq r$$

a contradiction. So, it must be that $a \in \overline{B(b_0,r)} \Rightarrow a \in A \cap \overline{B(b_0,r)} = A_0$
A_0 is proximinal. Finally, by Lemma 4.2, note that for any $(a_0,b_0) \in A_0 \times B_0$,

$$d(a_0,b_0) = r = d(A,B)$$

□

Corollary 4.4.1. *For (A,B) a nonempty spherically complete pair satisfying $\delta(B) \leq d(A,B)$, then (A_0,B_0) is also a nonempty spherically complete pair.*

Definition 4.1. *A map $f : A \cup B \to A \cup B$ is noncyclic iff:*

$$f(A) \subseteq A \text{ and } f(B) \subseteq B$$

Theorem 4.5. *Let A and B be nonempty spherically complete subspaces of an ultrametric space (X,d) with $\delta(B) \leq d(A,B)$. Suppose $f : A \cup B \to A \cup B$ is a noncyclic and a nonexpansive mapping such that for any $x \in X$ with $x \neq f(x)$:*

$$\liminf_{n\to\infty} d(f^n(x), f^{n+1}(x)) < d(x, f(x)) \tag{3}$$

Then, there exists $a \in A$ and $b \in B$ such that:

$$f(a) = a,\ f(b) = b \text{ and } d(a,b) = d(A,B)$$

Proof. By Corollary 4.4.1, we have that A_0 and B_0 are spherically complete.
claim: $f(A_0) \subseteq A_0$ and $f(B_0) \subseteq B_0$
Let $x \in A_0$ and $y \in B_0$, then by Theorem 4.4 (2) $\Rightarrow$ (3), we get :

$$d(x,y) = d(A_0,B_0) = d(A,B)$$

$$\Rightarrow d(f(x),f(y)) \leq d(A,B) \text{ (since f is nonexpansive)}$$

But, we also had f noncyclic (i.e. $f(A) \subseteq A$ and $f(B) \subseteq B$), so
$\Rightarrow f(x) \in A$ and $f(y) \in B \Rightarrow d(A,B) \leq d(f(x),f(y))$, which gives us:

$$d(f(x),f(y)) = d(A,B)$$

and so $f(x) \in A_0$ and $f(y) \in B_0$ which implies $f(A_0) \subseteq A_0$ and $f(B_0) \subseteq B_0$ which proves the claim.
Recall that for $a \in A$ and $b \in B$ with $r = d(A,B)$,

$$A_0 = A \cap \overline{B(b,r)} \text{ and } B_0 = B \cap \overline{B(a,r)}$$

By Theorem 2.4, A_0 and B_0 are closed balls in (A,d) and (B,d) respectively. So A_0 and B_0 are f-invariant closed balls in (A,d) and (B,d) respectively. Also, since f is noncyclic, $f : A \to A$ and $f : B \to B$ (all conditions for Theorem 3.6 are satisfied). By Theorem 3.6 there are $a \in A_0$ and $b \in B_0$ such that:

$$f(a) = a \text{ and } f(b) = b$$

and again, by Theorem 4.4, $d(a,b) = d(A,B)$ as desired. □

5 Examples

5.1 σ - ultrametric

Let (X, d) be a complete ultrametric space, then (X, σ) with

$$\sigma(x, y) := \inf\{2^{-n} : n \in \mathbb{Z}, d(x, y) \leq 2^{-n}\}$$

is spherically complete.

The following example motivates the condition of (X, d) being complete.

Example. *Let $X = \mathbb{Z}^+$ and $d : \mathbb{Z}^+ \times \mathbb{Z}^+ \to [0, 1]$ be defined as:*

$$d(n, m) = \begin{cases} 0 & \text{if } n = m \\ \max\left\{\frac{1}{n}, \frac{1}{m}\right\} & \text{otherwise} \end{cases}$$

Note that d is not complete, as for the sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n = n$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} \frac{1}{x_n} = 0$$

but $\lim_{n\to\infty} n$ does not exist, in particular is not in $\mathbb{Z}^+$.
We denote

$$n_x := \sup\left\{n \in \mathbb{Z} : \frac{1}{x} \leq 2^{-n}\right\}$$

Then, it can be shown that balls in the ultrametric space (X, σ) are:

$$B(a, r) = \begin{cases} \{x \in X, 2^{-n_x} < r\} & \text{if } 2^{-n_a} < r \\ \{a\} & \text{if } r \leq 2^{-n_a} \end{cases}$$

or

$$\overline{B(a, r)} = \begin{cases} \{x \in X, 2^{-n_x} \leq r\} & \text{if } 2^{-n_a} \leq r \\ \{a\} & \text{if } r < 2^{-n_a} \end{cases}$$

Consider the sequence $B_n = \overline{B(2^n, \frac{1}{2^n})}$, then

$$\bigcap_{n\in\mathbb{N}} B_n = \bigcap_{n\in\mathbb{N}} \{x \in X, x \geq 2^n\} = \emptyset$$

So (X, σ) is not spherically complete.

We now consider another metric d which is complete in order to apply Theorem 4.5.

Example. *Consider the ultrametric space (X, d) presented in Example $(\star)$ with $X = \mathbb{Z}^+$ and $d : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ as:*

$$d(n, m) = \begin{cases} 0 & \text{if } n = m \\ \max\left\{1 + \frac{1}{n}, 1 + \frac{1}{m}\right\} & \text{if } n \neq m \end{cases}$$

Recall that this ultrametric space is not spherically complete, but only complete. We thus consider (X, σ) as defined above. Note that

$$\forall m, n \in \mathbb{Z}^+, m \neq n \Rightarrow 1 < d(m,n) \leq 2$$

and so whenever $m \neq n$, $\sigma(m,n) = 2$, we thus get that balls in (X, σ) are of the form:

$$B(a,r) = \begin{cases} \mathbb{Z}^+ & \text{if } r > 2 \\ \{a\} & \text{if } r \leq 2 \end{cases}$$

Remark. *Here, for $r = 2$,*

$$B(a,r) = \{a\} \neq \mathbb{Z}^+ = \overline{B(a,r)}$$

Note that for any sequence of nested balls, $(B_n)_{n\in\mathbb{N}}$, where $B_n = B(a_n, r_n)$: if there exists $N \in \mathbb{N}$ such that $r_N < 2$, then $\cap_{n\in\mathbb{N}} B_n = \{a_N\}$, otherwise, $\cap_{n\in\mathbb{N}} B_n = \mathbb{Z}^+$.
In particular, (X, σ) is spherically complete. Theorem 4.5 can now be applied. Since $(\mathbb{Z}^+, \sigma)$ is spherically complete, it follows that any subspaces A and B are also spherically complete. The Theorem is not applicable only in the case where $A \cap B \neq \emptyset$ and $A \neq \{a\}$ and $B \neq \{b\}$ for any $a, b \in \mathbb{Z}^+$.

5.2 Levi-Civita field

Define:

$$\mathcal{R} := \{f : \mathbb{Q} \to \mathbb{R} : \mathit{supp}(f) \text{ is left-finite}\}$$

where, $\mathit{supp}(f) := \{q \in \mathbb{Q} : f(q) \neq 0\}$ and $\mathit{supp}(f)$ is said to be left-finite iff $\mathit{supp}(f) \cap]-\infty, q]$ is finite for all $q \in \mathbb{Q}$.
We also define the function $\lambda : \mathcal{R} \to \mathbb{Q} \cup \{\infty\}$:

$$\lambda(x) := \begin{cases} \min \mathit{supp}(x) & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

Note that $\min \mathit{supp}(x)$ exists as $\mathit{supp}(x)$ is left-finite. Then, we define an ultrametric valuation $|\cdot| : \mathcal{R} \to \mathbb{R}^+$:

$$|x| := e^{-\lambda(x)}$$

which induces an ultrametric $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}^+$:

$$d(x,y) := |x - y|$$

This ultrametric is complete, we check if $(\mathcal{R}, d)$ is spherically complete.

Proposition 5.1. *$(\mathcal{R}, d)$ is not spherically complete*

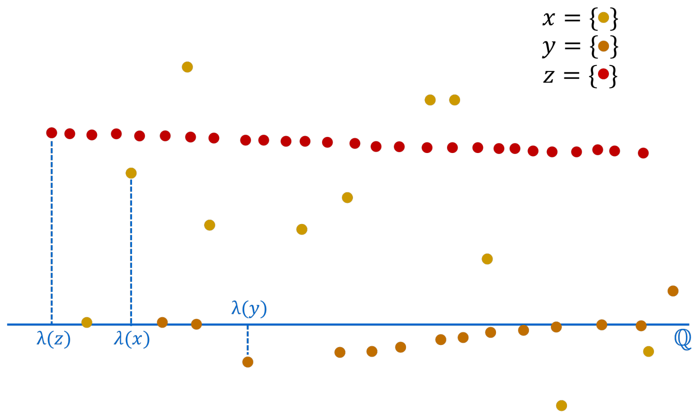

Figure 4: examples for the function λ

Proof. Let $B(a_n, r_n) = (B_n)_{n\in\mathbb{N}}$ be a decreasing sequence of nested balls in $(\mathcal{R}, d)$. So,

$$\begin{aligned} B(a_n, r_n) &= \{x \in \mathcal{R} : |x - a_n| < r_n\} \\ &= \{x \in \mathcal{R} : e^{-\lambda(x-a_n)} < r_n\} \\ &= \{x \in \mathcal{R} : -\ln r_n < \lambda(x - a_n)\} \end{aligned}$$

Then, consider r_n with $\lim_{n\to\infty} r_n = 0$. So $\lim_{n\to\infty} -\ln r_n = \infty$, but for any $a \in \mathcal{R}$ there is no $x \in \mathcal{R}$ such that $\infty < \lambda(x - a)$. In particular:

$$\bigcap_{n\in\mathbb{N}} B(a, r_n) = \emptyset$$

We thus have that $(\mathcal{R}, d)$ is not spherically complete. $\square$

Example. *Take $B(a_n, r_n)$ with $r_n = e^{-n}$ and $a_n \in \mathcal{R}$ defined, for $x \in \mathbb{Q}$, as:*

$$a_n[x] := \begin{cases} 1 & \text{if } x \in [1, n] \cap \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases}$$

Note that for fixed $n \in \mathbb{Z}^+$, $x \in B(a_n, r_n) \Rightarrow d(x, a_n) = e^{-\lambda(x-a_n)} < e^{-n} \Rightarrow n < \lambda(x-a_n)$. Now, suppose $x \in \bigcap_{n\in\mathbb{N}} B(a_n, e^{-n})$, then it must be that for all $n \in \mathbb{N}$, $n < \lambda(x - a_n) \in \mathbb{Q}$ but no such $\lambda(x - a_n) \in \mathbb{Q}$ exist, so $\bigcap_{n\in\mathbb{N}} B(a_n, e^{-n}) = \emptyset$

Remark. *If we had, instead, considered r_n with $\lim_{n\to\infty} > 0$, then*

$$\bigcap_{n\in\mathbb{N}} B(a_n, r_n) \neq \emptyset$$

To see this, let $(B(a_n, r_n)) = (B_n)_{n\in\mathbb{N}}$ be a decreasing sequence of nested balls in $(\mathcal{R}, d)$. Suppose $(r_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence for the radius. This can be done since:

$$\begin{aligned} B(a_n, r_n) &= B(a_{n+1}, r_n)\ ,\ \textit{because } a_{n+1} \in B(a_n, r_n) \\ &= B(a_{n+1}, r_{n+1})\ ,\ \textit{assuming } r_n = r_{n+1} \\ \Rightarrow B(a_n, r_n) &= B(a_{n+1}, r_{n+1}) \end{aligned}$$

$$B(a_n, r_n) = \{x \in \mathcal{R} : -\ln r_n < \min\{q \in \mathbb{Q} : x[q] \neq a_n[q]\} \textit{ or } x = a_n\}$$

In particular, $x \in B(a_n, r_n)$ if x is such that :

$$\forall q \in \left]-\infty, -\ln(r_n)\right]\ ,\ x[q] = a_n[q]$$

We thus get that for any decreasing sequence of nested balls $(B(a_n, r_n))_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} r_n > 0$*:*

For $a \in \mathcal{R}$ such that for any $q \in \mathbb{Q}$,

$$a[q] := \begin{cases} a_0[q] & \textit{if } q \leq -\ln r_0 \\ a_n[q] & \textit{if } -\ln r_{n-1} < q \leq -\ln r_n \\ 0 & \textit{otherwise} \end{cases}$$

$a \in \bigcap_{n\in\mathbb{N}} B(a_n, r_n)$.

Since $(\mathcal{R}, d)$ is not spherically complete, we consider instead (X, σ), where σ is from the previous example. We apply Theorem 4.5
Consider the subspaces:

$$A = \{x \in \mathcal{R} : \lambda(x) = 0\}$$
$$B = \{x \in \mathcal{R} : \lambda(x) > 0\}$$

Note that for any $x, y \in B$, $\lambda(x - y) > 0 \Rightarrow -\lambda(x - y) < 0 \Rightarrow e^{-\lambda(x-y)} < 1 \Rightarrow d(x, y) < 1 \Rightarrow \sigma(x - y) \leq 1 \Rightarrow \delta(B) \leq 1$.
Now, consider $a \in A$ and $b \in B$, then $\lambda(a - b) = 0 \Rightarrow d(a, b) = 1 \Rightarrow \sigma(a, b) = 1 \Rightarrow \sigma(A, B) = 1$. In particular, we have $\delta(B) \leq \sigma(A, B)$. Now, we need to find $f : A \cup B \to A \cup B$ which is noncyclic, nonexpansive and satisfies (3). Let f be the following truncation:

$$f(x)[y] = \begin{cases} x[y] & \text{if } y \leq g(x) := a\lambda(x) + b \\ 0 & \text{otherwise} \end{cases} \quad \text{where } y \in \mathbb{Q} \text{ and } a, b \in \mathbb{R}$$

Note that for any $x \in A$, we must have $\lambda(f(x)) = 0$ in order for f to be noncyclic and so $g(x) \geq 0$ for any $x \in A \cup B$. We will also take $g(x) \geq \lambda(x)$ for any $x \in \mathcal{R}$ so that

$$\lambda(x) = \lambda(f(x)) \text{ for any } x \in \mathcal{R}$$

Remark. *In order to satisfy the condition that $g(x) \geq \lambda(x) \geq 0$, we must have:*

$$g(x) = a\lambda(x) + b \textit{ where } a \geq 1, b \geq 0$$

So, $f(A) \subseteq A$ and $f(B) \subseteq B$ (i.e. f is noncyclic).
We now check that f is nonexpansive.

Proposition 5.2. *Let $x, y \in A \cup B$, then*

$$\lambda(f(x) - f(y)) \geq \lambda(x - y)$$

Proof. We have

$$\begin{aligned} \lambda(x-y) &= \min\{q \in \mathbb{Q} : x[q] \neq y[q]\} \\ \lambda(f(x) - f(y)) &= \min\{q \in \mathbb{Q} : f(x)[q] \neq f(y)[q]\} \end{aligned}$$

Suppose $\lambda(f(x) - f(y)) < \lambda(x - y)$, then there exists $q \in \mathbb{Q}$ such that $f(x)[q] \neq f(y)[q]$ while $x[q] = y[q]$ (either $f(x)[q] = 0$ or $f(y)[q] = 0$). So, assume W.L.O.G.

$$g(x) < q \text{ and } q \leq g(y)$$

and thus, $a\lambda(x) + b < a\lambda(y) + b \Rightarrow \lambda(x) < \lambda(y) \Rightarrow \lambda(x - y) = \lambda(x)$. Also, we have $\lambda(f(x)) = \lambda(x) < \lambda(y) = \lambda(f(y)) \Rightarrow \lambda(f(x) - f(y)) = \lambda(f(x))$. Then,

$$\lambda(f(x) - f(y)) = \lambda(f(x)) = \lambda(x) = \lambda(x - y)$$

which contradicts the assumption that $\lambda(f(x) - f(y)) < \lambda(x - y)$. We thus have $\lambda(f(x) - f(y)) \geq \lambda(x - y)$ as desired. $\square$

We then get:

$$\lambda(f(x) - f(y)) \geq \lambda(x - y) \Rightarrow d(f(x), f(y)) \leq d(x, y) \Rightarrow \sigma(f(x), f(y)) \leq \sigma(x, y)$$

as desired.
We check if inequality (3) holds. We have that, for any $x \in \mathcal{R}$, by our choice of f, $\lambda(f(x)) = \lambda(x)$, so $g(f(x)) = g(x)$ and $f(x) = f^n(x)$ for $n \geq 1$. And so if $x \neq f(x)$,

$$\liminf_{n \to \infty} \sigma(f^n(x), f^{n+1}(x)) = \sigma(f(x), f^2(x)) = 0 < \sigma(x, f(x))$$

So (3) holds as desired. And so by Theorem 4.5, f has at least one fixed point in A and at least one fixed point in B.

Remark. *f has infinitely many distinct fixed points in B*

Proof. for any $x \in B \setminus \{0\}$, $0 < \lambda(x) \leq g(x)$, so

$$x_g := x|_{]0, g(x)]} \neq 0 \text{ and } f(x_g) = x_g$$

$\{x_g : x \in B\} \cup \{0\} = \{x \in B : \mathit{supp}(x) \subset]0, g(x)]\} = \{x \in B : f(x) = x\}$ is infinite, as $g(x) > 0$. $\square$

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