The Coincidence that Wasn't

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The coincidence

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The coincidence

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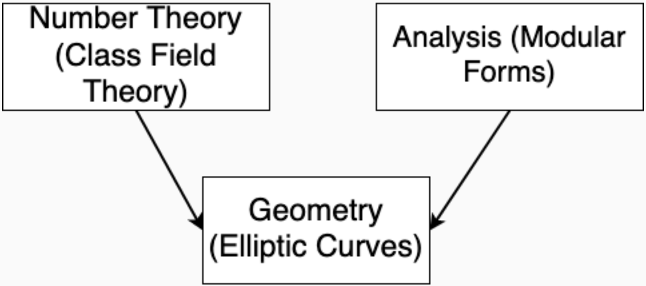
$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925\ldots$$

. . . that wasn't



Warning for experts: Many lies ahead!

Binary quadratic forms

A **binary quadratic form** is a function:

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where $a, b, c \in \mathbb{Z}$

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Question

How close is the discriminant to classifying quadratic forms?

Binary quadratic forms

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Question (Gauss; 1801)

For which D is $\mathrm{Cl}(D) = 1$?

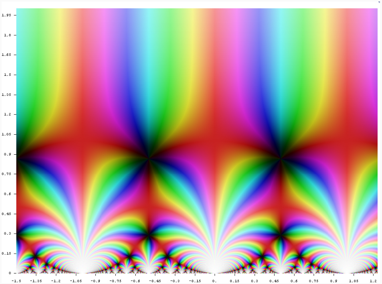
Theorem (Baker, Heegner, Stark; 1967)

If $D < 0$, then $\mathrm{Cl}(D) = 1$ if and only if

$$D = -1, -2, -3, -7, -11, -19, -43, -67, -163$$

The j-function

$$j(\tau+1) = j(\tau) \qquad j(-\tfrac{1}{\tau}) = j(\tau)$$



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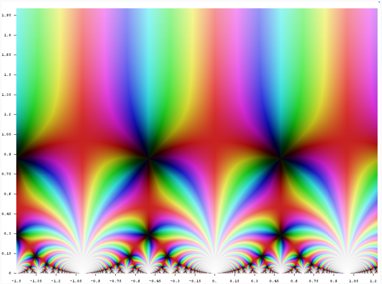

$$j(\tau) = 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}$$

where $g_2(\tau) = 60\sum_{m,n\in\mathbb{Z}}(m + n\tau)^{-4}$,

$$g_3(\tau) = 140\sum_{m,n\in\mathbb{Z}}(m + n\tau)^{-6}$$

Putting it all together

Given a quadratic form $f = ax^2 + bxy + cy^2$, let $\tau_f \in \mathbb{C}$ be the root of $ax^2 + bx + c$:

$$f = ax^2 + bxy + cy^2 \quad \Longrightarrow \quad \tau_f = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

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$$f = ax^2 + bxy + cy^2 \quad \Longrightarrow \quad \tau_f = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$f = x^2 + xy + 41y^2 \quad \Longrightarrow \quad \tau_f = \frac{-1 + \sqrt{-163}}{2}$$

Putting it all together

The j-function assigns a *class invariant* to quadratic forms:

$$j(\tau_f) = j(\tau_g) \quad \Longleftrightarrow \quad f \text{ and } g \text{ are the same (in a sense)}$$

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If $\mathrm{Cl}(D) = 1$, *then* $j(\tau_f)$ *is an integer!*

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Miracle

If $\mathrm{Cl}(D) = 1$*, then* $j(\tau_f)$ *is an integer!*

In fact, $j(\tau_f)$ *is a* ***algebraic integer*** *of degree* $\mathrm{Cl}(D)$*, where* $D = \mathrm{disc}(f)$*.*

Bringing it home

Consider $f = x^2 + xy + 41y^2$:

- $D = -163$
- $\mathrm{Cl}(D) = 1$
- $\tau_f = \frac{-1+\sqrt{-163}}{2}$

$\implies \quad j(\frac{-1+\sqrt{-163}}{2})$ is an integer: $= (-640320)^3$

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$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 \ldots$$

(where $q = e^{2\pi i \tau}$)

$q = e^{2\pi i \frac{-1+\sqrt{-163}}{2}} = -e^{-\pi\sqrt{163}} \qquad q^{-1} = -e^{\pi\sqrt{163}}$

Bringing it home

$$
\begin{aligned}
j(\frac{-1+\sqrt{-163}}{2}) &= (-640320)^3 \\
&= -e^{\pi\sqrt{163}} + 744 + c_2 e^{-2\pi\sqrt{163}} + c_3 e^{-3\pi\sqrt{163}} + \dots \\
&= -e^{\pi\sqrt{163}} + 744 + O\left(e^{-\pi\sqrt{163}}\right)
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At long last

$$e^{\pi\sqrt{163}} \approx (640320)^3 + 744$$