Gödel's Constructible Universe

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1 Introduction

The famous philosopher and logician Bertrand Russell once gave an analogy: if you have an infinite number of pair of socks, each composed with a left and a right sock, then a choice function is easily defined. We can just always pick the left sock in each pair of socks. Now let us remove the condition of the distinguishing feature. We then have an infinite number of non-empty sets. Can we say for sure that there exists a choice function? The axiom of choice allows us to do so. It is in fact used by most mathematicians since many accepted results require the Axiom of Choice in their proof. It is part of the ZFC, the Zermelo-Fraenkel set theory with Choice.

A natural question to ask is whether the Axiom of Choice is consistent with the other axioms of ZF. It has been proven consistent by Gödel in 1938 and independent by Cohen in 1963. Gödel showed that if ZF is consistent then ZF is consistent with the Axiom of Choice by constructing a model, called the Constructible Universe, that satisfies ZFC. That is, the negation of the Axiom of Choice is not provable with the axioms of ZF. Cohen, on the other hand, proved, using forcing, that ZF along with the negation of the Axiom of Choice is consistent. Thus, the Axiom of Choice cannot be proven to be true nor false in ZF, it is independent.

In this paper, we will go through the important notions to understand Gödel's Constructible Universe. To this end, we will explore topics such as the axioms of ZF, ordinal and cardinal numbers, models of set theory.

Before getting started, we would like to mention that in what follows, most theorems, proofs and definitions are taken from "Set Theory" (2006), by Thomas Jech.

2 The Axioms of Zermelo-Fraenkel and Choice

2.0.1 Set theory and its formulas

The language of set theory uses two predicates: = and \in . The first denotes equality and the second indicates the binary relation of membership.

Let x, y be sets. Atomic formulas are of the form

$$x = y, \ x \in y. \tag{1}$$

We can also construct more complex formulas using the following connectives: for φ, ψ formulas,

$$\varphi \land \psi, \varphi \lor \psi, \neg \varphi, \varphi \to \psi, \varphi \leftrightarrow \psi.$$

Furthermore, two quantifiers are used:

$$\forall x\varphi \text{ and } \exists x\varphi.$$

Variables that are not bounded by quantifiers are called **free variables**. They occur in formulas as such:

$$\varphi(x_1,\cdots,x_n) \tag{2}$$

where x_i are free variables.

A formula without free variables is called a **sentence**.

In ZFC, there is only one type of object, namely sets. However, in some cases, it is highly inconvenient to discuss an object with itself as a reference, we therefore introduce another type of object in subsection 2.10.

2.1 Axiom of Extensionality

If X and Y have the same elements, then X = Y:

$$\forall u(u \in X \leftrightarrow u \in Y) \to X = Y. \tag{3}$$

Note that the converse is an axiom in predicate calculus. We then have both directions.

2.2 Axiom of Pairing

For any a and b there exsists a set $\{a, b\}$ that contains exactly a and b:

$$\forall a \ \forall b \ \exists c \ \forall x \ (x \in c \leftrightarrow x = a \lor x = b). \tag{4}$$

2.3 Axiom Schema of Separation

If P is a property with parameter p, then for any X and p there exists a set $Y = \{u \in X : P(u, p)\}$ that contains all those $u \in X$ that have property P: Let $\varphi(u, p)$ be a formula,

$$\forall X \ \forall p \ \exists Y \ \forall u \ (u \in Y \leftrightarrow u \in X \land \varphi(u, p)). \tag{5}$$

Another way of looking at it is that the intersection of a class of the form $\{u: \varphi(u, p_1, \dots, p_n)\}$ with any set is a set. Here, notice that we replaced p by p_1, \dots, p_n , which is a more general version of the Separation Schema.

The condition that X must be a set in the Axiom Schema of Separation prevents situations like Russell's Paradox, whereas with a stronger version removing the condition, the Comprehension Schema, would cause $\{X : X \notin X\}$ to be a set.

2.4 Axiom of Union

For any X there exists a set $Y = \bigcup X$, the union of all elements of X:

$$\forall X \exists Y \forall u \ (u \in Y \leftrightarrow \exists z (z \in X \land u \in z)).$$
(6)

Note that here, $Y = \{u : (\exists z \in X) u \in z\} = \bigcup \{z : z \in X\} = \bigcup X$.

2.5 Axiom of Power Set

For any X there exists a set Y = P(X), the set of all subsets of X.

$$\forall X \exists Y \,\forall u \,(u \in Y \leftrightarrow u \subset X). \tag{7}$$

2.6 Axiom of Infinity

There exists an infinite set:

$$\exists S(\emptyset \in S \land (\forall v \in S)x \cup \{x\} \in S).$$
(8)

At first glance, this is not an intuitive way of describing an infinite set. The reason why the Axiom of Infinity is defined as such is to avoid using natural numbers as they are not defined yet.

2.7 Axiom Schema of Replacement

If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) : x \in X\}$: Let $\varphi(x, y, p)$ be a formula,

$$\forall x \forall y \forall z (\varphi(x, y, p) \land \varphi(x, z, p) \to y = z) \to \forall X \exists Y \forall y (y \in Y \leftrightarrow (\exists x \in X) \varphi(x, y, p)).$$
(9)

The first part of equation (7) describes the class F being a function, the second part identifies Y = F(X) as a set. Once again, there can be more than one parameter.

2.8 Axiom of Regularity

Every nonempty set has an \in -minimal element.

$$\forall S(S \neq \emptyset \to (\exists x \in S)S \cap x = \emptyset). \tag{10}$$

Especially, there cannot be infinite sequence of memberships. Particularly, we cannot have chains of the form $x_0 \in x_1 \in x_2 \in \cdots \in x_0$.

2.9 Axiom of Choice

Every family of nonempty sets has a choice function:

Let S be a family of sets. Then there exists a function $f(X) \in X$ for every $X \in S$. (11)

2.1 to 2.8 are the axioms of ZF. We obtain ZFC by adding 2.9.

2.10 Classes

Let $\varphi(x, p_1, \dots, p_n)$ be a formula. We then say that C defined as

$$C = \{x : \varphi(x, p_1, \cdots, p_n)\}\tag{12}$$

is a class.

One reason for which we make the distinction between classes and sets is because we cannot define C to be a set, but we still need to manipulate objects defined by formulas. In particular, we need this concept to define the class of all sets, also called the **universal class**, which is proper class (not a set):

$$V = \{x : x = x\}.$$
 (13)

Note that every set is a class.

The reason why V cannot be a set is as follows. Suppose the set of all sets S exists. Then S contradicts the Axiom of Regularity. This also leads to another issue, namely Russell's Paradox.

2.10.1 Russell's Paradox

Consider the set of all sets S. Subsection 2.3 states the Axiom Schema of Separation with equation (5) stating it formally. In the statement, we can define the set $Y = \{u \in X : P(u, x)\}$. Notice the condition $u \in X$, for X a set. With the existence of S, the set of all sets, replacing X with S, we get that for any property P there exists a set $Y = \{x \in S : P(x)\} = \{x : P(x)\}$. This is also commonly called the Axiom Schema of Comprehension. Now let $P(x) = x \notin x$ a property. Then $Y = \{X : X \notin X\} = \{X \in S : X \notin X\}$ is a set. But $Y \in Y$ and $Y \notin Y$ are both contradictions. Therefore, S cannot be a set.

3 Ordinals

In this section we introduce Ordinal numbers along with related concepts that will be necessary in the construction of L, the Constructible Universe. In addition, we prove the equivalence of the Axiom of Choice and Well-Ordering principle.

3.0.1 Well-Ordering

We say that a linear order < of a set P is a **well-ordering** if every nonempty subset of P has a least element.

A linear ordering on a set P is defined as follows, for any $p, q, r \in P$:

- 1. $p \neq p$
- 2. if p < q and q < r, then p < r
- 3. p < r or p = q or q < p.

Note that the first two conditions gives a **partial ordering** of the set P.

3.0.2 Well-founded relations

We say that a binary relation E on a set X is **well-founded** if every nonempty $x \subset X$ has an E-minimal element. That is, for every nonempty $x \subset X$, there is a $a \in x$ such that $\nexists y \in x$ with yEa. In particular, a well-ordering is a well-founded relation.

We can extend this definition to classes. Let ${\cal P}$ be a class. A binary relation E on ${\cal P}$ is then well-founded if

1. every nonempty subset x of P has an E-minimal element, and

2. $\operatorname{ext}_E(x)$ is a set, for every $x \in P$,

where

$$\operatorname{ext}_{E}(x) = \{ z \in P : zEx \}.$$
(14)

We call equation (14) the **extension** of x.

We say that a class M is **extensional** if the relation \in on M is extensional. In other words, for $X, Y \in M$, if $X \neq Y$, then $X \cap M \neq Y \cap M$, that is $\operatorname{ext}_E(X) \neq \operatorname{ext}_E(Y)$.

3.0.3 Transitive sets

We say that a set T is **transitive** if $\forall t \in T, t \subset T$. I.e. every element is a subset.

3.0.4 Ordinal number

Informally, we can view ordinals as order-types of well-ordered sets. In other words, Formally, we define an **ordinal number** to be a transitive set that is well-ordered by \in .

The class of all ordinals is denoted by *Ord*. In general, greek letters α, β, γ are used to denote ordinal numbers. Furthermore, we define

$$\alpha < \beta \text{ if and only if } \alpha \in \beta.$$
(15)

Here are some facts about ordinals:

- 1. $0 = \emptyset \in Ord$
- 2. If $\alpha \in Ord$ and $\beta \in \alpha$, then $\beta \in Ord$
- 3. If $\alpha \neq \beta$, both ordinals, and $\alpha \subset \beta$, then $\alpha \in \beta$
- 4. If $\alpha, \beta \in Ord$, then either $\alpha \subset \beta$ or $\beta \subset \alpha$.

There are two types of ordinals, successor and limit ordinals. Let us define $\beta + 1 = \beta \cup \{\beta\} = \inf\{\xi : \xi < \beta\}$. If $\alpha = \beta + 1$ for some $\beta \in Ord$, then we say that α is a **successor ordinal**. If an ordinal α is not a successor ordinal, i.e. there does not exist a β such that $\alpha = \beta + 1$, then α is then called a **limit ordinal** and is defined as $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup \alpha$.

We can remark that we construct natural numbers, and therefore all numbers, using ordinals. In fact, the first ones are defined as follows:

1. $0 = \{\}$ 2. $1 = \{\} \cup \{\{\}\} = \{\{\}\}$ 3. $2 = \{\{\}\} \cup \{\{\{\}\}\} = \{\{\}, \{\{\}\}\}\}$ 4. ...

We call a sequence **transfinite** if it is indexed by ordinals.

With the special structure that ordinals carry, we define a special type of induction called **transfinite induction**. To prove a property P on Ord, the class of all ordinals, it suffices to show:

- 1. P(0),
- 2. if $P(\alpha)$, then $P(\alpha + 1)$,
- 3. if α is a nonzero limit ordinal and $P(\beta)$ for all $\beta < \alpha$, then $P(\alpha)$.

Theorem 3.1 (Zermelo's Well-Ordering Theorem). Every set can be wellordered assuming the Axiom of Choice.

Proof. Let A be a set. To show that A can be well-ordered, we will define a bijection from A to a proper subset of the ordinals.

Let S be the set of nonempty subsets of A. Assuming the Axiom of Choice, let $f: S \to A$ be a choice function such that $f(X) \in X$ for every nonempty subset X of A.

Let θ be some ordinal, not fixed for now. We define the function $G: \theta \to A$ by induction:

G(0) = f(A) $G(1) = f(A - \{G(0)\})$... $G(\alpha) = f(A - \{G(\beta) : \beta < \alpha\} \text{ as long as } A - \{G(\beta) : \beta < \alpha\} \text{ is nonempty}$ We now show that there must exist a $\theta \in Ord$ for which $G^{-1}(A) = \theta$. We will prove it by contradiction. Notice that, by definition, G is injective, that is, different ordinal numbers map to different elements of A.

Suppose that $G(\theta)$ is a proper subset of A for every $\theta \in Ord$. Then we have an injection from the class Ord into A. Then the partial inverse function $G^{-1}(A) = Ord$ is not a set, which contradicts the Axiom Schema of Replacement, as the range of G is a set.

Hence, since Ord is transitive, there must be an ordinal number θ such that $A \subset G(\theta)$. Take such θ to be minimal. Then $\forall \alpha < \theta$, $G(\alpha)$ is a proper subset of A. By the transfinite definition of ordinal numbers, we have that $A = G(\theta)$.

An interesting result about this theorem is that it is equivalent to the Axiom of Choice. Indeed, a choice function can be defined using the well-ordering property. If all sets $X \subset S$ can be well-ordered, we then can choose the least element of each X to be our choice function. We then have $f(X) \in X \forall X \in S$.

3.0.5 The class of all sets

We define V, the **cumulative hierarchy of sets** by transfinite induction: for $\alpha, \beta \in Ord$

$$V_0 = \emptyset, V_{\alpha+1} = P(V_\alpha), V_\alpha = \bigcup_{\beta < \alpha} V_\beta \text{ if } \alpha \text{ is a limit ordinal }.$$
 (16)

Some facts on V_{α} : for $\alpha, \beta \in Ord$

- 1. V_{α} is transitive,
- 2. If $\alpha < \beta$, then $V_{\alpha} \subset V_{\beta}$,
- 3. $\alpha \subset V_{\alpha}$.

We can then define

$$V = \bigcup_{\alpha \in Ord} V_{\alpha}.$$
 (17)

Theorem 3.2. For any set x, $\exists \alpha \in Ord$ such that $x \in V_{\alpha}$, thus $x \in V$.

Proof. Recall the Axiom of Regularity: every nonempty set has an \in -minimal element.

First, we need to show that every nonempty class has an \in -minimal element, that is the Axiom of Regularity implies that it also applies to classes. Let C be a nonempty class. Then $\exists S \in C$. If $S \cap C = \emptyset$, we are done. Otherwise, if $S \cap C \neq \emptyset$, we need to find an x such that $x \in C$ and $x \cap C = \emptyset$. Consider the set $X = TC(S) \cap C$, where $TC(S) = \{T : T \supset S, T \text{ transitive}\}$ is the transitive closure of S. We know that TC(S) is not empty since we can define a T:

$$S_0 = S, \ S_{n+1} = \bigcup S_n, \ T = \bigcup_{n=0}^{\infty} S_n.$$
 (18)

X is nonempty since $S \subset TC(S)$ and $S \cap C \neq \emptyset$, so $S \cap C \subset S \subset X \neq \emptyset$. Since X is a nonempty set, the Axiom of Regularity gives that $\exists x \in X$ such that $x \cap X = \emptyset$. We claim that $x \cap C = \emptyset$. Suppose not, then $\exists y \in x$ and $y \in C$. Thus $y \in x \implies y \in T$ since T is transitive. But $x \cap X = \emptyset$, $y \in x \cap T \cap C \implies y \in x \cap X$ is a contradiction. We now have all nonempty classes have an \in -minimal element.

Let C be the class of all sets which are not in V. Suppose C is nonempty. Then C has an \in -minimal element x. $x \cap C = \emptyset \implies \forall y \in x, y \notin C$, thus $y \in V$. Hence $\exists \alpha \in Ord$ such that $y \in V_{\alpha}$. But $\{y : y \in x\} \subset V \implies x \subset V$. Therefore, $\exists \beta \in Ord$ with $x \in V_{\beta}$. We then have that the class C is empty and every set is in V.

4 Cardinals

Let X be a set. We denote |X| the cardinality of X.

Two sets X, Y are said to be of **same cardinality** if there exists a bijection between X and Y. It is denoted |X| = |Y|. We define **cardinal numbers** to be equivalence classes of this equivalence relation.

We say that an ordinal number is a cardinal if $|\alpha| \neq |\beta|$ for all $\beta < \alpha$. In particular, all finite cardinals are equal to the finite ordinals, i.e. the natural numbers, and all infinite cardinals are limit ordinals. In particular, we call infinite cardinals **alephs**, denoted \aleph . The alephs are defined as such:

$$\aleph_0 = \omega_0 = \omega, \ \aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_{\alpha}^+, \tag{19}$$

$$\aleph_{\alpha} = \omega_{\alpha} = \sup\{\omega_{\beta} : \beta < \alpha\}, \text{ if } \alpha \text{ is a limit ordinal.}$$
(20)

We say that $\aleph_{\alpha+1}$ is a successor cardinal. If the index α is a limit ordinal, then we call \aleph_{α} a limit cardinal.

We say that $|X| \leq |Y|$ if there is an injective mapping from X onto Y.

Note that the class of cardinals is well-ordered since it is defined based on the ordinals and so is a subclass of Ord.

4.0.1 Arithmetic operations on cardinals

We define the following cardinal operations: Let A, B be sets with $|A| = \kappa, |B| = \lambda$.

- 1. $\kappa + \lambda = |A \cup B|$, if A, B disjoint
- 2. $\kappa \cdot \lambda = |A \times B|$
- 3. $\kappa^{\lambda} = |A^B|$, where $A^B = \{f : B \to A\}$.

Theorem 4.1 (Cantor). Let X be a set. Then |X| < |P(X)|.

In other words, for any set, the cardinality of its power set is strictly greater than the cardinality of the set itself. *Proof.* Let f be a function from X to P(X). Clearly, $|X| \leq |P(X)|$. It then suffices to show that f cannot be a bijection.

Notice that it suffices to find an element of P(X) that is not in the range of f. Consider the set $Y = \{x \in X : x \notin f(x)\} \in P(X)$. We claim that Y is not in the range of f. Suppose for contradiction, that there exists $z \in X$ with f(z) = Y. Then either $z \in Y$ or $z \notin Y$. If $z \in Y$, then by definition of Y, $z \notin f(z) = Y$. If $z \notin Y = f(z)$, then $z \in Y$. ξ

Thus, $|X| \neq |P(X)|$ and we have |X| < |P(X)|.

Lemma 4.1. If $|A| = \kappa$, then $|P(A)| = 2^{\kappa}$.

Proof. Let A be a set with cardinality κ . We define a bijective map from P(A) to $\{0,1\}^A$. From subsection 4.0.1, we know that $|\{0,1\}^A| = 2^{\kappa}$.

Let $X \in P(A)$. Consider the function $\chi_X : A \to \{0, 1\}$ defined as:

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \in A - X \end{cases}$$

Define $f(X) = \chi_X$. Then $f: P(A) \to \{0, 1\}^A$ is bijective.

4.0.2 The Generalized Continuum Hypothesis (GCH)

Informally, the GCH says that there is no cardinal between the cardinality of a set and that of the power set of that set. Formally, it is to say, if X is a set with cardinality \aleph_{α} , then:

$$2^{\aleph_{\alpha}} = \aleph_{\alpha+1},\tag{21}$$

where $2^{\aleph_{\alpha}}$ is the cardinality of the power set of X and $\aleph_{\alpha+1}$ is the cardinal that comes right after \aleph_{α} .

5 Models

A language \mathcal{L} is defined as a set of symbols of relations P, functions F, and constants c:

$$\mathcal{L} = \{P, \cdots, F, \cdots, c, \cdots\}.$$
(22)

We define a **model** for a language \mathcal{L} to be a pair $\mathfrak{U} = (A, \mathcal{I})$, where A is the universe of \mathfrak{U} and \mathcal{I} an interpretation mapping symbols of \mathcal{L} to relations, functions and constants in A.

Let $\mathfrak{U} = (\mathcal{A}, \mathcal{I})$ be a model. We define a **submodel** \mathfrak{L} of \mathfrak{U} to be a subset $\mathcal{B} \subset \mathcal{A}$ with the interpretations of \mathfrak{L} be those of \mathfrak{U} restricted to the symbols of \mathcal{B} . In this case, we denote $\mathfrak{L} \subset \mathfrak{U}$.

We call a submodel \mathfrak{L} of a model \mathfrak{U} an **elementary submodel** of \mathfrak{U} if for every formula φ and every $b_1, \dots, b_n \in B$,

$$\varphi^{\mathfrak{L}}(b_1,\cdots,b_n) \leftrightarrow \varphi^{\mathfrak{U}}(b_1,\cdots,b_n),$$
 (23)

that is, $\varphi(b_1, \dots, b_n)$ is satisfied by \mathfrak{U} if and only if it is satisfied by \mathfrak{L} . In this case, we denote it as

$$\mathfrak{L} \prec \mathfrak{U}.$$
 (24)

We denote \mathfrak{U} satisfying a formula $\varphi(x)$ by

$$\mathfrak{U}\models\varphi(x).\tag{25}$$

5.0.1 Gödel's Completeness and Incompleteness Theorems

Gödel's theorems are famous results in set theory.

- 1. The Completeness Theorem states that a model exists for every set of consistent sentences. Note that a set of sentences Σ is said to be consistent if there is no provable contradiction from Σ .
- 2. The First Incompleteness Theorem states that no consistent extension of Peano Arithmetic is complete, that is, there exists undecidable statements. We say that a theory is incomplete if there exists sentences that cannot be proven true or false. In particular, if ZFC is consistent, then it will remain incomplete.
- 3. The Second Incompleteness Theorem states that sufficiently strong theories (as Peano Arithmetic or ZF, if consistent) cannot prove their own consistency. Particularly, we cannot prove in ZF that there exists a model of ZF.

5.0.2 Relativization to models of set theory

Let (M, E) be a model of set theory, where M is a class and E, a binary relation on M. Let φ be a formula in the language of set theory. We relativize φ to (M, E) as follows:

- 1. $(x \in y)^{M,E} \leftrightarrow x E y$
- 2. $(x = y)^{M,E} \leftrightarrow x = y$
- 3. $(\neg \varphi)^{M,E} \leftrightarrow \neg \varphi^{M,E}$
- 4. $(\varphi \land \psi)^{M,E} \leftrightarrow \varphi^{M,E} \land \psi^{M,E}$
- 5. $(\exists x\varphi)^{M,E} \leftrightarrow (\exists x \in M)\varphi^{M,E}$.

The relativization is defined similarly for the other connectives and the universal quantifier \forall . $\varphi(x_1, \dots, x_n)$ relativized to (M, E) is denoted

$$\varphi^{M,E}(x_1,\cdots,x_n). \tag{26}$$

In cases where $E = \in$, we omit E and just write φ^M .

We denote *Form* to be the set of all formulas of the language $\{\epsilon\}$.

5.0.3 Δ_0 Formulas

We say that a formula of set theory σ is a Δ_0 formula if one of the following holds:

- 1. σ contains no quantifier;
- 2. σ is of the form $\phi \land \psi, \phi \lor \psi, \neg \phi, \phi \to \psi$ or $\phi \leftrightarrow \psi$, when ϕ, ψ are Δ_0 formulas;
- 3. its only quantifiers are bounded, i.e. $(\exists x \in y)\phi$ or $(\forall x \in y)\phi$ for ϕ are Δ_0 formula.

5.0.4Absoluteness

We say that a formula φ is **absolute** if for all x_1, \dots, x_n

$$\varphi^M(x_1, \cdots, x_n) \leftrightarrow \varphi(x_1, \cdots, x_n).$$
 (27)

In particular, a formula is absolute if it is both upward absolute and downward absolute. That is, its truth value does not change from submodels to 'bigger' models and vice-versa.

5.0.5Transitive models

Let M be a transitive class. We then say that (M, ϵ) is a **transitive model** of set theory.

Theorem 5.1. Δ_0 formulas are absolute for transitive models of set theory.

Proof. The proof goes by induction on the complexity of formulas. First, atomic formulas are absolute.

Second, the logical connectives does not change the absoluteness of a formula.

Third, let $y \in M$ and suppose $\sigma = (\exists x \in y)\phi$, with ϕ a Δ_0 formula. If σ^M holds, then $((\exists x \in y)\phi)^M = (\exists x(x \in y \land \phi(x)))^M = (\exists x \in M)(x \in y \land \phi^M(x))$ holds. Since ϕ is absolute, $\phi^M \leftrightarrow \phi$. Thus, $(\exists x \in y)(\phi(x)) = \sigma$ holds.

Now suppose $\sigma = (\exists x \in y)\phi(x)$ holds. Since $y \in M$ and M is transitive, we have that $x \in M$. Thus $(\exists x \in M)(x \in y \land \phi^M(x))$ holds since $\phi^M(x) \leftrightarrow \phi(x)$. Hence σ^M holds.

The proof for $(\forall x \in y)\phi$ is very similar.

Theorem 5.2 (Reflection Principle).

1. Let $\varphi(x_1, \dots, x_n)$ be a formula. Then for any set of constants M_0 , there exists a set $M \supset M_0$ such that for all $x_1, \cdots, x_n \in M$,

> $\varphi^M(x_1,\cdots,x_n) \leftrightarrow \varphi(x_1,\cdots,x_n).$ (28)

In this case, we say that M reflects φ .

- 2. There is a transitive M that reflects φ for any M_0 . Furthermore, there is a limit ordinal α such that V_{α} reflects φ with $M_0 \subset V_{\alpha}$.
- 3. With the Axiom of Choice, we have case (1) with $|M| \leq |M_0| \cdot \aleph_0$. In particular, we can obtain a countable M that satisfies (1).

Note that we can have either case (2) or (3), but not both.

 φ needs not to be exactly one formula, Theorem 5.2 also applies to a finite number of formulas.

With the Axiom of Choice, we can get a countable transitive model M of a true sentence σ . Notice that this satisfies both (2) and (3).

The Reflection Principle, just as Gödel's Second Incompleteness Theorem does, implies that ZF is not finitely axiomatizable.

6 The Constructible Universe

Gödel introduced L, the class of all constructible sets, the smallest transitive model of ZF that contains all ordinal numbers, as a proof of consistency of the Axiom of Choice and the Generalized Continuum Hypothesis.

6.0.1 Definable sets

Let M be a set. We say that a set X is a **definable** over the model (M, \in)) if there exists a formula φ in the set of all formulas of the language $\{\in\}$ and some parameters $p_1, p_2, \cdots, p_n \in M$ such that $X = \{x \in M : (M, \in) \models \varphi[x, p_1, \cdots, p_n]\}$.

We can then define

$$def(M) = \{ X \subset M : X \text{ is definable over } (M, \epsilon) \}.$$
(29)

6.0.2 L, the class of all constructible sets

L is defined using transfinite induction.

- 1. $L_0 = \emptyset, L_{\alpha+1} = \operatorname{def}(L_\alpha),$
- 2. $L_{\alpha} = \bigcup_{\beta < \alpha} L_{\beta}$ if α is a limit ordinal, and
- 3. $L = \bigcup_{\alpha \in Ord} L_{\alpha}$.

Theorem 6.1. L is a transitive class.

Proof. First, by transfinite induction, we show that L_{α} is transitive for every α . The empty set is transitive.

Suppose L_{α} is transitive. Let $x \in L_{\alpha+1}$. Then $x \in def(L_{\alpha})$, so x = Y, for $Y \subset L_{\alpha}$ and Y is definable over (L_{α}, ϵ) . Hence, $Y = \{y \in L_{\alpha} : (L_{\alpha}, \epsilon) \models \varphi[y, p_1, \cdots, p_m]\}$. Thus, $x = Y \subset L_{\alpha} \subset L_{\alpha+1}$. $L_{\alpha+1}$ is transitive.

Now suppose L_{β} is transitive for every $\beta < \alpha$ for α a limit ordinal. Let $x \in L_{\alpha}$. Then $x \in L_{\beta}$ for some $\beta < \alpha$. So $x \subset L_{\beta} \implies x \subset L_{\alpha}$. L_{α} is transitive.

Let $x \in L$. Since $L = \bigcup_{\alpha \in Ord} L_{\alpha}$, we have $x \in L_{\alpha}$ for some ordinal α . From above, L_{α} is transitive which implies that $x \subset L_{\alpha}$. Thus $x \subset \bigcup_{\alpha \in Ord} L_{\alpha} = L$. L is transitive. \Box

Theorem 6.2. L contains all ordinal numbers.

Proof. Note that this is claiming that every ordinal is constructible.

Let α be an ordinal. This proof goes by transfinite induction on α . More precisely, we are showing that $\alpha \in L_{\alpha+1}$ for every α .

First, we notice that $\alpha \subset L_{\alpha}$. Assuming, for the induction, that $\beta \in L_{\beta+1}$ for every $\beta < \alpha$, we have that $\beta \in L_{\alpha}$. Thus $\alpha \subset L_{\alpha}$.

Second, we show that $\alpha \notin L_{\alpha}$ for all $\alpha \in Ord$. Otherwise, there exists a smallest limit ordinal α such that $L_{\alpha} = \bigcup_{\beta < \alpha} \operatorname{def}(L_{\beta})$ with $\alpha \in L_{\alpha}$. Then $\alpha \in L_{\beta}$ for some $\beta < \alpha$, thus $\alpha \subset L_{\beta}$. But this implies that $\beta \in L_{\beta}$, which contradicts the minimality of α . Hence, by transitivity of the L_{α} 's, for all $\beta \ge \alpha, \beta \notin L_{\alpha}$.

Third, we remark that since $\alpha \subset L_{\alpha}$ and $\alpha \in Ord$, $\alpha \subset L_{\alpha} \cap Ord$. We claim that $\alpha = L_{\alpha} \cap Ord$. If not, then there exists $\beta \subset L_{\alpha} \cap Ord$ such that $\beta > \alpha$, that is $\alpha \in \beta$. But that would imply that $\alpha \in L_{\alpha}$, which is a contradiction.

The third step gives us that $\alpha = \{x \in L_{\alpha} : x \text{ is an ordinal}\}$. We know that 'x is an ordinal' is a Δ_0 -formula. Thus by Theorem 5.1, $\alpha = \{x \in L_{\alpha} : x \text{ is an ordinal}\} = \{x \in L_{\alpha} : L_{\alpha} \models x \text{ is an ordinal}\}$. Hence, α is definable over L_{α} and $\alpha \in L_{\alpha+1}$.

Theorem 6.3. L is a model of ZF.

Proof. To show that L is a model of ZF, it suffices to show that σ^L holds for all axioms σ of ZF. By Theorem 6.1 we have that L is a transitive class. Thus, using Theorem 5.1, any Δ_0 -formula is absolute in L, that is its truth does not change from V to L.

1. Axiom of Extensionality

Transitiveness implies extensionality. In fact, let x, y be distinct in L. $x, y \in L \implies x, y \subset L$. Since x, y distinct, there exists $z \in x$ such that $z \notin y$ or the other way around. Since L is transitive, $z \in L$. Thus, $L \models \neg(\forall z)(z \in x \iff z \in y)$. Therefore, extensionality holds.

2. Axiom of Pairing

Let $a, b \in L$. Let $c = \{a, b\}$. There exists $\alpha \in Ord$ such that $a, b \in L_{\alpha}$. Then c is definable over L_{α} , i.e. $c \in L_{\alpha+1}$. $c = \{a, b\} \leftrightarrow a \in c \land b \in c \land (\forall d \in c)(d = a \lor d = b)$, thus c is Δ_0 .

3. Axiom Schema of Separation

Let φ be a formula, $X, p \in L$. Applying the Reflection Principle, there exists $\alpha \in Ord$ such that $X, p \in L_{\alpha}$ and $Y = \{u \in X : \varphi^{L}(u, p)\} = \{u \in U \in X : \varphi^{L}(u, p)\}$

 $X : \varphi^{L_{\alpha}}(u,p)$ }. Hence, $Y = \{u \in L_{\alpha} : L_{\alpha} \models u \in X \land \varphi(u,p)\}$. Thus we have that $Y \in L$. This shows that the Axiom of Separation with formula φ holds in L, using the definition of relativization of formulas (5.0.2), for any formula φ .

4. Axiom of Union

Let $X \in L$, $Y = \bigcup X$, the union of all elements of X. Since L is transitive, $X \subset L$ and hence $Y \subset L$. Let $\alpha \in Ord$ such that $X \in L_{\alpha}$ and $Y \subset L_{\alpha}$. Then $Y \in L_{\alpha+1}$, as Y is definable over L_{α} . Note that $Y = \bigcup X'$ is Δ_0 as it can be expressed as $(\forall z \in Z)(\exists x \in X)z \in x \land (\forall x \in X)(\forall z \in x)z \in Z$. Therefore, the Axiom of Union holds in L.

5. Axiom of Power Set

Let $X \in L$. We wish to show that the power set of X exists in L. Let $Y = P(X) \cap L$. Let $\alpha \in Ord$ such that $Y \subset L_{\alpha}$. Then Y is definable over L_{α} by the Δ_0 formula ' $x \subset X$ '. We need that $Y = P(X) \cap L = P^L(X)$, that is 'Y is the power set of X' holds in L. Recall the definition of Y being the power set of X: $\forall u(u \in Y \leftrightarrow u \subset X)$. Notice that this definition is satisfied by our construction of Y.

6. Axiom of Infinity

We know that $\omega \in L_{\omega+1}$, thus $\omega \in L$, by the proof of Theorem 5.2. Take $S = \omega$, the first limit ordinal. Then $\emptyset \in S$ and $(\forall x \in S)x \cup \{x\} \in S$. Thus the Axiom of Infinity holds in L.

7. Axiom Schema of Replacement

Let class F be a function in L. We want to show that for any $X \in L$, there exists $Y = F(X) = \{F(x) : x \in X\} \in L$. Using the Separation Schema, the Axiom Schema of Replacement can be proved from a weaker version, namely $\forall X \exists Y F(X) \subset Y$. Let $X \in L$. Then there exists $\alpha \in Ord$ such that $F(X) = \{F(x) : x \in X\} \subset L_{\alpha}$. This is sufficient to show that the Axiom Schema of Replacement holds in L.

8. Axiom of Regularity

Let $S \in L$ non-empty. If $\exists x \in S$ with $x \cap S = \emptyset$, then $x \in L$ by transitivity of L. Thus $\exists x \in L$ with $x \in S$ and $x \cap S = \emptyset$. Therefore the Axiom of Regularity holds in L.

We now have that L is a model of ZF.

6.0.3 Gödel Operations

- 1. $G_1(X, Y) = \{X, Y\}$ 2. $G_2(X, Y) = X \times Y$ 3. $G_3(X, Y) = \varepsilon(X, Y) = \{(u, v) : u \in X \land v \in Y \land u \in v\}$
 - 4. $G_4(X, Y) = X Y$

- 5. $G_5(X,Y) = X \cap Y$
- 6. $G_6(X) = \bigcup X$
- 7. $G_7(X) = \text{dom}(X)$
- 8. $G_8(X) = \{(u, v) : (v, u) \in X\}$
- 9. $G_9(X) = \{(u, v, w) : (u, w, v) \in X\}$
- 10. $G_{10}(X) = \{(u, v, w) : (v, w, u) \in X\}$

We call compositions of the above 10 operations Gödel operations.

Theorem 6.4 (Gödel's Normal Form Theorem). Every Δ_0 -formula can be expressed as a Gödel operation.

The proof of Theorem 6.4 is done by induction on the complexity of Δ_0 formulas.

Theorem 6.5. Gödel operations are absolute for transitive models.

In particular, the proof for this theorem shows that any property expressed by a Gödel operation can be rewritten as a Δ_0 formula, which we know are absolute for transitive models. Once again, the proof is on induction on the complexity of Gödel operations.

6.0.4 Inner Models of ZF

We say that a model M of ZF is an **inner model** if it is a transitive class containing all ordinals and it satisfies the axioms of ZF.

We remark that L is an inner model of ZF. We will show in Theorem 5.6 that it is in fact the smallest inner model of ZF.

6.0.5 The Lévy Hierarchy

We define a formula to be Σ_0 and Π_0 if all its quantifiers are bounded. Note that this corresponds to the definition given in 5.0.3, that of a Δ_0 formula.

By induction, we define Σ_{n+1} to be those formulas of the form $\exists x \varphi$ for φ a \prod_n formula.

Similarly, we define a Π_{n+1} formula to be of the form $\forall x \varphi$ for $\varphi \in \Sigma_n$ formula. We say that a property is Δ_n if it is both Σ_n and Π_n .

Theorem 6.6 (Gödel). 1. L satisfies V = L (Axiom of Constructibility)

2. L is the smallest inner model of ZF

Proof. 1. We want to show that 'every set is constructible' holds in L.

Let x be a set in L. First we prove that 'x is constructible' is absolute for inner models of ZF, in particular for L. Let M be an inner model of ZF. Then M contains all ordinals. Thus,

 $(x \text{ is constructible})^M \leftrightarrow \exists \alpha \in Mx \in L^M_\alpha \leftrightarrow \exists \alpha x \in L_\alpha \leftrightarrow x \text{ is constructible.}$

Note that we can go from $x \in L^M_{\alpha}$ to $x \in L_{\alpha}$ because the function $\alpha \mapsto L_{\alpha}$ is absolute. Indeed, induction step in the definition of L_{α} is Σ_1 and thus upward absolute. In other words, L^M_{α} is unique and is equal to L_{α} .

Now we have that $(x \text{ is constructible})^L$ if and only if x is constructible, that is $x \in L$, which is true by assumption. Thus L satisfies the Axiom of Constructibility.

2. We want to show that any inner model of ZF M contains L.

Let M be an inner model of ZF. Then the class of all constructible sets in $M L^M$ is $L (L^M = L)$, since $\alpha \mapsto L_\alpha$ is absolute by the argument used in part 1. Hence L is a submodel of M.

We would like to add some comments on part 1 of the above Theorem. V = L essentially says "every set is definable". And it makes sense that this holds in L since everything in L is definable.

Another way to look at it is to picture a model as a room. When you enter the room and close the door, there are walls around you and you don't see through them. If you enter the room of L, all you see is what is inside, that is, definable sets. You are unaware of what exists outside of L, and hence, have no knowledge of possibly the existence of non-definable sets. Therefore, from your point of view, every set is definable. Thus it is not surprising that V = L holds in L.

Theorem 6.7. Consistency of the Axiom of Choice.

Proof. We will prove that there is a well-ordering of the class L. With Theorem 5.6 part 1, we then have that the Axiom of Choice holds in L.

We define the well-ordering of L inductively. For each $\alpha \in Ord$, we define a well-ordering $<_{\alpha}$ of L_{α} . The idea is to have $<_{\alpha}$ extend $<_{\beta}$ for any $\alpha > \beta$. That is, if $x <_{\beta} y$, then $x <_{\alpha} y$ for any $\alpha > \beta$. Furthermore, if $x \in L_{\beta}$ and $y \in L_{\alpha} - L_{\beta}$, that is $\alpha > \beta$, then $x <_{\beta} y$.

First, let α be a limit ordinal. We then let $<_{\alpha} = \bigcup_{\beta < \alpha} <_{\beta}$.

Second, suppose α is not a limit ordinal. The definition of $<_{\alpha+1}$ is slightly more complicated. Let M be a transitive set. As $def(M) = cl(M \cup \{M\}) \cap P(X)$, we can also write $L_{\alpha+1}$ as follows:

$$L_{\alpha+1} = P(L_{\alpha}) \cap \operatorname{cl}(L_{\alpha} \cup \{L_{\alpha}\}) = P(L_{\alpha}) \cap \bigcup_{n=1}^{\infty} W_n^{\alpha},$$
(30)

where cl denotes closure under Gödel operations. In the above, we define W_n^{α} inductively:

$$W_0^{\alpha} = L_{\alpha} \cup \{L_{\alpha}\}, W_{n+1}^{\alpha} = \{G_i(X, Y) : X, Y \in W_n^{\alpha}, i = 1, \cdots, 10\}.$$
 (31)

Notice that the definition in equation (11) allows to take compositions of the operations defined in 5.0.3 so we obtain the full closure under Gödel operations with the infinite union.

We can then define $<_{\alpha+1}$. We do it by induction on n.

- 1. $<_{\alpha+1}^{0}$ is the well-ordering of W_{0}^{α} that extends $<_{\alpha}$.
- 2. $<_{\alpha+1}^{n+1}$ is the well-ordering of W_{n+1}^{α} such that $x <_{\alpha+1}^{n+1} y$ if and only if:
 - (a) $x <_{\alpha+1}^n y$,
 - (b) or $x \in W_n^{\alpha}$ and $y \notin W_n^{\alpha}$, or
 - (c) $x \notin W_n^{\alpha}$ and $y \notin W_n^{\alpha}$ and, letting i, j be the least such numbers with all parameters of x, y in W_n^{α} , :
 - i. i < j for $x = G_i(u, v)$ and $y = G_j(s, t)$, or

ii. i = j and $u <_{\alpha+1}^n s$, or

iii.
$$i = j, u = s$$
, and $v <_{\alpha+1}^n t$.

Now letting ;p/

$$<_{\alpha+1} = \bigcup_{n=1}^{\infty} <_{\alpha+1}^{n} \cap (P(L_{\alpha}) \times P(L_{\alpha}))$$
(32)

we have a well-ordering of $L_{\alpha+1}$ extending $<_{\alpha}$.

$$x <_L y$$
 if and only if $\exists \alpha x <_\alpha y$. (33)

 $<_L$ is then a well-ordering of the class L.

Now let X be any set. Theorem 5.6 gives us that V = L. Hence $X \in L \implies X \in L_{\alpha}$ for some $\alpha \in Ord$. Since L_{α} is transitive, $X \subset L_{\alpha}$. As L_{α} is well-orderable, as we have shown just above, X is also well-orderable. Thus, for any set X, X is well-orderable and the Axiom of Choice holds in L.

6.0.6 Transitive Collapse

We say that a map π is a **transitive collapse** if the range of π is transitive and

$$xEy \to \pi(x) \in \pi(y).$$
 (34)

Theorem 6.8 (Mostowski's Collapsing Theorem).

- 1. Let E be a well-founded extensional relation on a class P. Then there exists a transitive class M and an isomorphism $\pi : (P, E) \to (M, \epsilon)$ such that M and π are unique.
- 2. Every extensional class P is isomorphic to a unique class M, with a unique isomorphism.
- 3. In the previous case, if $T \subset P$ is transitive, then $\pi(x) = x, \forall x \in T$.

Proof. We note that proving (1) is sufficient to prove the theorem. Let P be a class with E a well-founded relation.

Since E is a well-founded relation on P, we can use well-founded induction to define the map π . That is, we can define $\pi(x)$ based on $\pi(z)$ for zEx. Let

$$\pi(x) = \{\pi(z) : zEx\}.$$
(35)

Remark that if x is E-minimal in P, that is there is no $z \in P$ with zEx, then $\pi(x) = \emptyset$. In particular, if $E = \epsilon$, then

$$\pi(x) = \{\pi(z) : z \in x \cap P\}.$$
(36)

We then define $M = \pi(P) = \{\pi(x) : x \in P\}$. We claim that M is transitive. Let $y \in M$. Then $y \in \pi(P) \implies \exists x \in P$ such that $y = \pi(x) = \{\pi(z) : zEx\} \subset M$. Clearly, π is surjective. It then suffices to show that π is injective. Let $x \in P$.

Then $\pi(x) = y \in M$.

We define the rank of a set x as such

$$rank(x) =$$
 the least α such that $x \in V_{\alpha+1}$. (37)

Now suppose that π is not injective. Then let $z \in M$ of least rank such that $z = \pi(x) = \pi(y)$ but $x \neq y$, for $x, y \in P$. Since x, y distinct, and as they are defined by the relation E, $\operatorname{ext}_E(x) \neq \operatorname{ext}_E(y)$. Thus, without loss of generality, we can assume that there exists $u \in \operatorname{ext}_E(x)$ but $u \notin \operatorname{ext}_E(y)$. Call $\pi(u) = t$. Since $\pi(x) = \pi(y), \pi(u) \in \pi(x) \Longrightarrow \exists v \neq u$ with $\pi(v) = \pi(v) \in \pi(y)$ such that vEy. But notice that this implies that $t \in z$, thus $\operatorname{rank}(t) < \operatorname{rank}(z)$, which is a contradiction. Thus we can conclude that π must be injective.

Next, we show that π is indeed a transitive collapse. M, the range of π , is transitive. We just need to show

$$xEy \to \pi(x) \in \pi(y).$$
 (38)

Let xEy, then equation (31) gives us that $\pi(x) \in \pi(y)$. Furthermore, since π is injective, we also have the other direction

$$\pi(x) \in \pi(y) \to xEy. \tag{39}$$

It remains to prove that M and π are unique. Suppose there are two different isomorphisms $\pi_1 : P \to M_1$ and $\pi_2 : P \to M_2$. Then M_1 must be isomorphic M_2 , and hence $\pi_1 = \pi_2$.

Finally, we show case (3). Let P be an extensional class isomorphic to a class M. Let $T \subset P$ be transitive. Then for any $x \in T$, $x \in P$. Thus $\pi(x) = {\pi(z) : z \in x \cap P}$. Observe that if $x = \emptyset$, then $\pi(x) = x$. Now suppose for ϵ -induction that $\forall z \in x, \pi(z) = z$. Then $\pi(x) = {\pi(z) : z \in x} = {z : z \in x} = x$.

Theorem 6.9 (Löwenheim-Skolem). Every infinite model for a countable language has a countable elementary submodel.

Note that Theorem 6.9 requires the use of the Axiom of Choice.

Lemma 6.1 (Gödel's Condensation Lemma). If γ is a limit ordinal and M an elementary submodel of (L_{δ}, \in) , then the transitive collapse of M is L_{γ} , with $\gamma \leq \delta$.

Theorem 6.10 (Gödel). Consistency of the Generalized Continuum Hypothesis.

Proof. Gödel proved that if V = L, then $2^{\aleph_{\alpha}} = \aleph_{\alpha+1}$. If the Generalized Continuum Hypothesis did not hold, there could be a cardinal strictly between $2^{\aleph_{\alpha}}$ and \aleph_{α} . Then, we would have $\aleph_{\alpha+1} \leq 2^{\aleph_{\alpha}}$. For our claim, it then suffices to show that $2^{\aleph_{\alpha}} \leq \aleph_{\alpha+1}$.

In order to do so, we will show that $P(\omega_{\alpha}) \subset L_{\omega_{\alpha+1}}$ for any $\alpha \in Ord$. Let $X \subset \omega_{\alpha}$ be a constructible set. If we can show that there exists a $\gamma < \omega_{\alpha+1}$ with $X \in L_{\gamma}$, then we are done. Indeed, by the definition of L_{β} 's, this would imply that $P(X) \subset L_{\theta}$ for any $\theta > \gamma$, in particular, $P^L(X) \subset L_{\omega_{\alpha+1}}$. Then $|P^L(X)| = 2^{\aleph_{\alpha}} \leq \aleph_{\alpha+1} = |L_{\omega_{\alpha+1}}|$.

Here is a subproof of $\aleph_{\alpha+1} = |L_{\omega_{\alpha+1}}|$. It suffices to show that $|\alpha| = |L_{\alpha}|$ for any $\alpha \ge \omega$. We do so using transfinite induction. At $\alpha = \omega$, $|\alpha| = |\bigcup_{n < \omega} n| =$ $\aleph_0 = |L_{\alpha}| = |\bigcup_{n < \omega} L_n|$ as $|L_n| \le \omega$ for every $n \in \omega$. Now suppose α is a limit ordinal. Recall $\alpha = \bigcup_{\beta < \alpha} \beta$ and $L_{\alpha} = \bigcup_{\beta < \alpha} L_{\beta}$. By induction hypothesis, if $\beta < \alpha$, $|\beta| = |L_{\beta}|$. Since the union operation does not change cardinality, we have that $|\alpha| = |L_{\alpha}|$. Finally, suppose $\alpha + 1$ is a successor cardinal. Recall $\alpha + 1 = \alpha \cup \{\alpha\}$ and $L_{\alpha+1} = \operatorname{def}(L_{\alpha}) \subset P(L_{\alpha})$. We remark that by the definition of successor ordinal, $|P(\alpha)| = |\alpha| = |\alpha + 1|$. Since induction hypothesis gives us $|\alpha| = |L_{\alpha}|$, we have that $|L_{\alpha+1}| = |\operatorname{def}(L_{\alpha})| \le |P(L_{\alpha})| = |\alpha| = |\alpha + 1|$.

Now going back to the main proof, let X be a constructible subset of ω_{α} . Suppose V = L. Then, by the Reflection Principle, there exists a limit ordinal $\delta > \omega_{\alpha}$ such that $X \in L_{\delta}$. Using the Löwenheim-Skolem theorem (L satisfying the Axiom of Choice is sufficient), we can find an elementary submodel M of L_{γ} such that $X \in M$, $\omega_{\alpha} \subset M$, and $|M| = \aleph_{\alpha}$. By Mostowski's Collapsing Theorem, we get that M is isomorphic to a transitive model N with collapsing map π . The Condensation Lemma says that N is L_{γ} for some $\gamma \leq \delta$, γ a limit ordinal. Notice that $\gamma < \omega_{\alpha+1}$ since $|N| = |\gamma| = \aleph_{\alpha}$. As ω_{α} is transitive, $\omega_{\alpha} \subset M \implies \pi(\omega_{\alpha}) = \omega_{\alpha}$ and $\pi(X) = X$, by Mostowski's Collapsing Theorem. Since $X \in M$, we have $X \in L_{\gamma}$, the desired result.

7 Conclusion

This was an introduction to set theory. We covered elementary notions in the Zermelo-Fraenkel set theory, ordinal and cardinal numbers, models and Gödel's Constructible Universe. In particular, we have presented the proofs of consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with L. This is half of the proof for the independence of those statements. The other half can be found in Paul Cohen's proof, which uses a self-invented technique, namely Forcing. This is, in fact, another interesting topic to explore.

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