Introduction to the Einstein Constraint Equations

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Abstract

Einstein constraint equations (ECE) is a well studied problem in the mathematical construction of general relativity. More specifically, solution to the ECE provide us with suitable initial data that can evolve into solutions of the spacetime Einstein equations. Despite its application in physics, mathematically, ECE problem relates with classical problems in Riemannian or semi-Riemannian geometry, such as scalar curvature prescription problems and related geometric partial differential equation (PDE) problems, which further motivates their analysis.

This report aims to do a brief introduction to Einstein constraint equations, and the method of solving it. We will mainly emphasize on the constant mean-curvature (CMC) case, some brief mention on near-CMC case, and some toy examples on non-CMC case.

Keywords: Einstein constraint equations, general relativity, manifold, differential geometry, partial differential equations

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1 Conventions and Notations

Throughout this report, we use Einstein summation notation for contracting repeated upper-lower index pairs, unless otherwise stated. Manifolds will be assumed to be Hausdorff and second countable. When specifying the dimension of a manifold M, we write M^d for a d-dimensional manifold.

I will adapt the following notations:

1. The Lie derivative with respect to a vector field v is denoted as $\mathscr{L}_v$; it satisfies

$$\mathscr{L}_v w^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu.$$

2. Our conventions for the extrinsic curvature are

$$\begin{aligned} &K(X,Y) \coloneqq \bar{g}(\mathrm{I\!I\!I}(X,Y), n), \text{ for all } X, Y \in \Gamma(TM), \\ &\mathrm{I\!I\!I}(X,Y) \coloneqq (\bar{\nabla}_{\bar{X}} \bar{Y})^\perp, \end{aligned}$$

   where $\bar{X}, \bar{Y}$ denotes arbitrary extension to V and $\mathrm{I\!I\!I} : \Gamma(TM) \times \Gamma(TM) \mapsto \Gamma(TM)^\perp$ denotes the second fundamental form of $M \hookrightarrow (V, \bar{g})$. K stands as the extrinsic curvature of $M \hookrightarrow (V, \bar{g})$.

3. We use the vector operator ∇ to denote covariant differentiation with respect to some given connection.

4. We use parentheses and square brackets respectively fro symmetrization and antisymmetrization of indices.

5. Our convention for the Riemann tensor may be expecified by

$$[\nabla_a, \nabla_b] = \mathcal{R}_{ab}.$$

6. We use the usual notation $\Gamma^\sigma_{\mu\nu}$ to denote the Christoffel symbols, where the Greek letters signify its indices.

7. We adopt the notation $\tau \coloneqq \mathrm{tr}_g K$ for the mean curvature.

2 Preliminaries on General Relativity and Differential Geometry

We first introduce some basic definitions. We refer to *The Einstein Constant Equations* [1] by Rodrigo Avalos and Jorge H. Lira for their basic concepts on the subject.

Definition 2.1. *A semi-Riemannian manifold (V, g) will be called* ***Lorentzian*** *if the metric g has constant index equal to 1.*

Definition 2.2. *Let (V, g) be a Lorentzian manifold and let $p \in V$. We will say that a vector $v \in T_pV$, $v \neq 0$, is* ***time-like*** *if $g_p(v, v) < 0$,* ***light-like*** *(or* ***null****) if $g_p(v, v) = 0$, and* ***space-like*** *if $g_p(v, v) > 0$. Along these lines, we define the* ***light-cone*** *(or* ***null-cone****) at p as the subset of T_pV formed by all the null-vectors.*

Definition 2.3. *The manifold $\mathbb{R}^{n+1}$ equipped with the Lorentzian metric η given by*

$$\eta = -dx^0 \otimes dx^0 + \sum_{i=1}^{n} dx^i \otimes dx^i,$$

where $\{x^\alpha\}_\alpha^n = 0$ stand for (global) canonical coordinates for $\mathbb{R}^n$, is referred to as the ***Minkowski space-time****, and we denote it by $\mathbb{M}^{n+1}$.*

As a subject to general relativity, we will always consider time-orientable Lorentzian manifolds, which we shall also refer to as space-times.

Definition 2.4. *Let (V, g) be a Lorentzian manifold. At each point $p \in V$, we have two null-cones in T_pV. A choice of one of these null-cones is a* ***time-orientation*** *for T_pV. A smooth function τ on V which assigns to each $p \in V$ a null-cone in T_pV is said to be a time-orientation for V. We say (V, g) is* ***time-orientable*** *if it admits such a time-orientation function.*

It is straightforward to see that a Lorentzian manifold is time-orientable if and only if it admits a global time-like vector field. Its general structure also inherit some properties reserving the causal structure for a physical problem. Here we introduce some relevant concepts.

Definition 2.5. *Let (V, g) be a time-orientable Lorentzian manifold and $p, q \in V$, we will write*

1. *$p \ll q$ if there is a future-pointing time-like curve in V from p to q.*

2. *$p < q$ if there is a future-pointing causal curve in V from p to q.*

3. *Given a subset $A \subseteq V$, we define the **chronological future** $\mathcal{I}^+(A)$ and **past** $\mathcal{I}^-(A)$ of A by*

$$\mathcal{I}^+(A) \coloneqq \{q \in V \,:\, \exists\, p \in A \text{ with } p \ll q\},$$
$$\mathcal{I}^-(A) \coloneqq \{q \in V \,:\, \exists\, p \in A \text{ with } p \gg q\},$$

*and the **causal future** $\mathcal{J}^+(A)$ and **past** $\mathcal{J}^-(A)$ of A by*

$$\mathcal{J}^+(A) \coloneqq \{q \in V \,:\, \exists\, p \in A \text{ with } p \leq q\},$$
$$\mathcal{J}^-(A) \coloneqq \{q \in V \,:\, \exists\, p \in A \text{ with } p \geq q\},$$

Definition 2.6. *Let (V, g) be a Lorentzian manifold. We will say that the **strong causality condition** holds at $p \in V$ if for any given neighbourhood $\mathcal{U}$ of p, there is a neighbourhood $V \subseteq U$ of p such that every causal curve with endpoints in V is entirely contained in $\mathcal{U}$.*

Given two points $p, q \in V$ and $p < q$, we use the notation $\mathcal{J}(p, q) \coloneqq \mathcal{J}^+(p) \cap \mathcal{J}^-(q)$, which is the smallest set containing all future-pointing causal curves from p to q.

Definition 2.7. *A Lorentzian manifold (V, g) is **globally hyperbolic** if:*

1. *The strong causality condition holds in V;*

2. *If $p, q \in V$ and $p < q$, then $\mathcal{J}(p, q)$ is compact.*

Definition 2.8. *A **Cauchy hypersurface** in a Lorentzian manifold (V, g) is a subset M that is met exactly once by every inextendible time-like curve in V.*

The following result links the two notions of global hyperbolicity and Cauchy surfaces:

Theorem 2.1. *Any globally hyperbolic space-time* (V, g) *admits a smooth space-like Cauchy hypersurface* M*. Furthermore,* V *is diffeomorphic to* $\mathbb{R} \times M$*.*

Furthermore, the above result can be strengthened, establishing that (V, g) is isometric to $(\mathbb{R} \times M, -N^2 d\mathcal{T}^2 + \bar{g})$, where $\mathcal{T} : \mathbb{R} \times M \mapsto \mathbb{R}$ is the natural projection, $N : \mathbb{R} \times M \mapsto (0, \infty)$ is a smooth function, and $\bar{g}$ is a symmetric $(0, 2)$-tensor field which, for each $\mathcal{T}$, restricts to a Riemannian metric on $\mathcal{T} \times M \cong M$. Note that $\nabla\mathcal{T}$ is time-like and past-pointing, i.e, $\mathcal{T}$ is a time-function. A further generalisation of these ideas can be obtained for globally hyperbolic manifolds with appropriate boundary.
There are a couple of interesting consequences of the above theorem. First, notice that any non-trivial topology in a globally-hyperbolic spacetime must be contained within its Cauchy surface. Second, a Cauchy hypersurface in a globally hyperbolic space-time is a suitable subset where we can pose initial conditions for evolution problems. In fact, our task will be to start with a Cauchy surface M and and initial data on it, and then show that we can evolve such initial data to create space-time solutions to the Einstein equations. Although general existence results only provide us with a slab $[0, T] \times M$ on which the spacetime solution is guaranteed to exist, whenever solutions are guaranteed to exist for all times, we recover a globally-hyperbolic spacetime by evolution.

3 Introduction to Einstein Constraint Equations

We will first state the Einstein field equation (EFE), which relates the geometry of a spacetime to the distribution of matter within it,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T(\bar{g}, \bar{\psi}), \tag{3.1}$$

where on the left hand side, $G_{\mu\nu}$ is the Einstein tensor, Λ denotes the cosmological constant. The Einstein tensor is defined as $G_{\mu\nu} := \mathrm{Ric}_{\bar{g}} - \frac{1}{2} R_{\bar{g}}\, \bar{g}$ with $\mathrm{Ric}_{\bar{g}}$ and $R_{\bar{g}}$ respectively denoting the Ricci tensor and scalar curvature associated to $\bar{g}$. On the right-hand side, T denotes the energy-momentum tensor field associated to the matter fields sourcing the gravitational field, which will typically depend on the space-time metric $\bar{g}$, and some collection of physical fields, here collectively denoted by $\bar{\psi}$.

Now we will introduce the initial value formulation to the problem. Let us start by considering globally hyperbolic vacuum $(n+1)$-dimensional spacetime $(V^{n+1} = \mathbb{R} \times M^n, \bar{g})$ so that the Einstein equations get reduced to

$$\mathrm{Ric}_{\bar{g}} = 0. \tag{3.2}$$

The objective is to be able to give initial data on M and guarantee that we can evolve it into such a solution. But there are some immediate subtleties in this procedure. First, notice that in this analysis we will have to make a clear spacetime splitting, and therefore, we will introduce a time parameter t along the $\mathbb{R}$ factor, and the global future pointing time-like vector-field ∂_t tangent to the time-curves $t \mapsto (t, x) \in V$. We also denote the tangential component of ∂_t to M_t by X, which is a time-dependent vector field tangent to M known as the shift vector, and the normal component to M_t will be denoted by a function $N > 0$ referred to as the lapse function. These objects allow us to build adapted local frames $\{e_\alpha\}_{\alpha=0}^n$ of the form

$$\begin{aligned} e_0 &= \partial_t - X \perp M_t, \\ e_i &= \partial_{x^i}, \end{aligned} \tag{3.3}$$

for for any coordinate system $\{x^i\}_{i=1}^n$ on M, and their dual co-frames $\{\theta^\alpha\}_{\alpha=0}^n$ are

$$\begin{aligned} \theta^0 &= dt, \\ \theta^i &= dx^i + X^i\, dt. \end{aligned} \tag{3.4}$$

Using such frames, the space-time metric can be locally put in the form

$$\bar{g} = -N^2 dt \otimes dt + \bar{g}_t, \tag{3.5}$$

where the induced metric $\bar{g}_t$ on M_t has the local form $\bar{g}_t = \bar{g}_{ij}\, \theta^i \otimes \theta^j$. Notice that the future pointing unit normal to each Mt can then be written as

$$n = \frac{1}{N}(\partial_t - X). \tag{3.6}$$

In the above space-time splitting, the choice of our family of time-like curves defined by the vector field ∂_t is uniquely determined by the choice of lapse and shift, since $\partial_t = Nn + X$. So, each choice of $N > 0$ and X satisfying $-N^2 + |X|^2_{\bar{g}_t} < 0$ determines a unique such family of space-time observers and vice-versa. So our choice of space-time splitting according to a preferred ∂_t should work merely as a gauge choice.

To make the problem of more clear, we attempt to prescribe a Riemannian manifold (M^n, g) equipped with a symmetric $(0,2)$-tensor field K and initial data for the lapse-shift $(N, X, \partial_t N, \partial_t X)|_{t=0}$, which determine the family of observers along whose integral curves we intend to evolve the initial data, and then find an isometric embedding $\iota : (M, g) \mapsto (V = I \times M, g)$ with $I \subseteq \mathbb{R}^n$ such that $\bar{g}$ solves the space-time Einstein equations. In the vacuum case given by $\mathrm{Ric}_{\bar{g}} = 0$. Our objective is to utilize the freedom in choosing the flow lines along which we evolve the data. This demands having enough freedom so as to guarantee that at the end of the problem $\partial_t|_{t=0} = (Nn + X)|_{t=0}$ is time-like. From the well-known Gauss–Codazzi equations for hypersurfaces, which for a space-like hypersurface (M, g, K) isometrically immersed in a Lorentzian manifold $(V, \bar{g})$ such that

$$\bar{g}(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W) - (K(X,Z)K(Y,W) - K(Y,Z)K(X,W)), \tag{3.7}$$

$$\bar{g}(\bar{R}(X,Y)Z,n) = (\nabla_X K)(Y,Z) - (\nabla_Y K)(X,Z). \tag{3.8}$$

Equations (3.7) and (3.8) are Gauss' and Codazzi's equations respectively. We have $X, Y, Z \in \Gamma(TM)$, n stands for the future-pointing unit normal vector field to M, and the quantities without a bar on top are constructed with the intrinsic induced Riemannian metric g on M. The above equations are a priori necessary conditions that (g, K) must satisfy

Proposition 3.1. *Let (M, g, K) be a space-like hypersurface isometrically immersed in a Lorentzian manifold $(V, \bar{g})$ satisfying the Einstein equations $G_{\bar{g}} + \Lambda\bar{g} = T$ for some energy-momentum tensor T. Then g and K satisfy the following* ***constraint*** *equations on M:*

$$\begin{aligned} R_g - |K|^2_{\bar{g}} + (\mathrm{tr}_g\, K)^2 - 2\Lambda &= 2\epsilon, \\ \mathrm{div}_g K - d\,(\mathrm{tr}_g\, K) &= J, \end{aligned} \tag{3.9}$$

where $\epsilon := T(n,n)$ and $J := -T(n,\cdot) \in \Gamma(TM)$ denote the energy and momentum densities *induced on M.*

We will then briefly introduce the procedure of deriving above Einstein constraint equations in the following proof.

Proof. Given any local orthonormal frame $\{n, e_i\}_{i=1}^n$from the Gauss equation we can compute that

$$\begin{aligned}\sum_{i,j=1}^n \bar{g}(\bar{R}(e_i,e_j)e_j,e_i) &= \sum_{i,j=1}^n g(R(e_i,e_j)e_j,e_i) - \sum_{i,j=1}^n (K(e_i,e_j)K(e_j,e_i) - K(e_j,e_j)K(e_i,e_i)) \\ &= R_g - \sum_{i,j=1}^n (K(e_i,e_j)K(e_j,e_i) - K(e_j,e_j)K(e_i,e_i)) \\ &= R_g - |k|_g^2 - (\operatorname{tr}_g K)^2.\end{aligned} \tag{3.10}$$

Since

$$\begin{aligned}\mathrm{Ric}_{\bar{g}}(e_i,e_j) &= \sum_{\alpha=0}^n \bar{g}(e_\alpha,e_\alpha)\bar{g}(\bar{R}(e_\alpha,e_i)e_j,e_\alpha) \\ &= -\bar{g}(\bar{R}(n,e_i)e_j,n) + \sum_{k=1}^n \bar{g}(\bar{R}(e_k,e_i)e_j,e_k),\end{aligned} \tag{3.11}$$

we get that

$$\begin{aligned}R_g - |k|_g^2 - (\operatorname{tr}_g K)^2 &= \mathrm{Ric}_{\bar{g}}(n,n) + \sum_{i=1}^n \mathrm{Ric}_{\bar{g}}(e_i,e_i) \\ &= 2\,\mathrm{Ric}_{\bar{g}}(n,n) + (-\mathrm{Ric}_{\bar{g}}(n,n) + \sum_{i=1}^n \mathrm{Ric}_{\bar{g}}(e_i,e_i)) \\ &= 2\,\mathrm{Ric}_{\bar{g}}(n,n) + R_{\bar{g}} = 2\left(\mathrm{Ric}_{\bar{g}} - \frac{1}{2}\bar{g}R_{\bar{g}}\right)(n,n) \\ &= 2(T - \Lambda\bar{g})(n,n) = 2T(n,n) + 2\Lambda.\end{aligned} \tag{3.12}$$

Thus, fromthedefinition $T(n,n) \coloneqq \epsilon$, we get

$$R_g - |K|_g^2 + (\operatorname{tr}_g K)^2 = 2(\epsilon + \Lambda). \tag{3.13}$$

Now considering the Codazzi equation,

$$\begin{aligned}\mathrm{Ric}_{\bar g}(n, e_i) &= \sum_{\alpha=0}^{n} \bar g(e_\alpha, e_\alpha)\bar g(\bar R(e_\alpha, n)e_i, e_\alpha) = \sum_{j=1}^{n} \bar g(\bar R(e_j, n)e_i, e_j) \\ &= \sum_{j=1}^{n} \bar g(\bar R(e_i, e_j)e_j, n) = \sum_{j=1}^{n}(\nabla_{e_i} K)(e_j, e_j) - \sum_{j=1}^{n}(\nabla_{e_j} K)(e_i, e_j) \\ &= \mathrm{tr}_g(\nabla_{e_i} K) - \mathrm{div}_g\, K(e_i) = \nabla_{e_i} \mathrm{tr}_g\, K - \mathrm{div}_g\, K(e_i).\end{aligned} \tag{3.14}$$

Since $\mathrm{Ric}_{\bar g}(n, e_i) = T(n, e_i)$, we get

$$d\,(\mathrm{tr}_g\, K)(e_i) - \mathrm{div}_g\, K(e_i) = T(n, e_i).$$

Finally, from the definition of the physical momentum density is $J \coloneqq -T(n, \cdot)$, we arrive at the momentum constraint:

$$\mathrm{div}_g K - d\,(\mathrm{tr}_g\, K) = J, \tag{3.15}$$

which finishes the derivations. □

To briefly describe the main steps in this construction, let us equip M with a some fixed smooth and complete Riemannian metric e, then trivially embed M into $V = \mathbb{R} \times M$ and fix a background Riemannian metric $\hat e = dt^2 + e$ on V. From now on, quantities constructed from $\hat e$ will be denoted with a hat on top. The idea is first to consider the reduced Einstein equations given by

$$\mathrm{Ric}^{(\hat e)}_{\bar g} = 0. \tag{3.16}$$

The advantage now is that this is a set of quasi-linear wave equations where some standard PDE theory theorems guarantee that, for appropriate initial data on $\bar g$, the system possesses one and only one solution. By appropriate initial data we mean (g, K, N) in some appropriate $\mathring{H}^s_{loc}$-Sobolev space and $K, \partial_t N|_{t=0}, \partial_t X|_{t=0}$ in the corresponding $\mathring{H}^{s-1}_{loc}$, with $s > \frac{n}{2} + 1$. The solution to this problem provides us with a Lorentzian metric $\bar g$ on $[0, T) \times M$ for some $T > 0$. We want to show that if our initial data set (M, g, K) solves the vacuum constraint equations (3.9) with $\epsilon = \Lambda = J = 0$, then an appropriate choice of the gauge data $\partial_t N|_{t=0}, \partial_t X|_{t=0}$ guarantees that $\hat F = 0$ and then $\mathrm{Ric}_{\bar g} = 0$ and $(V, \bar g)$ is therefore our desired

Cauchy development of (M, g, K). If $\hat{F}|_{t=0} = \partial_t \hat{F}|_{t=0} = 0$, then $\hat{F} = 0$. If

1. The initial data for the solution $\bar{g}$ to Equation (3.16) solves the vacuum constraints associated to Equations (3.9);
2. $\hat{F}|_{t=0} = 0$,

then $\partial_t \hat{F}|_{t=0} = 0$. Now, let us consider an adapted frame $\{e_\alpha\}_{\alpha=0}^n$ of the form of Equation (3.3) and assume that we have constructed $\bar{g}$ out of initial data M, g, K satisfying the constraints and with $N|_{t=0} = 1$ and $X|_{t=0} = 0$. Then a straightforward computation gives

$$\begin{aligned} F^0|_{t=0} &= -(\partial_t N|_{t=0} + g^{ij} K_{ij}), \\ F_i|_{t=0} &= -\partial_t X_i|_{t=0} + g_{ij} g^{kl} (\Gamma^j_{kl}(g) - \Gamma^j_{kl}(e)). \end{aligned}$$

Therefore, we can fix the initial conditions $\partial_t N|_{t=0}, \partial_t X_i|_{t=0}$ on M so as to satisfy $\hat{F}|_{t=0} = 0$. Such a solution $\bar{g}$ solves the full vacuum Einstein equations on V and is therefore an appropriate short-time Cauchy development of (M, g, K).

Notice that if we have two Cauchy developments $(V_i, \bar{g}_i), i = 1, 2$, of the same geometric data (M, g, K), and therefore implying that their initial data can differ only via the initial data of N, X which selects the space-time observers, then these developments are isometric and also geometrically unique. Also, there is unique, up to isometries, maximal globally hyperbolic development of any such vacuum initial data set. The solutions to these problems have the right causality behaviour, i.e., they exhibit the finite-speed propagation associated to solutions of wave equations inherited via hyperbolic theory applied to Equation (3.16). In particular, the limit speed of propagation is given by that of that of the null curves of $\bar{g}$.

Finally, the above discussion can be readily extended along the same lines to non-vacuum situations. These last cases which involve an electromagnetic field actually present one further subtlety, which is that the Maxwell equations of electromagnetism also impose constraints on the admissible initial data for the electromagnetic 2-form F. Due to the limited space of this report, I will not extend on this subject. But I suggest the readers to further read on their own if this topic interest you.

4 Constant Mean Curvature (CMC) Method and Some Classical Results

Now we will start our analysis of the constraint equations for the general relativistic initial data sets. The first objective is to cast the ECE as system of geometric elliptic PDEs. Notice that the ECE seen as equations for (g, K) on M^n are a highly under-determined system, and in particular, we have some freedom to look for a useful decomposition of (g, K) into prescribed data and unknowns which may turn it into a determined elliptic system. The ideal objective would be that such splitting is natural both from a geometric and a physical stand point. The best known method to the problem is the so called conformal method. This method splits g into a prescribed conformal class and an unknown conformal factor, while it splits K into a prescribed trace part (mean curvature) and unknown traceless part, which itself undergoes a further slitting allowing to write the momentum constraint as an elliptic equation on some vector field X. We will discuss how under special geometric conditions which involve a constant mean curvature (CMC) hypothesis, the conformal method decouples the Gauss–Codazzi constraints (3.9). It is in such situations that this method is most effective. In particular, we will present results which include the CMC vacuum classification of Isenberg (1995) as well as the more recent remarkable developments of Maxwell (2005).

4.1 The Conformal Method

Recall that the constraint Equations (3.9) stands as a highly under-determined system posed for (g, K), and we attempt to exploit this freedom to split (g, K) in some clever way into prescribed data and unknowns for the system following the the confor- mal method, which translates Equations (3.9) into a determined elliptic PDE system. In this context, the energy constraints have the form of a generalised scalar curvature prescription problem, and therefore the conformal deformations work quite nicely. We have the following computational result.

Proposition 4.1. *Let (M^n, g) be a Riemannian manifold with $n \geq 3$. Suppose that $g = \varphi^{\frac{4}{n-2}}\gamma$ for some other Riemannian metric γ on M. Then, the following transformation rule* *for the scalar curvature holds*

$$R_g = \varphi^{-\frac{n+2}{n-2}} \left(R_\gamma \varphi - \frac{4(n-1)}{n-2} \Delta_\gamma \varphi \right), \tag{4.1}$$

where Δ_γ stands for the negative Laplace operator.

In the above context we will denote by ∇ the Riemannian connection associated to g, and by D the corresponding connection associated to γ. Also, the second order linear operator appearing in the right hand side of Equation (4.1), given by $L_g \coloneqq \Delta_\gamma - c_n R_\gamma$ will be referred to as the conformal Laplacian. Equation (4.1) transforms the energy constraint into:

$$\Delta_\gamma \varphi - c_n R_\gamma \, \varphi + c_n \left(|K|_g^2 - \tau^2 + 2\epsilon \right) \varphi^{\frac{n+2}{n-2}} = 0, \tag{4.2}$$

where $c_n = \frac{1}{4} \frac{n-2}{n-1}$.

We now split extrinsic curvature into its trace and traceless parts, so that we can freely specify the trace as a parameter. The trace part $\tau = \operatorname{tr}_g K$ of K will naturally inherit some scaling under conformal deformations. We need to impose scaling for the traceless part under conformal transformations, following the York splitting as follows:

$$K = \varphi^{-2} \tilde{K} + \frac{\tau}{n} g,$$

where $\tilde{K}$ is a γ-traceless, and thus g-traceless, $(0,2)$-tensor field, where we take the convention that $\tilde{K}$ moves its indices with the conformal metric γ, while the physical extrinsic curvature K moves its indices with the physical metric g, i.e.,

$$\begin{aligned} K_{ij} &= \varphi^{-2} \tilde{K}_{ij} + \frac{\tau}{n} g_{ij}, \\ K^{ij} &= \varphi^{-2\frac{n+2}{n-2}} \tilde{K}^{ij} + \frac{\tau}{n} g^{ij}. \end{aligned} \tag{4.3}$$

This in particular implies that

$$|K|_g^2 = \varphi^{-\frac{4n}{n-2}} |\tilde{K}|_\gamma^2 + \frac{\tau^2}{n}.$$

To obtain a similar form for the momentum constraint and rewrite it as a determined elliptic

PDE system natually, we first consider the following computational result.

Proposition 4.2. *Consider the Riemannian manifold* (M, g)*, with* $g = \varphi^{\frac{4}{n-4}}\gamma$ *for some other Riemannian metric* γ *on* M*. Let* $K \in \Gamma(T^0_2 M)$ *be symmetric and split it as in Equations (4.3). The the* g *and* γ *divergences of* K *are related via the following expression*

$$\mathrm{div}_g K = \varphi^{-\frac{2n}{n-2}}\,\mathrm{div}_\gamma \tilde{K} + \frac{1}{n} d\,\tau. \tag{4.4}$$

The above proposition shows that the choice of scaling avoids the first order contributions. Now we can make a further decomposition to the conformally formulated energy and momentum constraints, Equations (4.2), and rewrite them as

$$\Delta_\gamma \varphi - c_n R_\gamma\,\varphi + c_n\,|\tilde{K}|^2_\gamma \varphi^{-\frac{3n-2}{n-2}} + c_n\,(\frac{1-n}{n}\tau^2 + 2\epsilon)\,\varphi^{\frac{n+2}{n-2}} = 0, \tag{4.5}$$

$$\mathrm{div}_\gamma \tilde{K} - \left(\frac{n-1}{n} d\,\tau + J\right)\varphi^{\frac{2n}{n-2}} = 0. \tag{4.6}$$

We refer to Equation (4.5) as the Lichnerowicz equation. Above equations take different forms depending of our physical model, which determines the form of ϵ and J, as well as the remaining geometric data, related to the extrinsic curvature. The input geometric data would be the metric γ, which fixes the conformal class of physical metric g, the mean curvature τ. They are posed for the conformal factor u and the traceless tensor $\tilde{K}$. In the case of vacuum ($\epsilon = J = 0$) maximal ($\tau = 0$), the system decouples. In such a case, we first find a traceless tensor which is γ-divergence free (such tensors are called TT-tensors, which stands for traceless and transverse tensors), which works as an input in the resulting equation for the conformal factor, and all of the analysis falls on the associated Lichnerowicz equation. On the other hand, for non-vacuum and/or non-maximal solutions, in general, we have coupled system.

Assuming that M is closed, let (M^n, γ) be a Riemannian manifold as above, with $n \leq 3$, and assume that $\gamma \in W^{2,p}$, with $p > \frac{n}{2}$. Then, define the conformal Killing Laplacian (CKL) operator

$$\begin{aligned} \Delta_{\gamma,\mathrm{comf}} : W^{2,p}(TM) &\mapsto L^p(T^*M), \\ X &\mapsto \mathrm{div}_\gamma(\mathscr{L}_{\gamma,\mathrm{comf}}X), \end{aligned} \tag{4.7}$$

where $\mathscr{L}_{\gamma,\mathrm{comf}}X := \mathscr{L}_X\gamma - \frac{2}{n}\gamma\mathrm{div}_\gamma X$ stands for the conformal Lie derivative, whose kernel is given by conformal Killing fields (CKF) of the metric γ. It is an elliptic operator.

Theorem 4.1. *Let (M^n, γ) be a smooth closed Riemannian manifold, $n \leq 3$. Then, for any $1 < p < \infty$, the following splitting holds*

$$W^{1,p}(\overset{\circ}{S_2}M) = \ker(L_1) \oplus \mathrm{Im}\{L_2\}, \tag{4.8}$$

where $L_1 : W^{1,p}(\overset{\circ}{S_2}M) \mapsto L^p(T^*M)$ is given by $L_1W := \mathrm{div}_\gamma W$, where $L_2 : W^{2,p}(TM) \mapsto W^{1,p}(\overset{\circ}{S_2}M)$ is given by $L_2X := \mathscr{L}_{\gamma,comf}X$. We denote by $\overset{\circ}{S_2}M$ the vector bundle whose fibres consist of traceless symmetric $(0,2)$-tensor fields on M.

Therefore, at least for smooth data γ on closed manifolds, we can always split the traceless-part of our extrinsic data via

$$\tilde{K} = \mathscr{L}_{\gamma,\mathrm{comf}}X + U, \tag{4.9}$$

where X is a vector field and U is the TT-tensor part associated to it by the above theorem.

Definition 4.1. *We will say that the physical sources (ϵ, J) in an initial data set (g, K, ϵ, J) are York-scaled if, under the conformal decomposition of (g, K) described above, their scaling on the initial data set induces a change in the momentum density of the form $J = \varphi^{-\frac{2n}{n-2}}\tilde{J}$, where $\tilde{J}$ is a 1-form constructed with the conformal data (γ, τ, U) plus additional prescribed data.*

The feature that makes York-scaled sources special is that, under an additional CMC-condition, they transform the conformally formulated momentum constraint into

$$\Delta_{\gamma,\mathrm{comf}}X = \tilde{J}, \tag{4.10}$$

which is completely decoupled from the associated Lichnerowicz equation. Therefore, in some sense, this generalises the CMC vacuum case mentioned above. In this case, we can deal with this linear PDE, solve for X, which completes all the information in $\tilde{K}$, and then, once more, the core of the analysis falls on the corresponding Lichnerowicz equation.

4.1.1 Conformal Covariance

For the relation between two different conformal initial data sets built from conformally related metrics γ and $\gamma' = \theta^{\frac{4}{n-2}}\gamma$, first notice that associated to conformal data ϑ, we have the solution $(\varphi, X, \tilde{E})$, from which we construct the physical initial data. Using primed or unprimed variables, the physical solution is the same in both cases. We find an action of the conformal group on the conformal data (ψ, ϑ), which makes it a kind of gauge group. Let us first present the following computational result.

Proposition 4.3. *Let us consider a Riemannian manifold $(M^n, \gamma), \gamma \in W^{2,p}, p > \frac{n}{2}$, and a conformally related Riemannian metric $\gamma' = \theta^{\frac{4}{n-2}}\gamma, \theta \in W^{2,p}$. The conformal Laplacian operators L_γ and $L_{\gamma'}$ associated to γ and γ' respectively satisfy the following conformal covariance property:*

$$L_\gamma \varphi = \theta^{\frac{n+2}{n-2}} L_{\gamma'}\varphi', \forall \varphi \in W^{2,p}, \tag{4.11}$$

where $\varphi' = \theta^{-1}\varphi \in W^{2,p}$.

We can find some preferred element in a conformal class which simplifies the problem, e.g., when we split the space of Riemannian metrics on M into its disjoint Yamabe classes. In such a case, our conformal class will belong to exactly one Yamabe class, and that allows us to select a conformal representative in $[\gamma]$ with fixed sign on the scalar curvature, which can be used to control the behaviour of L_γ as well as the existence of simple barriers for the Lichnerowicz equation. In such a case, we first fix a useful conformal representative to solve our problem, knowing that the final physical initial data will remain unaltered by these gauge choices.

4.2 CMC-Solutions on Closed Manifolds

With the physical initial data $\pi = \varphi^{-\frac{2n}{n-2}}\tilde{\pi}$ In the case we switch-off the fluid's contributions and adopt the CMC hypothesis, we obtain the decoupled system given by

$$\begin{aligned}
&\Delta_\gamma \varphi - r\varphi + a_{TT}\varphi^{-\frac{3n-2}{n-2}} - a_\tau \varphi^{\frac{n+2}{n-2}} + a_E\varphi^{-3} + a_{\tilde{F}}\varphi^{\frac{n-6}{n-2}} = 0,\\
&\Delta_{\gamma,\text{conf}} X = \omega_\phi - \tilde{E} \lrcorner \tilde{F},\\
&\text{div}_\gamma \tilde{E} = 0,\\
&d\,\tilde{F} = 0,
\end{aligned} \tag{4.12}$$

where we have introduced the additional notations

$$a_{TT} := c_n\left(|\tilde{K}|^2_\gamma\right) + \tilde{\pi}^2,\; a_E := c_n|\tilde{E}|,\; a_{\tilde{F}} = \frac{c_n}{2}|\tilde{F}|.$$

Note that System (4.12) is completely decoupled. Since the elec- tromagnetic constraints are completely decoupled, we assume that we have fixed a priori a closed 2-form $\tilde{F}$ and a γ-divergence-free vector field $\tilde{E}$ together with the remaining free data. Let us first concentrate on the decoupled momentum constraint given by

$$\tilde{J} := \Delta_{\gamma,\text{conf}} X = -\tilde{\pi} d\phi + \tilde{E} \lrcorner \tilde{F}. \tag{4.13}$$

Proposition 4.4. *Let $(M^n, \gamma), n \geq 3$, be a closed Riemannian manifold with $\gamma \in W^{2,p}$ and assume that $\phi \in W^{2,p}, \tilde{\pi}, \tilde{E}, \tilde{F} \in W^{1,p}$, with $p > \frac{n}{2}$. Then, $\tilde{J}$ in Equation (4.13) is L^p.*

The proof of the above proposition is a straightforward application of the Sobolev multiplication properties. We now aim to analyse a generic scalar equation on a closed Riemannian manifold $M^n, \gamma, \gamma \in W^{2,p}$, of the form

$$\Delta_\gamma \varphi = \sum_I a_I \varphi^I, \tag{4.14}$$

where the exponents I determine the type of non-linearities present in a specific problem, and we assume $a_I \in L^p$.

4.2.1 The Monotone Iteration Scheme

Here we describe an iterative method based on the existence of barrier functions. In particular, they were introduced by Isenberg (1995) to analyse the Lichnerowicz equa- tion associated to vacuum CMC initial data.

Lemma 4.1 (Weak Maximum Principle). *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$ and $p > \frac{n}{2}$. Consider a function $V \in L^p$ and assume that $V \geq 0$. Then, given $\varphi \in W^{2,p}$ the following implication holds*

$$\Delta_\gamma \varphi - V\varphi \geq 0 \rightarrow \varphi \leq 0. \tag{4.15}$$

The above maximum principle is robust enough to allow us to establish the monotone iteration scheme which is used in the analysis of CMC semi-linear equations. Nevertheless, for geometric problems, we sometimes need a stronger version which excludes the possibility of φ vanishing.

Lemma 4.2 (Strong Maximum Principle). *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$ and $p > \frac{n}{2}$. Consider a function $V \in L^p$ and assume that $V \geq 0$. Then, given $\varphi \in W^{2,p}$ satisfying the inequality*

$$\Delta_\gamma \varphi - V\varphi \geq 0, \tag{4.16}$$

if $\varphi(x) = 0$ for some $x \in M$, then $\varphi \equiv 0$.

Let us now introduce the following concepts concerning barriers of a CMC equation. First, let us define

$$\begin{aligned} f &: M \times \mathcal{I} \mapsto \mathbb{R}, \\ (x, y) &\mapsto f(x, y) := \sum_I a_I(x)\, y^I, \end{aligned} \tag{4.17}$$

where $\mathcal{I} \subseteq \mathbb{R}$ stands for an interval, and the coefficients $a_I \in L^p$. Also assume that $\partial_y f(x, y)$ exists and is continuous on $\mathcal{I}$. Notice that this is an imposition on $\mathcal{I}$ rather than on f, since, due to the form of f, this is satisfied by any interval $\mathcal{I} = [l, m] \subseteq \mathbb{R}^+$ with $l > 0$.

Definition 4.2. *Let (M^n, γ) be a Riemannian manifold with $\gamma \in W^{2,p}$ and $p > \frac{n}{2}$. We say that varphi$_- \in W^{2,p}$ is a **subsolution** of the equation $\Delta_\gamma \varphi = f(x, \varphi)$ if*

$$\Delta_\gamma \varphi_- \geq f(x, \varphi_-). \tag{4.18}$$

*Analogously, varphi$_+ \in W^{2,p}$ is a **supersolution** of the equation if*

$$\Delta_\gamma \varphi_+ \leq f(x, \varphi_+). \tag{4.19}$$

Theorem 4.2. *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$ and $p > \frac{n}{2}$. Consider the equation $\Delta_\gamma \varphi = f(x, \varphi)$ with f given in*

$$\Delta_\gamma \varphi_{k+1} - a\varphi_{k+1} = f(x, \varphi + k) - a\varphi_k. \tag{4.20}$$

If this equation admits a pair of $W^{2,p}$ sub and supersolutions $0 < l \leq \varphi_- \leq \varphi_+ \leq m$ with $[l, m] \in \mathcal{I}$, then there is a solution $\varphi \in W^{2,p}$ satisfying $\varphi_- \leq \varphi \leq \varphi_+$.

The above theorem will be our main tool when proving existence results for the Lichnerowicz equation. Therefore, we see that our task will be reduced to finding suitable barrier functions $\varphi_- \leq \varphi_+$ to our associated equation. In doing so, we will see that the behaviour of the linear term a_r in Equation (4.14) plays a particularly special role. Therefore, certain classification results concerning conformal deformations of scalar curvature are specially useful, which motivates the analysis presented in the next section concerning the Yamabe problem.

4.2.2 The Yamabe Classification

Lemma 4.3. *Let M^n, γ be a closed Riemannian manifold with $\gamma \in W^{2,p}, p > \frac{n}{2}$, and $n \geq 3$. Then, the functionals J_γ, q are all bounded from below for any $1 \leq q \leq \frac{n}{n-2}$.*

The following theorem is key in the low-regularity Yamabe classification.

Theorem 4.3. *Let M^n, γ be a closed Riemannian manifold with $\gamma \in W^{2,p}, p > \frac{n}{2}$, and $n \geq 3$. Then, there exists a $W^{2,p}$ function $\varphi > 0$ such that*

$$-a_n \Delta_\gamma \varphi + R_\gamma \varphi = \lambda_\gamma \varphi. \tag{4.21}$$

In particular, γ is conformal to a metric with continuous scalar curvature having the same sing as λ_γ.

We now state the Yamabe classification.

Theorem 4.4. *Let M^n, γ be a closed Riemannian manifold with $\gamma \in W^{2,p}, p > \frac{n}{2}$, and $n \geq 3$. Then, the following statements hold:*

1. *$\mathcal{Y}([\gamma]) > 0$ if and only if γ is conformal to a metric of continuous positive scalar curvature;*

2. *$\mathcal{Y}([\gamma]) = 0$ if and only if γ is conformal to a metric of continuous zero scalar curvature;*

3. *$\mathcal{Y}([\gamma]) < 0$ if and only if γ is conformal to a metric of continuous negative scalar curvature,*

where in the three cases above the conformal deformation is of the form $g = \varphi^{\frac{4}{n-2}}$, with $\varphi \in W^{2,p}$.

4.2.3 Non-Existence and Uniqueness

The following theorem concerns some straightforward non-existence results.

Theorem 4.5 (Non-Existence)**.** *Let M^n, γ be a closed Riemannian manifold with $\gamma \in W^{2,p}, p > \frac{n}{2}$, and $n \geq 3$. Consider the Lichnerowicz equation (4.5). If all the coefficients are in L^1, then, if either of the following situations*

1. *$a_r, a_\tau \geq 0$ and $a_{TT}, a_E, a_{\tilde{F}} \leq 0$;*

2. *$a_r, a_\tau \leq 0$ and $a_{TT}, a_E, a_{\tilde{F}} \geq 0$,*

and not all of these coefficients vanish identically. Then, the above equation admits no positive solutions.

Let us now present the following uniqueness result, which makes use of the geometric origin of Lichnerowicz's equation.

Theorem 4.6 (Uniqueness). *Let M^n, γ be a closed Riemannian manifold with $\gamma \in W^{2,p}, p > \frac{n}{2}$, and assume that the coefficients of Equation (4.5) satisfy the hypotheses of Proposition 4.4. Suppose, furthermore, that $n \leq 6$ and $a_\tau \geq 0$, and let φ_1 and φ_2 be two positive $W^{2,p}$-solution of Equation (4.5), then either $\varphi_1 \equiv \varphi_2$ or $a_{TT}, a_\tau, a_E, a_{\tilde{F}} \equiv 0, \mathcal{SY}([y]) = 0$ and $\varphi_1 = c\varphi_2$ for some constant $c > 0$.*

Remark 4.1. *Notice that in the above theorem the dimensional restriction $n \leq 6$ relates only to the magnetic term $|\tilde{F}|^2_\gamma$.*

5 Conclusion

In this article, we discussed the problem with Einstein constraint equations, with special investigation in vacuum Einstein constraint case with CMC methods. The far from CMC solutions for Einstein constraint equations is still a current area of research of many mathematicians and open for investigation.

References

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